Journal of Mathematical Inequalities
Volume 12, Number 1 (2018), 63–75 doi:10.7153/jmi-2018-12-05

NEW REFINEMENTS OF THE ERDÖS–MORDELL INEQUALITY

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we present further extensions of two known refinements of the Erdös-Mordell inequality. Several other new refinements of the Erdös-Mordell inequality are established as well. Some closely related interesting conjectures which have been checked by computer are also proposed.

1. Introduction

Given triangle ABC and its interior point P, let R_1, R_2, R_3 denote the distances of P from the vertices A, B, C and from the sides BC, CA, AB by r_1, r_2, r_3, respectively. Then the famous Erdös-Mordell inequality states that

$$\sum R_1 \geqslant 2\sum r_1, \tag{1.1}$$

where $\sum$ denote the cyclic sums over the triples (R_1, R_2, R_3) and (r_1, r_2, r_3). Equality in (1) holds if and only if $\triangle ABC$ is equilateral and P is its center.

Since this inequality posed by Erdös in [5] in 1935, various proofs are given. See [1], [3], [8], [9], [13], and [20] for example. On the other hand, many generalizations are established from different directions (cf. [2], [4], [6], [7], [14–19], [21–23] and the references therein). However, the refinements and sharpness of (1.1) rarely appeared in the literature. Recently, the author of this paper obtained some related results, see [9], [11] and [12].

In [12], Theorem 4.3 and 4.4 provide the following refinements of the Erdös-Mordell inequality:

$$\sum R_1 \geqslant \sqrt{\sum\left[R_1^2 + 2R_1r_1 + (r_2+r_3)^2\right]} \geqslant 2\sum r_1, \tag{1.2}$$

$$\begin{aligned}\sum R_1 &\geqslant \sqrt{\sum (R_2+R_3)(R_1+r_1)} \geqslant \sqrt{\frac{4}{3}\sum (R_2+r_2)(R_3+r_3)}\\ &\geqslant 2\sum r_1,\end{aligned} \tag{1.3}$$

respectively.

Mathematics subject classification (2010): 51M16.
Keywords and phrases: Erdös-Mordell inequality, triangle, interior point.

The author also obtained two refinements with one parameter for the Erdös-Mordell inequality (see [12, Theorem 4.1 and 4.2]). Moreover, some other refinements involving elements such as sides, altitudes, and medians etc. of a triangle are presented by the author in [12] and [9].

In this work we focus our attention on the refinements of the Erdös-Mordell inequality, which only concern six segments $R_1, R_2, R_3, r_1, r_2, r_3$. We shall give further extensions of (1.2) and (1.3) and establish several new refinements of the Erdös-Mordell. In the last section, we shall put forward some related interesting conjectures.

2. Lemmas

The following first lemma is well-known and important. For the proofs, see [1], [3], [4], and [8] for example.

LEMMA 1. *Let a,b,c denote the side lengths of the triangle ABC ($a = BC$, $b = CA$, $c = AB$) and let O denotes the circumcenter of ABC, then for an arbitrary point P inside triangle ABC*

$$aR_1 \geqslant br_3 + cr_2, \tag{2.1}$$

with equality if and only if P lies on the segment AO.

LEMMA 2. ([12]) *For any interior point P of the triangle ABC the following inequality holds:*

$$\sum R_1^2 \geqslant \sum R_1(r_2 + r_3). \tag{2.2}$$

Equality occurs only when $\triangle ABC$ is equilateral and P is its center.

LEMMA 3. ([18,22]) *For any interior point P of the triangle ABC the following inequality holds:*

$$\sum R_1 r_1 \geqslant 2\sum r_2 r_3, \tag{2.3}$$

with equality if and only if P coincide with one vertex of $\triangle ABC$ or $\triangle ABC$ is equilateral and P is its center.

LEMMA 4. ([12]) *For any interior point P of the triangle ABC the following double inequality holds:*

$$\sum R_2 R_3 \geqslant \frac{1}{2}\sum R_1(2r_1 + r_2 + r_3) \geqslant \sum (r_3 + r_1)(r_1 + r_2). \tag{2.4}$$

Equalities in (2.4) hold if and only if $\triangle ABC$ is equilateral and P is its center.

REMARK 1. It is easily verified the following identity:

$$\sum R_1(2r_1 + r_2 + r_3) = \sum (R_2 + R_3)(r_2 + r_3), \tag{2.5}$$

which means that (2.4) is equivalent to

$$\sum R_2R_3 \geqslant \frac{1}{2}\sum(R_2+R_3)(r_2+r_3) \geqslant \sum(r_3+r_1)(r_1+r_2). \tag{2.6}$$

LEMMA 5. ([10]) *Let P be an interior point of the triangle ABC, then*

$$R_2+R_3 \geqslant 2r_1+\frac{(r_2+r_3)^2}{R_1}, \tag{2.7}$$

with equality if and only if $CA=AB$ and P is the circumcenter of the triangle ABC.

LEMMA 6. *Let P be an interior point of the triangle ABC, then we have*

$$(R_2+R_3)(R_1+2r_1) \geqslant (2r_1+r_2+r_3)^2, \tag{2.8}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

Proof. Using inequality (2.7) and the Erdös-Mordell inequality (1.1), we have

$$\begin{aligned}(R_2+R_3)(R_1+2r_1) &= R_1(R_2+R_3)+2r_1(R_2+R_3)\\ &\geqslant 2R_1r_1+(r_2+r_3)^2+2r_1(R_2+R_3)\\ &= 2r_1\sum R_1+(r_2+r_3)^2\\ &\geqslant 4r_1\sum r_1+(r_2+r_3)^2\\ &= (2r_1+r_2+r_3)^2,\end{aligned}$$

as required.

In view of the equality conditions of (1.1) and (2.7), we conclude that the equality condition of (2.8) is the same as that of (1.1). The proof of Lemma 6 is completed. □

Incidentally, inequality (2.8) can also be easily proved by using Lemma 1.

REMARK 2. By Lemma 6 and the Erdös-Mordell inequality (1.1), we immediately obtain the following two inequalities:

$$\sum\frac{(2r_1+r_2+r_3)^2}{R_2+R_3} \leqslant 2\sum R_1, \tag{2.9}$$

$$\sum\frac{(2r_1+r_2+r_3)^2}{R_1+2r_1} \leqslant 2\sum R_1. \tag{2.10}$$

The first inequality is weaker than the conjectured inequality (4.1) in [11], till can be viewed as a sharpness of the Erdös-Mordell inequality (cf. [11]). The second inequality is a special case of Theorem 3 in [11].

In addition, by inequality (2.7) and its two analogues we immediately obtain the following symmetric inequality.

LEMMA 7. *For any interior point P of the triangle ABC*

$$\sum R_2R_3 \geqslant \sum R_1r_1 + \sum r_1^2 + \sum r_2r_3, \tag{2.11}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

REMARK 3. Inequality (2.11) is equivalent to the inequality (4.12) in [12] and equivalent to

$$\sum (R_1 - r_2 - r_3)(R_2 + R_3 + r_2 + r_3) \geqslant 0. \tag{2.12}$$

3. Main results

For simplicity, we shall not mention the equality conditions in the following theorems. But we wish to point out that all the inequalities in our theorems hold if and only if the $\triangle ABC$ is equilateral and P is its center (the equality conditions are in fact easily determined).

3.1. Further refinements of (1.2) and (1.3)

For inequality chain (1.2), we have the following further refinements:

THEOREM 1. *For any interior point P of the triangle ABC, we have*

$$\begin{aligned}\sum R_1 &\geqslant \sqrt{\sum \left[R_1^2 + 2R_1r_1 + (r_2+r_3)^2\right]} \geqslant \sum \sqrt{R_1(r_2+r_3)} \\ &\geqslant 2\sum r_1.\end{aligned} \tag{3.1}$$

Proof. Since the first inequality in (3.1) has already been proved in [12], we only need to prove the second inequality and third one in (3.1). We have the following identity:

$$\sum \left[R_1^2 + 2R_1r_1 + (r_2+r_3)^2\right] - 2\sum R_1 \sum r_1 = \sum (R_1 - r_2 - r_3)^2, \tag{3.2}$$

which is easily checked by expanding both sides. Then

$$\sqrt{\sum \left[R_1^2 + 2R_1r_1 + (r_2+r_3)^2\right]} \geqslant \sqrt{2\sum R_1 \sum r_1} \geqslant \sum \sqrt{R_1(r_2+r_3)},$$

where the last inequality is obtained by use of the Cauchy-Schwarz inequality. This completes the proof of the second inequality in (3.1).

In the sequel, we let the sign $\sum$ denote cyclic sums over the triples (R_1,R_2,R_3), (r_1,r_2,r_3) and (a,b,c).

Using Lemma 1, the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}\sum \sqrt{R_1(r_2+r_3)} &\geqslant \sum \sqrt{\frac{(r_2+r_3)(cr_2+br_3)}{a}} \geqslant \sum \frac{\sqrt{c}r_2 + \sqrt{b}r_3}{\sqrt{a}} \\ &= \sum \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right) r_1 \geqslant 2\sum r_1.\end{aligned}$$

This completes the proof of the last inequality in (3.1) and we finish the proof of Theorem 1. □

Another further refinement of (1.2) is as follows.

THEOREM 2. *For any interior point P of the triangle ABC, we have*

$$\sum R_1 \geqslant \sqrt{\sum\left[R_1^2+2R_1r_1+(r_2+r_3)^2\right]} \geqslant \sqrt{\sum(r_2+r_3)(R_2+R_3+r_2+r_3)}$$
$$\geqslant \sum\sqrt{\frac{1}{2}(r_2+r_3)(R_1+r_2+r_3)} \geqslant 2\sum r_1. \tag{3.3}$$

Proof. Since the double inequality (1.2) holds, we only need to prove the remaining inequalities of (3.3).

A short calculation gives

$$\sum\left[R_1^2+2R_1r_1+(r_2+r_3)^2\right]-\sum(r_2+r_3)(R_2+R_3+r_2+r_3)=\sum R_1^2-\sum R_1(r_2+r_3). \tag{3.4}$$

Thus, by Lemma 2, we know that the second inequality in (3.3) holds.

We now prove the third inequality in (3.3). Applying the Cauchy-Schwarz inequality, we get

$$\sum\sqrt{\frac{1}{2}(r_2+r_3)(R_1+r_2+r_3)} \leqslant \sqrt{\sum r_1\left(\sum R_1+2\sum r_1\right)}. \tag{3.5}$$

On the other hand, we have the following identity:

$$\sum(r_2+r_3)(R_2+R_3+r_2+r_3)-\sum r_1\left(\sum R_1+2\sum r_1\right)=\sum R_1r_1-2\sum r_2r_3, \tag{3.6}$$

which is easily obtained by expanding. Thus, by Lemma 3 we deduce that

$$\sum(r_2+r_3)(R_2+R_3+r_2+r_3) \geqslant \sum r_1\left(\sum R_1+2\sum r_1\right), \tag{3.7}$$

which together with (3.5) implies that the third inequality of (3.3) is true.

Finally, we prove the last inequality in (3.3). Applying Lemma 1, the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
\sum\sqrt{\frac{1}{2}(r_2+r_3)(R_1+r_2+r_3)} &\geqslant \frac{1}{\sqrt{2}}\sum\sqrt{(r_2+r_3)\left(\frac{br_3+cr_2}{a}+r_2+r_3\right)}\\
&= \frac{1}{\sqrt{2}}\sum\frac{\sqrt{(r_2+r_3)\left[(a+c)r_2+(a+b)r_3\right]}}{\sqrt{a}}\\
&\geqslant \frac{1}{\sqrt{2}}\sum\frac{r_2\sqrt{a+c}+r_3\sqrt{a+b}}{\sqrt{a}}\\
&= \frac{1}{\sqrt{2}}\sum\sqrt{b+c}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)r_1\\
&\geqslant \frac{1}{\sqrt{2}}\sum\left(2\sqrt{bc}\right)^{1/2}\cdot 2\left(\sqrt{bc}\right)^{-1/2}r_1\\
&= 2\sum r_1,
\end{aligned}$$

as required. The proof of Theorem 2 is completed. □

REMARK 4. We have know that there is no strict comparison between the third term of (3.1) and the third one of (3.3). But it seems likely that the third term of (3.1) is greater or equal to the fourth one of (3.3) (see Conjecture 1 in the last section).

In the following, we state and prove a further refinement of (1.3).

THEOREM 3. *For any interior point P of the triangle ABC, we have*

$$\begin{aligned}\sum R_1 &\geqslant \sqrt{\sum (R_2+R_3)(R_1+r_1)} \geqslant \sqrt{\frac{4}{3}\sum (R_2+r_2)(R_3+r_3)}\\ &\geqslant \frac{2}{3}\sum\sqrt{(R_2+r_2)(R_3+r_3)} \geqslant 2\sqrt[3]{(R_1+r_1)(R_2+r_2)(R_3+r_3)}\\ &\geqslant 2\sum r_1. \end{aligned} \tag{3.8}$$

Proof. Note that inequality chain (1.3) has been proved in [12]. Also, it is obvious that the third inequality and the fourth one of (3.8) immediately follow from the power mean inequality and the arithmetic-geometric mean inequality, respectively. So, it remains to prove the last inequality which is equivalent to

$$(R_1+r_1)(R_2+r_2)(R_3+r_3) \geqslant \left(\sum r_1\right)^3. \tag{3.9}$$

For proving this inequality, by Lemma 1 it is suffices to prove that

$$\left(\frac{br_3+cr_2}{a}+r_1\right)\left(\frac{cr_1+ar_3}{b}+r_2\right)\left(\frac{ar_2+br_1}{c}+r_3\right)-\left(\sum r_1\right)^3 \geqslant 0,$$

or equivalently

$$(ar_1+br_3+cr_2)(br_2+cr_1+ar_3)(cr_3+ar_2+br_1)-abc\left(\sum r_1\right)^3 \geqslant 0.$$

Expanding out and rearranging, one further knows again that it is equivalent to

$$\begin{aligned}&\sum\left[(ab^2+bc^2+ca^2-3abc)r_2+(ba^2+cb^2+ac^2-3abc)r_3\right]r_1^2\\ &+\sum(a^3+b^3+c^3-3abc)r_1r_2r_3 \geqslant 0.\end{aligned} \tag{3.10}$$

But, from the arithmetic-geometric mean inequality, we have

$$\begin{aligned}ab^2+bc^2+ca^2-3abc &\geqslant 0,\\ ba^2+cb^2+ac^2-3abc &\geqslant 0,\\ a^3+b^3+c^3-3abc &\geqslant 0.\end{aligned}$$

These three inequalities and the fact that $r_1 \geqslant 0$ etc., shows that (3.10) holds. So we have finished the proofs of (3.9) and the last inequality in (3.8). The proof of Theorem 3 is completed. □

3.2. Some new results

We first point out that the following refinement of the Erdös-Mordell inequality can be obtained by the previous inequalities. That is

THEOREM 4. *For any interior point P of the triangle ABC, we have*

$$\begin{aligned}\sum R_1 &\geqslant \sqrt{\sum R_1(R_2+R_3+r_2+r_3)}\\ &\geqslant \sqrt{\frac{1}{2}\sum(R_1+r_2+r_3)(R_2+R_3+r_2+r_3)}\\ &\geqslant \sqrt{\sum(r_2+r_3)(R_2+R_3+r_2+r_3)} \geqslant \sum\sqrt{\frac{1}{2}(r_2+r_3)(R_1+r_2+r_3)}\\ &\geqslant 2\sum r_1. \end{aligned} \tag{3.11}$$

Proof. It is clear that

$$\sum(R_2+R_3)(R_1+r_1) = \sum R_1(R_2+R_3+r_2+r_3). \tag{3.12}$$

Thus, the first inequality in (3.11) is equivalent with the first one of (1.3) and valid. From the preceding inequality (2.12), one sees that the second inequality and the third one of (3.11) hold. The last two inequalities of (3.11) are the same as in (3.3) and have been already proved. The proof of Theorem 4 is completed. □

As a direct consequence of Theorem 4, one has

$$\sum R_1 \geqslant \sqrt{\frac{1}{2}\sum(R_1+r_2+r_3)(R_2+R_3+r_2+r_3)} \geqslant 2\sum r_1. \tag{3.13}$$

Moreover, we find that this double inequality can be extended to the following case with one parameter.

THEOREM 5. *For any interior point P of the triangle ABC and non-negative number k, we have*

$$\sum R_1 \geqslant \sqrt{\frac{1}{k+1}\sum(kR_1+r_2+r_3)(R_2+R_3+r_2+r_3)} \geqslant 2\sum r_1. \tag{3.14}$$

Proof. Note that the first inequality in (3.11). For proving the first inequality in (3.14), it is suffices to prove that

$$\sqrt{\sum R_1(R_2+R_3+r_2+r_3)} \geqslant \sqrt{\frac{1}{k+1}\sum(kR_1+r_2+r_3)(R_2+R_3+r_2+r_3)}.$$

Since $k \geqslant 0$, we only need to prove that

$$(k+1)\sum R_1(R_2+R_3+r_2+r_3) - \sum(kR_1+r_2+r_3)(R_2+R_3+r_2+r_3) \geqslant 0,$$

which is simplified to the previous inequality (2.12). Hence the first inequality in (3.14) is proved.

In order to prove the second inequality of (3.14), we have to show that for $k \geqslant 0$

$$\sum (kR_1 + r_2 + r_3)(R_2 + R_3 + r_2 + r_3) - 4(k+1)\left(\sum r_1\right)^2 \geqslant 0,$$

which can be written in the equivalent form

$$\begin{aligned} &\sum R_1(2r_1 + r_2 + r_3) - 2\sum (r_3 + r_1)(r_1 + r_2) \\ &+k\left[\sum (R_2 + R_3)(R_1 + r_1) - 4\left(\sum r_1\right)^2\right] \geqslant 0. \end{aligned} \tag{3.15}$$

Now, from the previous inequality chain (1.3), we deduce that

$$\sum (R_2 + R_3)(R_1 + r_1) \geqslant 4\left(\sum r_1\right)^2, \tag{3.16}$$

which together with the second inequality of (2.4) imply that (3.15) holds for $k \geqslant 0$. Thus, the second inequality in (3.14) is proved and the proof of Theorem 5 is completed. □

The special case $k = 1$ of (3.14) reduces to (3.13). For $k = 0$ in (3.14), we get

$$\sum R_1 \geqslant \sqrt{\sum (r_2 + r_3)(R_2 + R_3 + r_2 + r_3)} \geqslant 2\sum r_1, \tag{3.17}$$

which can also be obtained from Theorem 2 or Theorem 4.

Motivated by inequality chain (3.17), the author finds the following similar refinement of the Erdös-Mordell inequality:

$$\sum R_1 \geqslant \sqrt{\frac{1}{2}\sum (R_2 + R_3)(R_2 + R_3 + r_2 + r_3)} \geqslant 2\sum r_1. \tag{3.18}$$

At first, the author gave a complicated proof for the second inequality of (3.18). Later, two extensions of (3.18) are found and then the simpler proofs of (3.18) are obtained. Now, we give the first extension of (3.18) as follows.

THEOREM 6. *For any interior point P of the triangle ABC, we have*

$$\begin{aligned} \sum R_1 &\geqslant \sqrt{\frac{1}{2}\sum (R_2 + R_3)(R_2 + R_3 + r_2 + r_3)} \\ &\geqslant \frac{1}{2}\sum \sqrt{(R_2 + R_3)(R_1 + 2r_1)} \geqslant 2\sum r_1. \end{aligned} \tag{3.19}$$

Proof. Firstly, one may check the following identity:

$$\begin{aligned} &2\left(\sum R_1\right)^2 - \sum (R_2 + R_3)(R_2 + R_3 + r_2 + r_3) \\ &= 2\sum R_2R_3 - \sum (R_2 + R_3)(r_2 + r_3). \end{aligned} \tag{3.20}$$

Thus, by the first inequality of (2.6) we see that the first inequality (3.19) holds.

Applying the Cauchy-Schwarz inequality, we obtain

$$\sum\sqrt{(R_2+R_3)(R_1+2r_1)} \leqslant \sqrt{2\sum R_1\left(\sum R_1+2\sum r_1\right)}. \tag{3.21}$$

So, to prove the second inequality in (3.19) it is sufficient to prove that

$$\sum R_1\left(\sum R_1+2\sum r_1\right) \leqslant \sum(R_2+R_3)(R_2+R_3+r_2+r_3). \tag{3.22}$$

Again, one may check that

$$\begin{aligned}&\sum(R_2+R_3)(R_2+R_3+r_2+r_3)-\sum R_1\left(\sum R_1+2\sum r_1\right)\\&=\sum R_1^2-\sum R_1(r_2+r_3).\end{aligned} \tag{3.23}$$

Thus, by Lemma 2 we deduce that (3.22) holds and then the second inequality in (3.19) is proved.

According to Lemma 6, we immediately obtain

$$\sum\sqrt{(R_2+R_3)(R_1+2r_1)} \geqslant 2\sum(2r_1+r_2+r_3)=4\sum r_1,$$

which shows that third inequality in (3.19) is proved. The proof of Theorem 6 is completed. □

Finally, we give another extension of (3.18) as follows.

THEOREM 7. *For any interior point P of the triangle ABC, we have*

$$\begin{aligned}\sum R_1 &\geqslant \sqrt{\frac{1}{2}\sum(R_2+R_3)(R_2+R_3+r_2+r_3)}\\&\geqslant \sum\sqrt{\frac{1}{2}R_1(R_1+r_2+r_3)} \geqslant \sqrt{\frac{1}{2}\left(\sum R_1\right)^2+2\left(\sum r_1\right)^2}\\&\geqslant \sum\sqrt{\frac{1}{2}(r_2+r_3)(R_1+r_2+r_3)} \geqslant 2\sum r_1.\end{aligned} \tag{3.24}$$

Proof. As we have shown the first inequality of (3.24) in (3.19), it is left to prove the remaining inequalities of (3.24). Applying the Cauchy-Schwarz inequality, we obtain the following identity:

$$\sum\sqrt{\frac{1}{2}R_1(R_1+r_2+r_3)} \leqslant \sqrt{\frac{1}{2}\sum R_1\left(\sum R_1+2\sum r_1\right)}. \tag{3.25}$$

which is similar to (3.22). Coupling (3.25) with (3.21) yields the second inequality in (3.24).

We now prove the third inequality in (3.24), namely

$$\sum\sqrt{\frac{1}{2}R_1(R_1+r_2+r_3)} \geqslant \sqrt{\frac{1}{2}\left(\sum R_1\right)^2+2\left(\sum r_1\right)^2}. \tag{3.26}$$

Squaring both sides gives the next equivalent inequality (required to prove):

$$\frac{1}{2}\sum R_1(r_2+r_3)+\sum\sqrt{R_2R_3(R_2+r_3+r_1)(R_3+r_1+r_2)} \\ \geqslant \sum R_2R_3+2\left(\sum r_1\right)^2. \tag{3.27}$$

By the Cauchy-Schwarz inequality we have

$$\sqrt{(R_2+r_3+r_1)(R_3+r_1+r_2)} \geqslant \sqrt{R_2R_3}+\sqrt{(r_3+r_1)(r_1+r_2)}.$$

Multiplying both sides by $\sqrt{R_2R_3}$ and then summing yields

$$\sum\sqrt{R_2R_3(R_2+r_3+r_1)(R_3+r_1+r_2)} \geqslant \sum R_2R_3+\sum\sqrt{R_2R_3(r_3+r_1)(r_1+r_2)}.$$

Hence, the proof of (3.27) becomes

$$\frac{1}{2}\sum R_1(r_2+r_3)+\sum\sqrt{R_2R_3(r_3+r_1)(r_1+r_2)} \geqslant 2\left(\sum r_1\right)^2. \tag{3.28}$$

Using Lemma 1, the Cauchy-Schwarz inequality, and the arithmetic-geometric mean inequality we have

$$\begin{aligned}
&\frac{1}{2}\sum R_1(r_2+r_3)+\sum\sqrt{R_2R_3(r_3+r_1)(r_1+r_2)} \\
&\geqslant \frac{1}{2}\sum\frac{(br_3+cr_2)(r_3+r_2)}{a}+\sum\frac{\sqrt{(cr_1+ar_3)(ar_2+br_1)(r_1+r_3)(r_2+r_1)}}{\sqrt{bc}} \\
&\geqslant \frac{1}{2}\sum\frac{(\sqrt{b}r_3+\sqrt{c}r_2)^2}{a}+\sum\frac{(\sqrt{c}r_1+\sqrt{a}r_3)(\sqrt{a}r_2+\sqrt{b}r_1)}{\sqrt{bc}} \\
&= \frac{1}{2}\sum\frac{br_3^2+cr_2^2}{a}+\sum\frac{\sqrt{bc}r_2r_3}{a}+\sum\frac{\sqrt{bc}r_1^2+ar_2r_3+\sqrt{ca}r_1r_2+\sqrt{ab}r_3r_1}{\sqrt{bc}} \\
&= \frac{1}{2}\sum\left(\frac{b}{c}+\frac{c}{b}\right)r_1^2+\sum\left(\frac{\sqrt{bc}}{a}+\frac{a}{\sqrt{bc}}\right)r_2r_3+\sum r_1^2+\sum\frac{b+c}{\sqrt{bc}}r_2r_3 \\
&\geqslant \sum r_1^2+2\sum r_2r_3+\sum r_1^2+2\sum r_2r_3 \\
&= 2\left(\sum r_1\right)^2,
\end{aligned}$$

which proves inequality (3.28).

Obviously, by the Erdös-Mordell inequality we have

$$\frac{1}{2}\left(\sum R_1\right)^2+2\left(\sum r_1\right)^2 \geqslant \sum r_1\left(\sum R_1+2\sum r_1\right).$$

Combining this with (3.26) and the previous inequality (3.5), we conclude that fourth inequality in (3.24) holds. Finally, we notice that the last inequality in (3.24) has been proved in Theorem 2. This completes the proof of Theorem 7. □

REMARK 5. There is no strict comparison between the third term of (3.19) and the third one of (3.24), and the fourth one of (3.24).

REMARK 6. If we replace the fifth term of (3.24) by the third one of (3.1), then the inequality chain still holds. In deed, by the simplest power mean inequality and the simplest arithmetic-geometric mean inequality we have

$$\begin{aligned}&\sum\sqrt{\frac{1}{2}\left(\sum R_1\right)^2+2\left(\sum r_1\right)^2}\geqslant\frac{1}{2}\left(\sum R_1+2\sum r_1\right)\\&=\frac{1}{2}\left[\sum R_1+\sum(r_2+r_3)\right]\geqslant\sum\sqrt{R_1(r_2+r_3)}.\end{aligned}$$

Thus, by (3.26) we know that the statement above is true.

REMARK 7. From inequality chain (3.24) we conclude that

$$\sum\left(\sqrt{R_1}-\sqrt{r_2+r_3}\right)\sqrt{R_1+r_2+r_3}\geqslant 0. \tag{3.29}$$

It seems very difficult to give a direct proof of this inequality.

4. Open problems

It is worth noting that there are many interesting inequalities related to the results of this paper should be studied. We introduce some of them in the following.

Motivated by (1.3) and inspired by the proved theorems in this paper, we propose the following inequality chain.

CONJECTURE 1. *If P is any interior point of the triangle ABC, then*

$$\begin{aligned}&\sqrt{\sum(R_2+R_3)(R_1+r_1)}\geqslant\frac{1}{2}\sum\sqrt{(R_2+R_3)(R_1+2r_1)}\\&\geqslant\frac{2}{3}\sum\sqrt{(R_2+r_2)(R_3+r_3)}\geqslant\sum\sqrt{R_1(r_2+r_3)}\\&\geqslant\sum\sqrt{\frac{1}{2}(r_2+r_3)(R_1+r_2+r_3)}.\end{aligned} \tag{4.1}$$

If (4.1) holds true, then we could obtain some new refinements of the Erdös-Mordell inequality by our results.

CONJECTURE 2. *If P is any interior point of the triangle ABC, then*

$$\sum\sqrt{\frac{1}{2}R_1(R_1+r_2+r_3)}\geqslant\sqrt{\frac{4}{3}\sum(R_2+r_2)(R_3+r_3)}. \tag{4.2}$$

If (4.2) is true, then we know that the second term of (3.8) can be replaced by the left hand side of (4.2).

Comparing the middle terms of (3.17) with the one of (3.18), we propose the following conjecture similar to the previous inequality (2.12).

CONJECTURE 3. *If P is any interior point of the triangle ABC, then*

$$\sum(R_2+R_3-2r_2-2r_3)(R_2+R_3+r_2+r_3)\geqslant 0. \tag{4.3}$$

It is well-known that the Erdös-Mordell inequality can be generalized to the case with weights (*cf.* [15, p.318, Theorem 15]):

$$\sum R_1x^2\geqslant 2\sum yzr_1, \tag{4.4}$$

where x,y,z are arbitrary real numbers and $\sum$ denote the cyclic sums over the triples $(R_1,R_2,R_3),(r_1,r_2,r_3)$ and (x,y,z).

We here propose a weighted inequality as follows.

CONJECTURE 4. *For any interior point P of the triangle ABC and all real numbers x,y,z, the following inequality holds:*

$$\sum(R_1+r_1)x^2\geqslant \sum r_1\sum yz. \tag{4.5}$$

Obviously, the special case $x=y=z=1$ of (4.5) implies the Erdös-Mordell inequality. And moreover, a property of ternary quadratic inequalities shows that inequality (3.9) could be obtained immediately from (4.5). It can be stated that if the following ternary inequality (with real coefficients p_1,p_2,p_3,q_1,q_2,q_3):

$$p_1x^2+p_2y^2+p_3z^2\geqslant q_1yz+q_2zx+q_3xy \tag{4.6}$$

holds for all real numbers x,y,z, then

$$p_1p_2p_3\geqslant q_1q_2q_3. \tag{4.7}$$

According to this conclusion, inequality (3.9) follows at once from (4.5) if it is right.

Another similar conjecture should be more studied is the following.

CONJECTURE 5. *For any interior point P of the triangle ABC and all real numbers x,y,z, the following inequality holds:*

$$\sum(R_1+2r_1)x^2\geqslant 2\sum yz(r_2+r_3). \tag{4.8}$$

We observe that if inequality (4.8) is valid then inequality (4.5) is obtained by adding (4.8) with (4.4) and then dividing both sides by 2.

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(Received October 10, 2016)

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