

## A MORE ACCURATE MULTIDIMENSIONAL HARDY–HILBERT TYPE INEQUALITY WITH A GENERAL HOMOGENEOUS KERNEL

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*Abstract.* In this paper, by the use of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of real analysis, a more accurate multidimensional Hardy-Hilbert-type inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of some published results. Moreover, the equivalent forms, the operator expressions and some particular examples are considered.

### 1. Introduction

If  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $a_m, b_n \geq 0$ ,  $a = \{a_m\}_{m=1}^{\infty} \in l^p$ ,  $b = \{b_n\}_{n=1}^{\infty} \in l^q$ ,  $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$ ,  $\|b\|_q > 0$ , then we have the following well known Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q, \quad (1)$$

and the following more accurate Hardy-Hilbert's inequality with the same best possible constant factor  $\frac{\pi}{\sin(\pi/p)}$  (cf. [1], Theorem 315, Theorem 323):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q, \quad (2)$$

Inequalities (1) and (2) are important in analysis and its applications (cf. [1], [2], [3]).

Assuming that  $\{\mu_m\}_{m=1}^{\infty}$  and  $\{v_n\}_{n=1}^{\infty}$  are positive sequences, such that

$$U_m = \sum_{i=1}^m \mu_i, V_n = \sum_{j=1}^n v_j \quad (m, n \in \mathbf{N} = \{1, 2, \dots\}),$$

we have the following Hardy-Hilbert-type inequality (cf. [1], Theorem 321):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} \frac{a_m^p}{m^{p-1}} \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \frac{b_n^q}{n^{q-1}} \right)^{\frac{1}{q}}. \quad (3)$$

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For  $\mu_i = \nu_j = 1$  ( $i, j \in \mathbf{N}$ ), inequality (3) reduces to (1).

In 2015, by using the transfer formula, Yang [4] gave the following multidimensional Hilbert-type inequality: For  $i_0, j_0 \in \mathbf{N}$ ,  $\alpha, \beta > 0$ ,

$$\begin{aligned} \|x\|_\alpha &:= \left( \sum_{k=1}^{i_0} |x^{(k)}|^\alpha \right)^{\frac{1}{\alpha}} \quad (x = (x^{(1)}, \dots, x^{(i_0)}) \in \mathbf{R}^{i_0}), \\ \|y\|_\beta &:= \left( \sum_{k=1}^{j_0} |y^{(k)}|^\beta \right)^{\frac{1}{\beta}} \quad (y = (y^{(1)}, \dots, y^{(j_0)}) \in \mathbf{R}^{j_0}), \end{aligned}$$

$0 < \lambda_1 \leq i_0, 0 < \lambda_2 \leq j_0, \lambda_1 + \lambda_2 = \lambda, a_m, b_n \geq 0$ , we have

$$\begin{aligned} &\sum_n \sum_m \frac{1}{\|m\|_\alpha^\lambda + \|n\|_\beta^\lambda} a_m b_n \\ &< K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \left[ \sum_m \|m\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p \right]^{\frac{1}{p}} \left[ \sum_n \|n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q \right]^{\frac{1}{q}}, \end{aligned} \tag{4}$$

where,  $\Sigma_m = \sum_{m_{i_0}=1}^\infty \dots \sum_{m_1=1}^\infty, \Sigma_n = \sum_{n_{j_0}=1}^\infty \dots \sum_{n_1=1}^\infty$ , the series in the right hand side of (4) are positive values, and the best possible constant factor  $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$  is indicated by

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})}.$$

For  $i_0 = j_0 = \lambda = \alpha = \beta = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ , inequality (4) reduces to (1). Some other results on this type of inequalities and multiple inequalities were provided by [5]–[25].

Recently, by using the weight coefficients, Yang [26] gave an extension of (3) as follows: For  $\eta > 0, 0 < \lambda_1 \leq 1, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda, a_m, b_n \geq 0$ ,

$$\begin{aligned} &\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(U_m^\eta + V_n^\eta)^{\lambda/\eta}} \\ &< \frac{1}{\eta} B\left(\frac{\lambda_1}{\eta}, \frac{\lambda_2}{\eta}\right) \left[ \sum_{m=1}^\infty \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{m^{p-1}} \right]^{\frac{1}{p}} \left[ \sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1} b_n^q}{n^{q-1}} \right]^{\frac{1}{q}}, \end{aligned} \tag{5}$$

where, the constant factor  $\frac{1}{\eta} B(\frac{\lambda_1}{\eta}, \frac{\lambda_2}{\eta})$  is the best possible (the series in the right hand side of (5) are positive values). Another results on Hardy-Hilbert-type inequalities and Hilbert-type inequalities were given by [27]–[37].

In this paper, by the use of the weight coefficients, the transfer formula, Hermite-Hadamard’s inequality and the technique of real analysis, a more accurate multidimensional Hardy-Hilbert’s inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of (4) and (5). Moreover, the equivalent forms, the operator expressions and some particular examples are considered.

2. Some lemmas

If  $\mu_i^{(k)} > 0, 0 \leq \tilde{\mu}_i^{(k)} \leq \frac{1}{2}\mu_i^{(k)}$  ( $k = 1, \dots, i_0; i = 1, \dots, m$ ),  $\nu_j^{(l)} > 0, 0 \leq \tilde{\nu}_j^{(l)} \leq \frac{1}{2}\nu_j^{(l)}$  ( $l = 1, \dots, j_0; j = 1, \dots, n$ ), then we set

$$U_m^{(k)} := \sum_{i=1}^m \mu_i^{(k)}, \quad \tilde{U}_m^{(k)} := U_m^{(k)} - \tilde{\mu}_m^{(k)} \quad (k = 1, \dots, i_0),$$

$$V_n^{(l)} := \sum_{j=1}^n \nu_j^{(l)}, \quad \tilde{V}_n^{(l)} := V_n^{(l)} - \tilde{\nu}_n^{(l)} \quad (l = 1, \dots, j_0),$$

$$U_m := (U_m^{(1)}, \dots, U_m^{(i_0)}), \quad \tilde{\mu}_m := (\tilde{\mu}_m^{(1)}, \dots, \tilde{\mu}_m^{(i_0)}),$$

$$\tilde{U}_m := (\tilde{U}_m^{(1)}, \dots, \tilde{U}_m^{(i_0)}) = U_m - \tilde{\mu}_m,$$

$$V_n := (V_n^{(1)}, \dots, V_n^{(j_0)}), \quad \tilde{\nu}_n := (\tilde{\nu}_n^{(1)}, \dots, \tilde{\nu}_n^{(j_0)}),$$

$$\tilde{V}_n := (\tilde{V}_n^{(1)}, \dots, \tilde{V}_n^{(j_0)}) = V_n - \tilde{\nu}_n \quad (m, n \in \mathbf{N}). \tag{6}$$

We also set functions  $\mu_k(t) := \mu_m^{(k)}, t \in (m - \frac{1}{2}, m + \frac{1}{2}]$  ( $m \in \mathbf{N}$ );  $\nu_l(t) := \nu_n^{(l)}, t \in (n - \frac{1}{2}, n + \frac{1}{2}]$  ( $n \in \mathbf{N}$ ), and

$$U_k(x) := \int_{\frac{1}{2}}^x \mu_k(t) dt \quad (k = 1, \dots, i_0),$$

$$V_l(y) := \int_{\frac{1}{2}}^y \nu_l(t) dt \quad (l = 1, \dots, j_0), \tag{7}$$

$$U(x) := (U_1(x), \dots, U_{i_0}(x)),$$

$$V(y) := (V_1(y), \dots, V_{j_0}(y)) \quad (x, y \geq \frac{1}{2}). \tag{8}$$

It follows that

$$U_k(m) = \int_{\frac{1}{2}}^m \mu_k(t) dt = \int_{\frac{1}{2}}^{m+\frac{1}{2}} \mu_k(t) dt - \frac{1}{2}\mu_m^{(k)}$$

$$\leq \tilde{U}_m^{(k)} \leq U_k\left(m + \frac{1}{2}\right) \quad (k = 1, \dots, i_0; m \in \mathbf{N}),$$

$$V_l(n) \leq \tilde{V}_n^{(l)} \leq V_l\left(n + \frac{1}{2}\right) \quad (l = 1, \dots, j_0; n \in \mathbf{N}),$$

and for  $x \in (m - \frac{1}{2}, m + \frac{1}{2})$ ,  $U'_k(x) = \mu_k(x) = \mu_m^{(k)}$  ( $k = 1, \dots, i_0; m \in \mathbf{N}$ ); for  $y \in (n - \frac{1}{2}, n + \frac{1}{2})$ ,  $V'_l(y) = \nu_l(y) = \nu_n^{(l)}$  ( $l = 1, \dots, j_0; n \in \mathbf{N}$ ).

LEMMA 1. (cf. [31]) Suppose that  $g(t) (> 0)$  is strictly decreasing and strictly convex in  $(\frac{1}{2}, \infty)$ , satisfying  $\int_{\frac{1}{2}}^\infty g(t) dt \in \mathbf{R}_+$ . We have the following Hermite-Hadamard's inequality

$$\int_n^{n+1} g(t) dt < g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t) dt \quad (n \in \mathbf{N}), \tag{9}$$

and then

$$\int_1^\infty g(t)dt < \sum_{n=1}^\infty g(n) < \int_{\frac{1}{2}}^\infty g(t)dt. \tag{10}$$

LEMMA 2. If  $i_0 \in \mathbf{N}$ ,  $\alpha, M > 0$ ,  $\Psi(u)$  is a non-negative measurable function in  $(0, 1]$ , and

$$D_M := \left\{ x = (x_1, \dots, x_{i_0}) \in \mathbf{R}_+^{i_0}; u = \sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha \leq 1 \right\}, \tag{11}$$

then we have the following transfer formula (cf. [5]):

$$\int \dots \int_{D_M} \Psi \left( \sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha \right) dx_1 \dots dx_{i_0} = \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 \Psi(u) u^{\frac{i_0}{\alpha}-1} du. \tag{12}$$

LEMMA 3. For  $i_0, j_0 \in \mathbf{N}$ ,  $\alpha, \beta, \varepsilon > 0$ ,  $\mu_m^{(k)} \geq \mu_{m+1}^{(k)}$  ( $m \in \mathbf{N}$ ;  $k = 1, \dots, i_0$ ),  $\mathbf{v}_n^{(l)} \geq \mathbf{v}_{n+1}^{(l)}$  ( $n \in \mathbf{N}$ ;  $l = 1, \dots, i_0$ ),  $c = \min_{1 \leq i \leq i_0, 1 \leq j \leq j_0} \{\mu_1^{(i)}, \mathbf{v}_1^{(j)}\} (> 0)$ , we have

$$\sum_m \|\tilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^\varepsilon \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + O(1), \tag{13}$$

$$\sum_n \|\tilde{V}_n\|_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \mathbf{v}_n^{(k)} \leq \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon c^\varepsilon j_0^\varepsilon \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \tilde{O}(1) \ (\varepsilon \rightarrow 0^+). \tag{14}$$

*Proof.* For  $M > ci_0^{1/\alpha}$ , we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{c^\alpha i_0}{M^\alpha}, \\ \frac{1}{(Mu^{1/\alpha})^{i_0+\varepsilon}}, & \frac{c^\alpha i_0}{M^\alpha} \leq u \leq 1. \end{cases}$$

By (12), it follows that

$$\begin{aligned} \int_{\{x \in \mathbf{R}_+^{i_0}; x_i \geq c\}} \frac{dx}{\|x\|_\alpha^{i_0+\varepsilon}} &= \lim_{M \rightarrow \infty} \int \dots \int_{D_M} \Psi \left( \sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha \right) dx_1 \dots dx_{i_0} \\ &= \lim_{M \rightarrow \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_{c^\alpha i_0/M^\alpha}^1 \frac{u^{\frac{i_0}{\alpha}-1}}{(Mu^{1/\alpha})^{i_0+\varepsilon}} du \\ &= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^\varepsilon \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})}. \end{aligned}$$

Then by (10) and the above result, in view of  $U_k(m) \leq \tilde{U}_m^{(k)}$ , we find

$$\begin{aligned}
 0 &< \sum_{\{m \in \mathbf{N}^{i_0}; m_i \geq 2\}} \|\tilde{U}_m\| \alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
 &\leq \sum_{\{m \in \mathbf{N}^{i_0}; m_i \geq 2\}} \int_{\{x \in \mathbf{R}_+^{i_0}; m_i - \frac{1}{2} \leq x_i < m_i + \frac{1}{2}\}} \|U(m)\| \alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} dx \\
 &< \sum_{\{m \in \mathbf{N}^{i_0}; m_i \geq 2\}} \int_{\{x \in \mathbf{R}_+^{i_0}; m_i - \frac{1}{2} \leq x_i < m_i + \frac{1}{2}\}} \|U(x)\| \alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_k(x) dx \\
 &= \int_{\{x \in \mathbf{R}_+^{i_0}; x_i \geq \frac{3}{2}\}} \|U(x)\| \alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_k(x) dx \\
 &\stackrel{v=U(x)}{=} \int_{\{v \in \mathbf{R}_+^{i_0}; v_i \geq \mu_1^{(i)}\}} \|v\| \alpha^{-i_0 - \varepsilon} dv \leq \int_{\{v \in \mathbf{R}_+^{i_0}; v_i \geq c\}} \|v\| \alpha^{-i_0 - \varepsilon} dv \\
 &= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})}.
 \end{aligned}$$

For  $i_0 = 1$ ,  $0 < \sum_{\{m \in \mathbf{N}^{i_0}; m_i = 1\}} \|\tilde{U}_m\| \alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leq (\mu_1^{(1)})^{-\varepsilon} < \infty$ ; for  $i_0 \geq 2$ , we set

$$H_i := \sum_{\{m \in \mathbf{N}^{i_0}; m_i = 1\}} \|\tilde{U}_m\| \alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \quad (i = 1, \dots, i_0).$$

Without lose of generality, we estimate  $H_{i_0}$  as follows:

$$\begin{aligned}
 H_{i_0} &\leq \sum_{\{m \in \mathbf{N}^{i_0}; m_{i_0} = 1\}} \|U(m)\| \alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
 &= \mu_1^{(i_0)} \sum_{m \in \mathbf{N}^{i_0 - 1}} \frac{\prod_{k=1}^{i_0 - 1} \mu_m^{(k)}}{[\sum_{i=1}^{i_0 - 1} U_i^\alpha(m) + (\frac{1}{2} \mu_1^{(i_0)}) \alpha]^{\frac{1}{\alpha}(i_0 + \varepsilon)}} \\
 &< \sum_{m \in \mathbf{N}^{i_0 - 1}} \int_{\{x \in \mathbf{R}_+^{i_0 - 1}; m_i - \frac{1}{2} \leq x_i < m_i + \frac{1}{2}\}} \frac{\mu_1^{(i_0)} \prod_{k=1}^{i_0 - 1} \mu_k(x) dx}{[\sum_{i=1}^{i_0 - 1} U_i^\alpha(x) + (\frac{1}{2} \mu_1^{(i_0)}) \alpha]^{\frac{1}{\alpha}(i_0 + \varepsilon)}} \\
 &= \mu_1^{(i_0)} \int_{\{x \in \mathbf{R}^{i_0 - 1}; x_i \geq \frac{1}{2}\}} \frac{\prod_{k=1}^{i_0 - 1} \mu_k(x)}{[\sum_{i=1}^{i_0 - 1} U_i^\alpha(x) + (\frac{1}{2} \mu_1^{(i_0)}) \alpha]^{\frac{1}{\alpha}(i_0 + \varepsilon)}} dx \\
 &\stackrel{v=U(x)}{=} \mu_1^{(i_0)} \int_{\{v \in \mathbf{R}^{i_0 - 1}\}} \frac{1}{[M^\alpha \sum_{i=1}^{i_0 - 1} (\frac{v_i}{M})^\alpha + (\frac{1}{2} \mu_1^{(i_0)}) \alpha]^{\frac{1}{\alpha}(i_0 + \varepsilon)}} dv.
 \end{aligned}$$

By (12), we find

$$\begin{aligned}
 H_{i_0} &\leq \mu_1^{(i_0)} \lim_{M \rightarrow \infty} \frac{M^{i_0-1} \Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0-1}{\alpha})} \int_0^1 \frac{u^{\frac{i_0-1}{\alpha}-1}}{[M^\alpha u + (\frac{1}{2}\mu_1^{(i_0)})^\alpha]^{\frac{1}{\alpha}(i_0+\varepsilon)}} du \\
 &\stackrel{t = \frac{M^\alpha u}{(\frac{1}{2}\mu_1^{(i_0)})^\alpha}}{=} \frac{1}{(\frac{1}{2})^{1+\varepsilon} (\mu_1^{(i_0)})^\varepsilon} \frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0-1}{\alpha})} \int_0^1 \frac{t^{\frac{i_0-1}{\alpha}-1} dt}{(t+1)^{\frac{1}{\alpha}(i_0+\varepsilon)}} \\
 &= \frac{1}{(\frac{1}{2})^{1+\varepsilon} (\mu_1^{(i_0)})^\varepsilon} \frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0-1}{\alpha})} B\left(\frac{i_0-1}{\alpha}, \frac{1+\varepsilon}{\alpha}\right) < \infty,
 \end{aligned}$$

namely,  $H_{i_0} = O_{i_0}(1)$ . Then we have

$$\begin{aligned}
 \sum_m \|\tilde{U}_m\| \alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} &\leq \sum_{\{m \in \mathbf{N}^{i_0}; m_i \geq 2\}} \|\tilde{U}_m\| \alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} + \sum_{i=1}^{i_0} H_i \\
 &\leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c^\varepsilon i_0^\varepsilon \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \sum_{i=1}^{i_0} O_i(1) \quad (\varepsilon \rightarrow 0^+),
 \end{aligned}$$

and then (13) follows. In the same way, we have (14).  $\square$

DEFINITION 1. If  $0 < \alpha, \beta \leq 1$ ,  $\lambda_1, \lambda_2 \in \mathbf{R}$ ,  $\lambda_1 + \lambda_2 = \lambda$ ,  $k_\lambda(x, y)$  is a positive continuous homogeneous function in  $\mathbf{R}^2$ , satisfying  $k_\lambda(vx, vy) = v^{-\lambda} k_\lambda(x, y) > 0$ ,

$$\begin{aligned}
 \frac{\partial}{\partial x} k_\lambda(x, y) &< 0, & \frac{\partial}{\partial y} k_\lambda(x, y) &< 0, \\
 \frac{\partial^2}{\partial x^2} k_\lambda(x, y) &> 0, & \frac{\partial^2}{\partial y^2} k_\lambda(x, y) &> 0 \quad (v, x, y > 0),
 \end{aligned}$$

and

$$k(\lambda_1) := \int_0^\infty k_\lambda(u, 1) u^{\lambda_1-1} du \in \mathbf{R}_+,$$

then we define two weight coefficients  $w(\lambda_1, n)$  and  $W(\lambda_2, m)$  as follows:

$$w(\lambda_1, n) := \sum_m k_\lambda(\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) \frac{\|\tilde{V}_n\|_\beta^{\lambda_2}}{\|\tilde{U}_m\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)}, \tag{15}$$

$$W(\lambda_2, m) := \sum_n k_\lambda(\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) \frac{\|\tilde{U}_m\|_\alpha^{\lambda_1}}{\|\tilde{V}_n\|_\beta^{j_0-\lambda_2}} \prod_{l=1}^{j_0} \nu_n^{(l)}. \tag{16}$$

NOTE 1. With regards to the assumptions of Definition 1, (i) for  $\lambda_1 \leq i_0, \lambda_2 \leq j_0$ , we still can find that

$$\begin{aligned}
 \frac{\partial}{\partial x} \left( k_\lambda(x, y) \frac{1}{x^{i_0-\lambda_1}} \right), & \quad \frac{\partial}{\partial y} \left( k_\lambda(x, y) \frac{1}{y^{j_0-\lambda_2}} \right) < 0; \\
 \frac{\partial^2}{\partial x^2} \left( k_\lambda(x, y) \frac{1}{x^{i_0-\lambda_1}} \right), & \quad \frac{\partial^2}{\partial y^2} \left( k_\lambda(x, y) \frac{1}{y^{j_0-\lambda_2}} \right) > 0 \quad (x, y > 0).
 \end{aligned}$$

(ii) If  $(-1)^i h^{(i)}(t) > 0$  ( $t > 0; i = 0, 1, 2$ ),  $b > 0, 0 < \alpha \leq 1$ , then we have

$$\begin{aligned} \frac{d}{dx} h((b+x^\alpha)^{\frac{1}{\alpha}}) &= h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-1} x^{\alpha-1} < 0, \\ \frac{d^2}{dx^2} h((b+x^\alpha)^{\frac{1}{\alpha}}) &= h''((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{2}{\alpha}-2} x^{2\alpha-2} \\ &\quad + (1-\alpha)h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-2} x^{2\alpha-2} \\ &\quad + (\alpha-1)h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-1} x^{\alpha-2} \\ &= h''((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{2}{\alpha}-2} x^{2\alpha-2} \\ &\quad + b(\alpha-1)h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-2} x^{\alpha-2} > 0 \quad (x > 0). \end{aligned}$$

Hence, by the assumptions and (9), for  $m_i - \frac{1}{2} < x_i < m_i + \frac{1}{2}$  ( $i = 1, \dots, i_0; m \in \mathbf{N}$ ), we have  $\prod_{k=1}^{i_0} \mu_m^{(k)} = \prod_{k=1}^{i_0} \mu_k(x)$  and

$$\begin{aligned} &k_\lambda (\|U(m)\|_\alpha, \|\tilde{V}_n\|_\beta) \|U(m)\|_\alpha^{\lambda_1 - i_0} \prod_{k=1}^{i_0} \mu_m^{(k)} \\ &< \int_{\{x \in \mathbf{R}_+^{i_0}; m_i - \frac{1}{2} < x_i < m_i + \frac{1}{2}\}} k_\lambda (\|U(x)\|_\alpha, \|\tilde{V}_n\|_\beta) \|U(x)\|_\alpha^{\lambda_1 - i_0} \prod_{k=1}^{i_0} \mu_k(x) dx. \end{aligned}$$

LEMMA 4. With regards to the assumptions of Definition 1, then (i) we have

$$w(\lambda_1, n) < K_\beta(\lambda_1) \quad (\text{for } \lambda_1 \leq i_0; n \in \mathbf{N}^{j_0}), \tag{17}$$

$$W(\lambda_2, m) < K_\alpha(\lambda_1) \quad (\text{for } \lambda_2 \leq j_0; m \in \mathbf{N}^{i_0}), \tag{18}$$

where,

$$K_\beta(\lambda_1) = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} k(\lambda_1), K_\alpha(\lambda_1) = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} k(\lambda_1); \tag{19}$$

(ii) for  $\mu_m^{(k)} \geq \mu_{m+1}^{(k)}$  ( $m \in \mathbf{N}$ ),  $v_n^{(l)} \geq v_{n+1}^{(l)}$  ( $n \in \mathbf{N}$ ),  $U_\infty^{(k)} = V_\infty^{(l)} = \infty$  ( $k = 1, \dots, i_0, l = 1, \dots, j_0$ ), we have

$$0 < K_\alpha(\lambda_1)(1 - \theta_\lambda(n)) < w(\lambda_1, n) \quad (\text{for } \lambda_1 \leq i_0; n \in \mathbf{N}^{j_0}), \tag{20}$$

where, for  $b := \max_{1 \leq k \leq i_0} \{\mu_1^{(k)}\} (> 0)$ ,

$$\theta_\lambda(n) := \frac{1}{k(\lambda_1)} \int_0^{b^{1/\alpha} / \|\tilde{V}_n\|_\beta} k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in (0, 1). \tag{21}$$

Proof. (i) Since  $\|\tilde{U}_m\|_\alpha \geq \|U(m)\|_\alpha$ , by (10), (12) and Note 1(ii), for  $\lambda_1 \leq i_0$ , it

follows that

$$\begin{aligned}
w(\lambda_1, n) &= \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) \frac{\|\tilde{V}_n\|_\beta^{\lambda_2}}{\|\tilde{U}_m\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
&\leq \sum_m k_\lambda (\|U(m)\|_\alpha, \|\tilde{V}_n\|_\beta) \frac{\|\tilde{V}_n\|_\beta^{\lambda_2}}{\|U(m)\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
&< \sum_m \int_{\{x \in \mathbf{R}_+^{i_0}; m_i - \frac{1}{2} < x_i \leq m_i + \frac{1}{2}\}} k_\lambda (\|U(x)\|_\alpha, \|\tilde{V}_n\|_\beta) \frac{\|\tilde{V}_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_k(x) dx \\
&= \int_{\{x \in \mathbf{R}_+^{i_0}; x_i > \frac{1}{2}\}} k_\lambda (\|U(x)\|_\alpha, \|\tilde{V}_n\|_\beta) \|U(x)\|_\alpha^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} \prod_{k=1}^{i_0} \mu_k(x) dx \\
&\stackrel{v=U(x)}{=} \int_{\mathbf{R}_+^{i_0}} k_\lambda (\|v\|_\alpha, \|\tilde{V}_n\|_\beta) \|v\|_\alpha^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} dv \\
&= \lim_{M \rightarrow \infty} \int_{\mathbf{D}_M} k_\lambda (M [\sum_{i=1}^{i_0} (\frac{v_i}{M})^\alpha]^\frac{1}{\alpha}, \|\tilde{V}_n\|_\beta) M^{\lambda_1-i_0} [\sum_{i=1}^{j_0} (\frac{v_i}{M})^\alpha]^\frac{\lambda_1-i_0}{\alpha} \|\tilde{V}_n\|_\beta^{\lambda_2} dv \\
&= \lim_{M \rightarrow \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 k_\lambda (Mu^{1/\alpha}, \|\tilde{V}_n\|_\beta) M^{\lambda_1-i_0} u^{(\lambda_1-i_0)/\alpha} \|\tilde{V}_n\|_\beta^{\lambda_2} u^{\frac{i_0}{\alpha}-1} du \\
&= \lim_{M \rightarrow \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 k_\lambda (Mu^{1/\alpha}, \|\tilde{V}_n\|_\beta) (Mu^{1/\alpha})^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} u^{\frac{i_0}{\alpha}-1} du \\
&\stackrel{t=\frac{Mu^{1/\alpha}}{\|\tilde{V}_n\|_\beta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \int_0^\infty k_\lambda(t, 1) t^{\lambda_1-1} dt = \frac{\Gamma^{i_0}(\frac{1}{\alpha}) k_\lambda(\lambda_1)}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} = K_\alpha(\lambda_1).
\end{aligned}$$

Hence, we have (17). In the same way,  $\lambda_2 \leq j_0$ , we have (18).

(ii) Since for  $m_i \leq x_i < m_i + \frac{1}{2}$ ,  $\mu_m^{(k)} \geq \mu_{m+1}^{(k)} = \mu^{(k)}(x + \frac{1}{2})$ ; for  $m_i + \frac{1}{2} \leq x_i < m_i + 1$ ,  $\mu_m^{(k)} = \mu^{(k)}(x + \frac{1}{2})$ , by (10) and in the same way, for  $b = \max_{1 \leq k \leq i_0} \{\mu_1^{(k)}\} (> 0)$ ,  $\lambda_1 \leq i_0$ , we have

$$\begin{aligned}
w(\lambda_1, n) &\geq \sum_m k_\lambda (\|U(m + \frac{1}{2})\|_\alpha, \|\tilde{V}_n\|_\beta) \|U(m + \frac{1}{2})\|_\alpha^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} \prod_{k=1}^{i_0} \mu_m^{(k)} \\
&> \sum_m \int_{\{x \in \mathbf{R}_+^{i_0}; m_i \leq x_i < m_i + 1\}} k_\lambda (\|U(x + \frac{1}{2})\|_\alpha, \|\tilde{V}_n\|_\beta) \|U(x + \frac{1}{2})\|_\alpha^{\lambda_1-i_0} \\
&\quad \times \|\tilde{V}_n\|_\beta^{\lambda_2} \prod_{k=1}^{i_0} \mu_k(x + \frac{1}{2}) dx \\
&= \int_{[1, \infty)^{i_0}} k_\lambda (\|U(x + \frac{1}{2})\|_\alpha, \|\tilde{V}_n\|_\beta) \|U(x + \frac{1}{2})\|_\alpha^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} \prod_{k=1}^{i_0} \mu_k(x + \frac{1}{2}) dx \\
&\geq \int_{[b, \infty)^{i_0}} k_\lambda (\|v\|_\alpha, \|\tilde{V}_n\|_\beta) \|v\|_\alpha^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} dv.
\end{aligned}$$



For  $M > bi_0^{1/\alpha}$ , we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{b^\alpha i_0}{M^\alpha}, \\ k_\lambda(Mu^{1/\alpha}, \|\tilde{V}_n\|_\beta)(Mu^{1/\alpha})^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2}, & \frac{b^\alpha i_0}{M^\alpha} \leq u \leq 1. \end{cases}$$

By (12), it follows that

$$\begin{aligned} & \int_{\{x \in \mathbf{R}_+^{i_0}; x_i \geq b\}} k_\lambda(\|x\|_\alpha, \|\tilde{V}_n\|_\beta) \|x\|_\alpha^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} dx \\ &= \lim_{M \rightarrow \infty} \int \cdots \int_{D_M} \Psi \left( \sum_{i=1}^{i_0} \left( \frac{x_i}{M} \right)^\alpha \right) dx_1 \cdots dx_{i_0} \\ &= \lim_{M \rightarrow \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_{b^\alpha i_0/M^\alpha}^1 k_\lambda(Mu^{\frac{1}{\alpha}}, \|\tilde{V}_n\|_\beta) (Mu^{\frac{1}{\alpha}})^{\lambda_1-i_0} \|\tilde{V}_n\|_\beta^{\lambda_2} u^{\frac{i_0}{\alpha}-1} du \\ &\stackrel{t = \frac{Mu^{1/\alpha}}{\|\tilde{V}_n\|_\beta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \int_{bi_0^{1/\alpha}/\|\tilde{V}_n\|_\beta}^\infty k_\lambda(t, 1) t^{\lambda_1-1} dt. \end{aligned}$$

Hence, in view of (21), we have

$$w(\lambda_1, n) > \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \int_{bi_0^{1/\alpha}/\|\tilde{V}_n\|_\beta}^\infty k_\lambda(t, 1) t^{\lambda_1-1} dt = K_\alpha(\lambda_1)(1 - \theta_\lambda(n)),$$

and then (20) follows.  $\square$

NOTE 2. If there exist constants  $a \geq 0$  and  $\delta > -\lambda_1$ , satisfying  $\lim_{t \rightarrow 0^+} \frac{k_\lambda(t, 1)}{t^\delta} = a$ , then, there exists a constant  $M > 0$ , such that  $\frac{k_\lambda(t, 1)}{t^\delta} \leq M$  ( $t \in (0, bi_0^{1/\alpha}/\|\tilde{V}_1\|_\beta]$ ), and

$$0 < \theta_\lambda(n) \leq \frac{M}{k(\lambda_1)} \int_0^{bi_0^{1/\alpha}/\|\tilde{V}_n\|_\beta} t^{\lambda_1+\delta-1} dt = \frac{1}{(\lambda_1 + \delta)k(\lambda_1)} \left( \frac{bi_0^{1/\alpha}}{\|\tilde{V}_n\|_\beta} \right)^{\lambda_1+\delta},$$

namely,  $\theta_\lambda(n) = O\left(\frac{1}{\|\tilde{V}_n\|_\beta^{\lambda_1+\delta}}\right)$  ( $\delta > -\lambda_1$ ).

### 3. Main results

Setting functions

$$\begin{aligned} \tilde{\Phi}(m) &:= \frac{\|\tilde{U}_m\|_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \quad (m \in \mathbf{N}^{i_0}), \\ \tilde{\Psi}(n) &:= \frac{\|\tilde{V}_n\|_\beta \| \beta \|_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0} v_n^{(l)})^{q-1}} \quad (n \in \mathbf{N}^{j_0}), \end{aligned}$$

and the following normed spaces

$$\begin{aligned}
 l_{p,\tilde{\Phi}} &:= \left\{ a = \{a_m\}; \|a\|_{p,\tilde{\Phi}} := \left\{ \sum_m \tilde{\Phi}(m) |a_m|^p \right\}^{\frac{1}{p}} < \infty \right\}, \\
 l_{q,\tilde{\Psi}} &:= \left\{ b = \{b_n\}; \|b\|_{q,\tilde{\Psi}} := \left\{ \sum_n \tilde{\Psi}(n) |b_n|^q \right\}^{\frac{1}{q}} < \infty \right\}, \\
 l_{p,\tilde{\Psi}^{1-p}} &:= \left\{ c = \{c_n\}; \|c\|_{p,\tilde{\Psi}^{1-p}} := \left\{ \sum_n \tilde{\Psi}^{1-p}(n) |c_n|^p \right\}^{\frac{1}{p}} < \infty \right\},
 \end{aligned}$$

we have

**THEOREM 1.** *With regards to the assumptions of Definition 1, if  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\lambda_1 \leq i_0$ ,  $\lambda_2 \leq j_0$ , then for  $a_m, b_n \geq 0$ ,  $a = \{a_m\} \in l_{p,\tilde{\Phi}}$ ,  $b = \{b_n\} \in l_{q,\tilde{\Psi}}$ ,  $\|a\|_{p,\tilde{\Phi}}, \|b\|_{q,\tilde{\Psi}} > 0$ , we have the following equivalent inequalities*

$$I := \sum_n \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) a_m b_n < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\tilde{\Phi}} \|b\|_{q,\tilde{\Psi}}, \quad (22)$$

$$\begin{aligned}
 J &:= \left\{ \sum_n \frac{\prod_{k=1}^{j_0} \nu_n^{(k)}}{\|\tilde{V}_n\|_\beta^{j_0-p\lambda_2}} \left[ \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) a_m \right]^p \right\}^{\frac{1}{p}} \\
 &< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\tilde{\Phi}}. \quad (23)
 \end{aligned}$$

where,

$$K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1). \quad (24)$$

*Proof.* By Hölder’s inequality with weight (cf. [38]), we have

$$\begin{aligned}
 I &= \sum_n \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) \left[ \frac{\|\tilde{U}_m\|_\alpha^{\frac{i_0-\lambda_1}{q}} (\prod_{l=1}^{j_0} \nu_n^{(l)})^{\frac{1}{p}} a_m}{\|\tilde{V}_n\|_\beta^{\frac{j_0-\lambda_2}{p}} (\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}}} \right] \\
 &\times \left[ \frac{\|\tilde{V}_n\|_\beta^{\frac{j_0-\lambda_2}{p}} (\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}} b_n}{\|\tilde{U}_m\|_\alpha^{\frac{i_0-\lambda_1}{q}} (\prod_{l=1}^{j_0} \nu_n^{(l)})^{\frac{1}{p}}} \right] \\
 &\leq \left[ \sum_m W(\lambda_2, m) \frac{\|\tilde{U}_m\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \right]^{\frac{1}{p}} \left[ \sum_n w(\lambda_1, n) \frac{\|\tilde{V}_n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q}{(\prod_{l=1}^{j_0} \nu_n^{(l)})} \right]^{\frac{1}{q}}.
 \end{aligned}$$

Then by (17) and (18), we have (22). We set

$$b_n := \frac{\prod_{l=1}^{j_0} \nu_n^{(l)}}{\|\tilde{V}_n\|_\beta^{j_0-p\lambda_2}} \left[ \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) a_m \right]^{p-1}, \quad n \in \mathbf{N}^{j_0}.$$

Then we have  $J = \|b\|_{q, \tilde{\Psi}}^{q-1}$ . Since the right hand side of (23) is finite, it follows that  $J < \infty$ . If  $J = 0$ , then (23) is trivially valid; if  $J > 0$ , then by (22), we have

$$\begin{aligned} \|b\|_{q, \tilde{\Psi}}^q &= J^p = I < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p, \tilde{\Phi}} \|b\|_{q, \tilde{\Psi}}, \\ \|b\|_{q, \tilde{\Psi}}^{q-1} &= J < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p, \tilde{\Phi}}, \end{aligned}$$

namely, (23) follows. On the other hand, assuming that (23) is valid, by Hölder’s inequality (cf. [38]), we have

$$I = \sum_n \left[ \frac{(\prod_{l=1}^{j_0} \nu_n^{(l)})^{1/p}}{\|\tilde{V}_n\|_\beta^{(j_0/p)-\lambda_2}} \sum_m k_\lambda(\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) a_m \right] \left[ \frac{\|\tilde{V}_n\|_\beta^{(j_0/p)-\lambda_2}}{(\prod_{l=1}^{j_0} \nu_n^{(l)})^{1/p}} b_n \right] \leq J \|b\|_{q, \tilde{\Psi}}. \tag{25}$$

Then by (23), we have (22), which is equivalent to (23).  $\square$

**THEOREM 2.** *With regards to the assumptions of Theorem 1, if  $\mu_m^{(k)} \geq \mu_{m+1}^{(k)}$  ( $m \in \mathbf{N}$ ),  $\nu_n^{(l)} \geq \nu_{n+1}^{(l)}$  ( $n \in \mathbf{N}$ ),  $U_\infty = V_\infty = \infty$  ( $k = 1, \dots, i_0, l = 1, \dots, j_0$ ), there exist constants  $a \geq 0$  and  $\delta > -\lambda_1$ , satisfying  $\lim_{t \rightarrow 0^+} \frac{k_\lambda(t, 1)}{\delta} = a$ , then the constant factor  $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$  in (22) and (23) is the best possible.*

*Proof.* For  $\varepsilon > 0$ ,  $\tilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} (< i_0)$ ,  $\tilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ , we set

$$\begin{aligned} \tilde{a} &= \{\tilde{a}_m\}, \tilde{a}_m := \|\tilde{U}_m\|_\alpha^{-i_0 + \tilde{\lambda}_1} \prod_{k=1}^{i_0} \mu_m^{(k)} \quad (m \in \mathbf{N}^{i_0}), \\ \tilde{b} &= \{\tilde{b}_n\}, \tilde{b}_n := \|\tilde{V}_n\|_\beta^{-j_0 + \tilde{\lambda}_2 - \varepsilon} \prod_{l=1}^{j_0} \nu_n^{(l)} \quad (n \in \mathbf{N}^{j_0}). \end{aligned}$$

Then by (13) and (14), we obtain

$$\begin{aligned} \|\tilde{a}\|_{p, \tilde{\Phi}} \|\tilde{b}\|_{q, \tilde{\Phi}} &= \left[ \sum_m \frac{\|\tilde{U}_m\|_\alpha^{p(i_0 - \lambda_1) - i_0} \tilde{a}_m^p}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \right]^{\frac{1}{p}} \left[ \sum_n \frac{\|\tilde{V}_n\|_\beta^{q(j_0 - \lambda_2) - j_0} \tilde{b}_n^q}{(\prod_{l=1}^{j_0} \nu_n^{(l)})^{q-1}} \right]^{\frac{1}{q}} \\ &= \left( \sum_m \|\tilde{U}_m\|_\alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \right)^{\frac{1}{p}} \left( \sum_n \|\tilde{V}_n\|_\beta^{-j_0 - \varepsilon} \prod_{l=1}^{j_0} \nu_n^{(l)} \right)^{\frac{1}{q}} \\ &\leq \frac{1}{\varepsilon} \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{c^\varepsilon i_0^\varepsilon \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1) \right)^{\frac{1}{p}} \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{c^\varepsilon j_0^\varepsilon \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \tilde{O}(1) \right)^{\frac{1}{q}}. \end{aligned}$$

By (20) and (21), we find

$$\begin{aligned} \tilde{I} &:= \sum_n \left[ \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) \tilde{a}_m \right] \tilde{b}_n \\ &= \sum_n w(\tilde{\lambda}_1, n) \|\tilde{V}_n\|_\beta^{-j_0 - \varepsilon} \prod_{l=1}^{j_0} \mathbf{v}_n^{(l)} \\ &> K_\alpha(\tilde{\lambda}_1) \sum_n \left( 1 - O\left(\frac{1}{\|\tilde{V}_n\|_\beta^{\lambda_1 + \delta}}\right) \right) \|\tilde{V}_n\|_\beta^{-j_0 - \varepsilon} \prod_{l=1}^{j_0} \mathbf{v}_n^{(l)} \\ &= K_\alpha(\tilde{\lambda}_1) \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon c^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \tilde{O}(1) - O_1(1) \right). \end{aligned}$$

If there exists a constant  $K \leq K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ , such that (22) is valid when replacing  $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$  by  $K$ , then we have  $\varepsilon \tilde{I} < \varepsilon K \|\tilde{a}\|_{p, \tilde{\Phi}} \|\tilde{b}\|_{q, \tilde{\Phi}}$ , namely,

$$\begin{aligned} &K_\alpha\left(\lambda_1 - \frac{\varepsilon}{p}\right) \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{c^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \tilde{O}(1) - \varepsilon O_1(1) \right) \\ &< K \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{c^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1) \right)^{\frac{1}{p}} \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{c^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \tilde{O}(1) \right)^{\frac{1}{q}}. \end{aligned}$$

For  $\varepsilon \rightarrow 0^+$ , since  $k(\lambda_1 - \frac{\varepsilon}{p}) \rightarrow k(\lambda_1)$  (cf. [3]), we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \frac{\Gamma^{i_0}(\frac{1}{\alpha}) k(\lambda_1)}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \leq K \left[ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{q}},$$

and then  $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \leq K$ . Hence,  $K = K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$  is the best possible constant factor of (22). The constant factor in (23) is still the best possible. Otherwise, we would reach a contradiction by (25) that the constant factor in (22) is not the best possible.  $\square$

### 4. Operator expressions and examples

With regards to the assumptions of Theorem 2, in view of

$$c_n := \frac{\prod_{k=1}^{j_0} \mathbf{v}_n^{(k)}}{\|\tilde{V}_n\|_\beta^{j_0 - p\lambda_2}} \left[ \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) a_m \right]^{p-1}, \quad n \in \mathbf{N}^{j_0}$$

$$c = \{c_n\}, \|c\|_{p, \tilde{\Psi}^{1-p}} = J < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p, \tilde{\Phi}} < \infty,$$

we can set the following definition:

DEFINITION 2. Define a multidimensional Hardy-Hilbert-type operator  $T : l_{p, \tilde{\Phi}} \rightarrow l_{p, \tilde{\Psi}^{1-p}}$  as follows: For any  $a \in l_{p, \tilde{\Phi}}$ , there exists a unique representation  $Ta = c \in l_{p, \tilde{\Psi}^{1-p}}$ , satisfying

$$Ta(n) := \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) a_m \quad (n \in \mathbf{N}^{j_0}). \tag{26}$$

For  $b \in l_{q, \tilde{\Psi}}$ , we define the following formal inner product of  $Ta$  and  $b$  as follows:

$$(Ta, b) := \sum_n \left[ \sum_m k_\lambda (\|\tilde{U}_m\|_\alpha, \|\tilde{V}_n\|_\beta) a_m \right] b_n. \tag{27}$$

Then by Theorem 1, we have the following equivalent inequalities:

$$(Ta, b) < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p, \tilde{\Phi}} \|b\|_{q, \tilde{\Psi}}, \tag{28}$$

$$\|Ta\|_{p, \tilde{\Psi}^{1-p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p, \tilde{\Phi}}. \tag{29}$$

It follows that  $T$  is bounded with

$$\|T\| := \sup_{a(\neq \theta) \in l_{p, \tilde{\Phi}}} \frac{\|Ta\|_{p, \tilde{\Psi}^{1-p}}}{\|a\|_{p, \tilde{\Phi}}} \leq K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1). \tag{30}$$

Since by Theorem 2, the constant factor  $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$  in (29) is the best possible, we have

$$\|T\| = K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1). \tag{31}$$

EXAMPLE 1. Setting  $k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda}$  ( $0 < \lambda \leq 1$ ), we can find

$$\begin{aligned} \frac{\partial}{\partial x} k_\lambda(x, y) &< 0, & \frac{\partial}{\partial y} k_\lambda(x, y) &< 0, \\ \frac{\partial^2}{\partial x^2} k_\lambda(x, y) &> 0, & \frac{\partial^2}{\partial y^2} k_\lambda(x, y) &> 0 \quad (x, y > 0), \end{aligned}$$

and for  $0 < \lambda_1 < 1$  ( $\leq i_0$ ),  $0 < \lambda_2 < 1$  ( $\leq j_0$ ),

$$k(\lambda_1) = \int_0^\infty \frac{u^{\lambda_1-1}}{u^{\lambda_1} + 1} du = \frac{1}{\lambda} \int_0^\infty \frac{1}{v+1} v^{\frac{\lambda_1}{\lambda}-1} dv = \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})} \in \mathbf{R}_+.$$

We set  $\delta = 0 (> -\lambda_1)$  and  $a = 1$ , in view of (31), we have

$$\|T\| = \left[ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})}.$$

EXAMPLE 2. Setting  $k_\lambda(x, y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$  ( $0 < \lambda \leq 1$ ), we can find (cf. [3], Example 2.2.2)

$$\begin{aligned} \frac{\partial}{\partial x} k_\lambda(x, y) &< 0, & \frac{\partial}{\partial y} k_\lambda(x, y) &< 0, \\ \frac{\partial^2}{\partial x^2} k_\lambda(x, y) &> 0, & \frac{\partial^2}{\partial y^2} k_\lambda(x, y) &> 0 \quad (x, y > 0), \end{aligned}$$

and for  $0 < \lambda_1 < 1$  ( $\leq i_0$ ),  $0 < \lambda_2 < 1$  ( $\leq j_0$ ),

$$k(\lambda_1) = \int_0^\infty \frac{\ln u}{u^\lambda - 1} u^{\lambda_1 - 1} du = \frac{1}{\lambda^2} \int_0^\infty \frac{\ln v}{v - 1} v^{\frac{\lambda_1}{\lambda} - 1} dv = \left[ \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})} \right]^2 \in \mathbf{R}_+.$$

We set  $\delta = 0$  ( $> -\lambda_1$ ) and  $a = 1$ , in view of (31), we have

$$\|T\| = \left[ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \left[ \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})} \right]^2.$$

EXAMPLE 3. Setting  $k_\lambda(x, y) = \frac{1}{(x+y)^\lambda}$  ( $\lambda > 0$ ), then we find

$$\begin{aligned} \frac{\partial}{\partial x} k_\lambda(x, y) &< 0, & \frac{\partial}{\partial y} k_\lambda(x, y) &< 0, \\ \frac{\partial^2}{\partial x^2} k_\lambda(x, y) &> 0, & \frac{\partial^2}{\partial y^2} k_\lambda(x, y) &> 0 \quad (x, y > 0), \end{aligned}$$

and for  $0 < \lambda_1 \leq i_0$ ,  $0 < \lambda_2 \leq j_0$ ,

$$k(\lambda_1) = \int_0^\infty \frac{u^{\lambda_1 - 1}}{(u + 1)^\lambda} du = B(\lambda_1, \lambda_2) \in \mathbf{R}_+.$$

We set  $\delta = 0$  ( $> -\lambda_1$ ) and  $a = 1$ , in view of (31), we have

$$\|T\| = \left[ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} B(\lambda_1, \lambda_2).$$

REMARK 1. For  $\tilde{\mu}_i^{(k)} = 0$  ( $k = 1, \dots, i_0$ ;  $i = 1, \dots, m$ ),  $\tilde{\nu}_j^{(l)} = 0$  ( $l = 1, \dots, j_0$ ;  $j = 1, \dots, n$ ), setting

$$\Phi(m) := \frac{\|U_m\| \alpha^{p(i_0 - \lambda_1) - i_0}}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}}, \quad \Psi(n) := \frac{\|V_n\| \beta^{q(j_0 - \lambda_2) - j_0}}{(\prod_{l=1}^{j_0} \nu_n^{(l)})^{q-1}} \quad (m \in \mathbf{N}^{i_0}, n \in \mathbf{N}^{j_0}),$$

(22) and (23) reduce the following equivalent inequalities with the same best possible constant factor  $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_2)$ :

$$\sum_n \sum_m k_\lambda(\|U_m\|_\alpha, \|V_n\|_\beta) a_m b_n < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_2) \|a\|_{p, \Phi} \|b\|_{q, \Psi}, \tag{32}$$

$$\left\{ \sum_n \frac{\prod_{k=1}^{j_0} \nu_n^{(k)}}{\|V_n\|_\beta^{j_0 - p \lambda_2}} \left[ \sum_m k_\lambda(\|U_m\|_\alpha, \|V_n\|_\beta) a_m \right]^p \right\}^{\frac{1}{p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_2) \|a\|_{p, \Phi}. \tag{33}$$

Hence, (22) and (23) are more accurate extensions of (32) and (33).

In particular, for  $k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda}$  ( $0 < \lambda_1 \leq i_0$ ,  $0 < \lambda_2 \leq j_0$ ),  $\mu_i = \nu_j = 1$  ( $i, j \in \mathbb{N}$ ), (32) reduces to (4); for  $k_\lambda(x, y) = \frac{1}{(x^\eta + y^\eta)^{\lambda/\eta}}$  ( $0 < \eta \leq 1$ ,  $0 < \lambda_1 \leq 1 = i_0$ ,  $0 < \lambda_2 \leq 1 = j_0$ ), (32) reduces to (5).

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#### REFERENCES

- [1] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, USA, 1934.
- [2] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Boston, USA, 1991.
- [3] B. YANG, *Discrete Hilbert-type inequalities*, Bentham Science Publishers Ltd., The United Arab Emirates, 2011.
- [4] B. YANG, *On a more accurate multidimensional Hilbert-type inequality with parameters*, Mathematical Inequalities and Applications, vol. 18, no. 2, pp. 429–441, 2015.
- [5] Y. HONG, *On Hardy-Hilbert integral inequalities with some parameters*, J. Ineq. in Pure & Applied Math., vol. 6, no. 4, Art. 92, pp. 1–10, 2005.
- [6] B. YANG, M. KRNIĆ, *On the Norm of a Multidimensional Hilbert-type Operator*, Sarajevo Journal of Mathematics, vol. 7, no. 20, pp. 223–243, 2011.
- [7] M. KRNIĆ, J. E. PEČARIĆ, P. VUKOVIĆ, *On some higher-dimensional Hilbert's and Hardy-Hilbert's type integral inequalities with parameters*, Math. Inequal. Appl., vol. 11, pp. 701–716, 2008.
- [8] M. KRNIĆ, P. VUKOVIĆ, *On a multidimensional version of the Hilbert-type inequality*, Analysis Mathematica, vol. 38, pp. 291–303, 2012.
- [9] M. TH. RASSIAS, B. YANG, *A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function*, Applied Mathematics and Computation, vol. 225, pp. 263–277, 2013.
- [10] B. YANG, *A multidimensional discrete Hilbert-type inequality*, Int. J. Nonlinear Anal. Appl. vol. 5, no. 1, pp. 80–88, 2014.
- [11] M. TH. RASSIAS, B. YANG, *On a multidimensional half – discrete Hilbert – type inequality related to the hyperbolic cotangent function*, Applied Mathematics and Computation, vol. 242, pp. 800–813, 2014.
- [12] M. TH. RASSIAS, B. YANG, *On a multidimensional Hilbert-type integral inequality associated to the gamma function*, Applied Mathematics and Computation, vol. 249, pp. 408–418, 2014.
- [13] Q. CHEN, B. YANG, *On a more accurate multidimensional Mulholland-type inequality*, Journal of Inequalities and Applications 2014, 2014:322.
- [14] B. YANG, Q. CHEN, *A multidimensional discrete Hilbert-type inequality*, Journal of Mathematical Inequalities, vol. 8, no. 2, pp. 267–277, 2014.
- [15] B. YANG, *On a more accurate reverse multidimensional half-discrete Hilbert-type inequalities*, Mathematical Inequalities and Applications, vol. 18, no. 2, pp. 589–605, 2015.
- [16] T. LIU, B. YANG, L. HE, *On a multidimensional Hilbert-type integral inequality with logarithm function*, Mathematical Inequalities and Applications, vol. 18, no. 4, pp. 1219–1234, 2015.
- [17] Z. HUANG, B. YANG, *A multidimensional Hilbert-type integral inequality*, Journal of Inequalities and Applications (2015), 2015:151.
- [18] Y. SHI, B. YANG, *On a multidimensional Hilbert-type inequality with parameters*, Journal of Inequalities and Applications (2015), 2015:371.
- [19] J. ZHONG, B. YANG, *An extension of a multidimensional Hilbert-type inequality*, Journal of Inequalities and Applications (2017) 2017:78.

- [20] V. ADIYASUREN, T. BATBOLD, M. M. KRNIĆ, *On several new Hilbert-type inequalities involving means operators*, Acta Math. Sin. Engl. Ser. vol. 29, pp. 1493–1514, 2013.
- [21] Y. HONG, *On multiple Hardy-Hilbert integral inequalities with some parameters*, Journal of Inequalities and Applications, vol. 2006, Article ID 94960, 11 pages.
- [22] Q. L. HUANG, *On a multiple Hilbert's inequality with parameters*, Journal of Inequalities and Applications, vol. 2010, Article ID 309319, 12 pages.
- [23] I. PERIĆ, P. VUKOVIĆ, *Multiple Hilbert's type inequalities with a homogeneous kernel*, Banach Journal of Mathematical Analysis, vol. 5, no. 2, pp. 33–43, 2011.
- [24] B. HE, *A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor*, Journal of Mathematical Analysis and Applications, vol. 431, pp. 990–902, 2015.
- [25] V. ADIYASUREN, T. BATBOLD, M. KRNIĆ, *Multiple Hilbert-type inequalities involving some differential operators*, Banach J. Math. Anal., vol. 10, no. 2, pp. 320–337, 2016.
- [26] B. YANG, *An extension of a Hardy-Hilbert-type inequality*, Journal of Guangdong University of Education, vol. 35, no. 3, pp. 1–7, 2015.
- [27] Y. SHI, B. YANG, *A new Hardy-Hilbert-type inequality with multi-parameters and a best possible constant factor*, Journal of Inequalities and Applications (2015), 2015: 380.
- [28] Q. HUANG, *A new extension of Hardy-Hilbert-type inequality*, Journal of Inequalities and Applications (2015), 2015: 397.
- [29] A. WANG, Q. HUANG, B. YANG, *A strengthened Mulholland-type inequality with parameters*, Journal of Inequalities and Applications (2015), 2015: 329.
- [30] M. TH. RASSIAS, B. YANG, *On a Hardy-Hilbert-type inequality with a general homogeneous kernel*, Int. J. Nonlinear Anal. Appl. vol. 7 no. 1, pp. 249–269, 2016.
- [31] B. YANG, Q. CHEN, *On a more accurate Hardy-Mulholland-type inequality*, Journal of Inequalities and Applications (2016), 2016: 82.
- [32] I. BRNETIĆ, J. E. PEČARIĆ, *Generalization of Hilbert's integral inequality*, Mathematical inequalities and applications, vol. 7, no. 2, pp. 199–205, 2004.
- [33] M. KRNIĆ, J. E. PEČARIĆ, *General Hilbert's and Hardy's inequalities*, Mathematical inequalities and applications. vol. 8, no. 1, pp. 29–51, 2005.
- [34] M. KRNIĆ, M. GAO, J. E. PEČARIĆ et. al., *On the best constant in Hilbert's inequality*, Mathematical inequalities and applications. vol. 8, no. 2, pp. 317–329, 2005.
- [35] Y. LI, J. WANG, B. HE, *On further analogs of Hilbert's inequality*, Journal of Inequalities and Applications, vol. 2007, Article ID 76329, 6 pages.
- [36] E. A. LAITH, *On some extensions of Hardy-Hilbert's inequality and applications*, Journal of Inequalities and Applications, vol. 2008, Article ID 546828, 14 pages.
- [37] R. P. AGARWAL, D. O'REGAN, S. H. SAKER, *Some Hardy-type inequalities with weighted functions via Opial type inequalities*, Advances in Dynamical Systems and Applications, vol. 10, pp. 1–9, 2015.
- [38] J. KUANG, *Applied inequalities*, Shangdong Science Technic Press, Jinan, China, 2004.

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