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THE MOMENT OF MAXIMUM NORMED SUMS OF RANDOMLY WEIGHTED PAIRWISE NQD SEQUENCES

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Abstract. This paper investigates the moment of maximum normed sums of randomly weighted pairwise negative quadrant dependent (NQD) random variables. A sufficient condition to the moment of this stochastic process is obtained, which extends the existing results.

1. Introduction

The aim of this paper is to investigate the moment of maximum normed sums of randomly weighted pairwise negative quadrant dependent (NQD) random variables stochastically dominated by a random variable. Now, let's recall some definitions.

DEFINITION 1.1. Two random variables X and Y are said to be NQD if for all real numbers x and y,

$$P(X<x,Y<y)\leqslant P(X<x)P(Y<y).$$

A sequence of random variables $\{X_n,n\geqslant 1\}$ is said to be pairwise NQD if X_i and X_j are NQD for any $i,\ j\in N^+$ and $i\neq j$.

DEFINITION 1.2. The sequence $\{X_n,n\geqslant 1\}$ is stochastically dominated by a random variable X if

$$\sup_{n\geqslant 1}P(|X_n|>t)\leqslant CP(|X|>t)$$

for some positive constant C and all $t\geqslant 0$.

The pairwise NQD is introduced by Lehmann [13] and it contains many dependence structures such as negatively associated (NA) sequences, negatively orthant dependent (NOD) sequences, negatively superadditive dependent (NSD) sequences, extended negatively dependent (END) sequences (see [5, 11, 19, 21] for the related definitions). Many authors pay attention to the research of pairwise NQD random variables. For example, Matula [15] established the strong law of large numbers for pairwise NQD

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sequences and the three series theorem for NA sequences; Wang et al. [23] studied the strong stability and Wu [25] investigated the maximal moment inequality for pairwise NQD sequences; Gan and Chen [8] studied some limit theorems and Sung [17, 18] studied the convergence in r-mean for pairwise NQD sequences; Wu and Jiang [28], Ko [12], Wu and Shen [29], Shen et al. [16], Wu and Guo [30] and Yang et al. [32] obtained some convergence results such as the strong law of large numbers, complete convergence and complete moment convergence for non-weighted or weighted pairwise NQD sequences; Yang and Hu [33] investigated the moving average process based on the pairwise NQD sequences, etc. In addition, for more works on the negative dependent sequences, one can refer to [4, 6, 7, 10, 20, 22, 26, 27, 31] and the references therein.

The conception of stochastical domination can be found in Adler and Rosalsky [1] and Adler et al. [2]. This assumption is very weak condition. For example, Hanson and Wright [9] and Wright [22] studied that a bound on tail probabilities for quadratic forms in independent random variables $\{X_n, n \geqslant 1\}$ by using the following condition: there exist $C > 0$ and $\gamma > 0$ such that $P(|X_n| \geqslant x) \leqslant C\int_x^\infty e^{-\gamma t^2}dt$ for all $n \geqslant 1$ and all $x \geqslant 0$.

Chen and Gan [3] studied the moment of maximum normed sums of ρ-mixing random variables and obtained the results of (2.3) and (2.4) in the following Section 2. Yao and Lin [34] extended the results of Chen and Gan [3] to the case of randomly weighted martingale differences. Recently, Li et al. [14] studied the properties of randomly weighted pairwise NQD sequences and gave an application to the limit theory. So this paper will investigate the moment of maximum normed sums of randomly weighted pairwise NQD random variables and obtain a sufficient moment condition to (2.3) and (2.4). We extend the results of Chen and Gan [3] and Yao and Lin [34] to the case of randomly weighted pairwise NQD random variables. Throughout the paper, $I(A)$ is the indicator function of set A and C, C_1, $C_2, \ldots$, denote some positive constants not depending on n.

LEMMA 1.1. (Lehmann [13]) *If random variables X and Y are NQD, then*

(i) $EXY \leqslant EXEY$;

(ii) $P(X > x, Y > y) \leqslant P(X > x)P(Y > y)$, $\forall\, x,\ y \in R$;

(iii) If f and g are both nondecreasing (or nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.

LEMMA 1.2. (Wu [25]) *Let $\{X_n, n \geqslant 1\}$ be a pairwise NQD sequence with $EX_n = 0$ and $EX_n^2 < \infty$ for all $n \geqslant 1$. Then*

$$E\Big(\max_{1\leqslant k\leqslant n}\Big(\sum_{i=1}^{k} X_i\Big)^2\Big) \leqslant C\log^2 n\sum_{i=1}^{n} EX_i^2,\ \ n \geqslant 1.$$

LEMMA 1.3. (Adler and Rosalsky [1] and Adler et al. [2]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables which are stochastically dominated by a random variable X. Then, for any $p > 0$ and $b > 0$, the following two statements hold:*

$$E[|X_n|^p I(|X_n| \leqslant b)] \leqslant C_1\{E[|X|^p I(|X| \leqslant b)] + b^p P(|X| > b)\},$$
$$E[|X_n|^p I(|X_n| > b)] \leqslant C_2 E[|X|^p I(|X| > b)].$$

So we obtain that $E|X_n|^p \leqslant C_3 E|X|^p$ *for all* $n \geqslant 1$.

2. The main result and its proof

THEOREM 2.1. *Let* $0<r<2$ *and* $0<p<2$. *Assume that* $\{X_n, n \geqslant 1\}$ *is a mean zero sequence of pairwise NQD random variables which are stochastically dominated by a random variable* X *such that*

$$\begin{cases} \text{for } p<r, & \begin{cases} E|X|<\infty, & \text{if } 0<r<1, \\ E[|X|\log^3(1+|X|)]<\infty, & \text{if } r=1, \\ E[|X|^r\log^2(1+|X|)]<\infty, & \text{if } r>1, \end{cases} \\ \text{for } p=r, & \begin{cases} E|X|<\infty, & \text{if } 0<r<1, \\ E[|X|\log^4(1+|X|)]<\infty, & \text{if } r=1, \\ E[|X|^r\log^3(1+|X|)]<\infty, & \text{if } r>1, \end{cases} \\ \text{for } p>r, & \begin{cases} E|X|<\infty, & \text{if } 0<p<1, \\ E[|X|\log(1+|X|)]<\infty, & \text{if } p=1, \\ E|X|^p<\infty, & \text{if } p>1. \end{cases} \end{cases} \tag{2.1}$$

Suppose that $\{A_n, n \geqslant 1\}$ *is an independent sequence of random variables which is also independent of the sequence* $\{X_n, n \geqslant 1\}$. *Let*

$$\sum_{i=1}^{n} EA_i^2 = O(n). \tag{2.2}$$

Denote $S_n = \sum_{i=1}^n A_i X_i$, $n \geqslant 1$. *Then it has*

$$E\left(\sup_{n\geqslant 1}\left|\frac{S_n}{n^{1/r}}\right|^p\right) < \infty, \tag{2.3}$$

which implies

$$E\left(\sup_{n\geqslant 1}\left|\frac{X_n}{n^{1/r}}\right|^p\right) < \infty. \tag{2.4}$$

Proof. Combining Lemma 1.1 with Remark 3.1 of Li et al. [14], for all fixed n, we obtain that $\{A_i^+X_i, 1 \leqslant i \leqslant n\}$, $\{A_i^-X_i, 1 \leqslant i \leqslant n\}$ are also pairwise NQD random variables. In view of $A_iX_i = A_i^+X_i - A_i^-X_i$, without loss of generality, we assume that $A_i \geqslant 0$, *a.s.*, in the proof. Since $S_n = \sum_{i=1}^n A_iX_i$, $n \geqslant 1$, it can be argued that

$$\begin{aligned} E\left(\sup_{n\geqslant 1}\left|\frac{S_n}{n^{1/r}}\right|^p\right) &\leqslant 2^{p/r} + \int_{2^{p/r}}^{\infty} \sum_{k=1}^{\infty} P\left(\max_{2^{k-1}\leqslant n<2^k}\left|\frac{S_n}{n^{1/r}}\right| > t^{1/p}\right) dt \\ &\leqslant 2^{p/r} + 2^{p/r}\sum_{k=1}^{\infty} 2^{-kp/r}\int_{2^{kp/r}}^{\infty} P\left(\max_{1\leqslant n\leqslant 2^k}|S_n| > s^{1/p}\right) ds. \end{aligned} \tag{2.5}$$

For $s^{1/p} > 0$ and $1 \leqslant i \leqslant n$, denote

$$X_{si} = -s^{1/p} I(X_i < -s^{1/p}) + X_i I(|X_i| \leqslant s^{1/p}) + s^{1/p} I(X_i > s^{1/p}) \text{ and } X_{si}^* = X_i - X_{si}.$$

For $1 \leqslant i \leqslant n$, it has

$$A_i X_i = [A_i X_{si} - E(A_i X_{si})] + A_i X_{si}^* + E(A_i X_{si}).$$

So it can be seen that

$$\begin{aligned}
&\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\Big(\max_{1 \leqslant n \leqslant 2^k} |S_n| > s^{1/p}\Big) ds \\
\leqslant\ & \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\Big(\max_{1 \leqslant n \leqslant 2^k} \Big|\sum_{i=1}^{n} A_i X_{si}^*\Big| > s^{1/p}/2\Big) ds \\
&+ \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\Big(\max_{1 \leqslant n \leqslant 2^k} \Big|\sum_{i=1}^{n} [A_i X_{si} - E A_i X_{si}]\Big| > s^{1/p}/4\Big) ds \\
&+ \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\Big(\max_{1 \leqslant n \leqslant 2^k} \Big|\sum_{i=1}^{n} E(A_i X_{si})\Big| > s^{1/p}/4\Big) ds \\
=:\ & H_1 + H_2 + H_3.
\end{aligned} \tag{2.6}$$

For any $1 \leqslant q \leqslant 2$, we get by Hölder's inequality and (2.2) that

$$\sum_{i=1}^{n} E|A_i|^q \leqslant \Big(\sum_{i=1}^{n} E A_i^2\Big)^{q/2} \Big(\sum_{i=1}^{n} 1\Big)^{1-q/2} = O(n). \tag{2.7}$$

By the fact that $\{A_n, n \geqslant 1\}$ is independent of $\{X_n, n \geqslant 1\}$, we obtain by Markov's inequality, Lemma 1.3 and (2.7) with $q = 1$ that

$$\begin{aligned}
H_1 &\leqslant 4 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} \Big(\sum_{i=1}^{2^k} E|A_i| E|X_i| I(|X_i| > s^{1/p})\Big) ds \\
&\leqslant C_1 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} \int_{2^{mp/r}}^{2^{(m+1)p/r}} s^{-1/p} E[|X| I(|X| > s^{1/p})] ds \\
&\leqslant C_2 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} 2^{mp/r - m/r} E[|X| I(|X| > 2^{m/r})] \\
&= C_2 \sum_{m=1}^{\infty} 2^{mp/r - m/r} E[|X| I(|X| > 2^{m/r})] \sum_{k=1}^{m} 2^{k-kp/r} \\
&\leqslant \begin{cases}
C_3 \sum_{m=1}^{\infty} 2^{m-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p < r, \\
C_4 \sum_{m=1}^{\infty} m 2^{m-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p = r, \\
C_5 \sum_{m=1}^{\infty} 2^{mp/r - m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p > r.
\end{cases}
\end{aligned} \tag{2.8}$$

First, we consider the case of $p < r$. If $0 < r < 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \\ &= \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \sum_{m=1}^{k} 2^{m(1-1/r)} \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \\ &\leqslant C_1 E|X|.\end{aligned}$$

If $r = 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{m=1}^{\infty} E[|X| I(|X| > 2^{m})] \\ &= \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E[|X| I(2^{k} < |X| \leqslant 2^{k+1})] \\ &= \sum_{k=1}^{\infty} k E[|X| I(2^{k} < |X| \leqslant 2^{k+1})] \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X| \log(1+|X|) I(2^{k} < |X| \leqslant 2^{k+1})] \\ &\leqslant C_1 E[|X| \log(1+|X|)].\end{aligned}$$

If $r > 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \sum_{m=1}^{k} 2^{m-m/r} \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X|^r I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \leqslant C_1 E|X|^r.\end{aligned}$$

Second, we consider the case $p = r$. Similarly, if $0 < r < 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} m 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \sum_{m=1}^{k} m 2^{m(1-1/r)} \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \leqslant C_2 E|X|.\end{aligned}$$

If $r = 1$, then it has

$$\begin{aligned}\sum_{m=1}^{\infty} m 2^{m-m/r} E[|X| I(X > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X| I(2^{k} < |X| \leqslant 2^{k+1})] \sum_{m=1}^{k} m \\ &\leqslant C_2 \sum_{k=1}^{\infty} E[|X| \log^2(1+|X|) I(2^{k} < |X| \leqslant 2^{k+1})] \\ &\leqslant C_2 E[|X| \log^2(1+|X|)].\end{aligned}$$

If $r > 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} m2^{m-m/r}E[|X|I(|X|>2^{m/r})] &\leqslant \sum_{k=1}^{\infty} E[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]k\sum_{m=1}^{k}2^{m-m/r}\\ &\leqslant C_1\sum_{k=1}^{\infty} k2^{k-k/r}E[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]\\ &\leqslant C_2E[|X|^r\log(1+|X|)].\end{aligned}$$

Third, it is time to consider the case $p > r$. If $0 < p < 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{mp/r-m/r}E[|X|I(|X|>2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]\sum_{m=1}^{k}2^{m(p-1)/r}\\ &\leqslant C_1\sum_{k=1}^{\infty} E[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]\\ &\leqslant C_1E|X|.\end{aligned}$$

If $p = 1$, we obtain

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{mp/r-m/r}E[|X|I(|X|>2^{m/r})] &= \sum_{m=1}^{\infty}\sum_{k=m}^{\infty} E[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]\\ &= \sum_{k=1}^{\infty} kE[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]\\ &\leqslant C_1E|X|\log(1+|X|).\end{aligned}$$

For $p > 1$, it has

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{mp/r-m/r}E[|X|I(|X|>2^{m/r})] &= \sum_{m=1}^{\infty} 2^{m(p-1)/r}\sum_{k=m}^{\infty} E[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]\\ &\leqslant C_1\sum_{k=1}^{\infty} 2^{k(p-1)/r}E[|X|I(2^{k/r}<|X|\leqslant 2^{(k+1)/r})]\\ &\leqslant C_1E|X|^p.\end{aligned}$$

Together with (2.8), we obtain that

$$H_1 \leqslant \begin{cases} C_1\sum_{m=1}^{\infty} 2^{m-m/r}E[|X|I(|X|>2^{m/r})], & \text{if } p<r,\\ C_2\sum_{m=1}^{\infty} m2^{m-m/r}E[|X|I(|X|>2^{m/r})], & \text{if } p=r,\\ C_3\sum_{m=1}^{\infty} 2^{mp/r-m/r}E[|X|I(|X|>2^{m/r})], & \text{if } p>r,\end{cases}$$

$$\leqslant \begin{cases} \text{for } p<r, & \begin{cases} C_4E|X|<\infty, & \text{if } 0<r<1, \\ C_5E[|X|\log(1+|X|)]<\infty, & \text{if } r=1, \\ C_6E|X|^r<\infty, & \text{if } r>1, \end{cases} \\ \text{for } p=r, & \begin{cases} C_7E|X|<\infty, & \text{if } 0<r<1, \\ C_8E[|X|\log^2(1+|X|)]<\infty, & \text{if } r=1, \\ C_9E[|X|^r\log(1+|X|)]<\infty, & \text{if } r>1, \end{cases} \\ \text{for } p>r, & \begin{cases} C_{10}E|X|<\infty, & \text{if } 0<p<1, \\ C_{11}E[|X|\log(1+|X|)]<\infty, & \text{if } p=1, \\ C_{12}E|X|^p<\infty, & \text{if } p>1. \end{cases} \end{cases} \tag{2.9}$$

From Lemma 1.1, it follows that $\{X_{si}, 1\leqslant i\leqslant n\}$ are pairwise NQD random variables. Moreover, by Remark 3.1 of Li et al. [14], we obtain that $\{A_iX_{si}-E(A_iX_{si}), 1\leqslant i\leqslant n\}$ are also pairwise NQD random variables with mean zero. Therefore, by Markov's inequality, C_r inequality, (2.2) and Lemmas 1.2 and 1.3, we establish that

$$\begin{aligned} H_2 &\leqslant C_1\sum_{k=1}^{\infty}2^{-kp/r}\int_{2^{kp/r}}^{\infty}s^{-2/p}E\Big\{\max_{1\leqslant n\leqslant 2^k}\Big(\sum_{i=1}^{n}[A_iX_{si}-E(A_iX_{si})]\Big)^2\Big\}ds \\ &\leqslant C_2\sum_{k=1}^{\infty}2^{-kp/r}\int_{2^{kp/r}}^{\infty}s^{-2/p}k^2\Big(\sum_{i=1}^{2^k}EA_i^2EX_{si}^2\Big)ds \\ &\leqslant C_3\sum_{k=1}^{\infty}k^22^{-kp/r+k}\int_{2^{kp/r}}^{\infty}s^{-2/p}E[X^2I(|X|\leqslant s^{1/p})]ds \\ &\quad +C_4\sum_{k=1}^{\infty}k^22^{-kp/r+k}\int_{2^{kp/r}}^{\infty}P(|X|>s^{1/p})ds \\ &=: C_3H_{21}+C_4H_{22}. \end{aligned} \tag{2.10}$$

For H_{21}, it can be found that

$$\begin{aligned} H_{21} &= \sum_{k=1}^{\infty}k^22^{-kp/r+k}\sum_{m=k}^{\infty}\int_{2^{mp/r}}^{2^{(m+1)p/r}}s^{-2/p}E[X^2I(|X|\leqslant s^{1/p})]ds \\ &\leqslant \sum_{k=1}^{\infty}k^22^{-kp/r+k}\sum_{m=k}^{\infty}2^{mp/r-2m/r}E[X^2I(|X|\leqslant 2^{(m+1)/r})] \\ &= \sum_{m=1}^{\infty}2^{m(p-2)/r}E[X^2I(|X|\leqslant 2^{(m+1)/r})]\sum_{k=1}^{m}k^22^{k(1-p/r)}. \end{aligned}$$

For all $\alpha>0$ and positive integer $k\geqslant 1$, it can be argued that

$$\sum_{n=m}^{\infty}\frac{n^k}{2^{\alpha n}}\leqslant C\frac{m^k}{2^{\alpha m}}, \tag{2.11}$$

where $m\geqslant 1$ and C is a positive constant not depending on m.

Now, we consider the case $p<r$. By $r<2$ and (2.11), it follows

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(|X| \leqslant 2^{(m+1)/r})] \sum_{k=1}^{m} k^2 2^{k(1-p/r)} \\
\leqslant\ & C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} E[X^2 I(|X| \leqslant 2^{(m+1)/r})] \\
=\ & C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} E[X^2 I(|X| \leqslant 2^{1/r})] \\
&+ C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} \sum_{i=1}^{m} E[X^2 I(2^{i/r} < |X| \leqslant 2^{(i+1)/r})] \\
\leqslant\ & C_2 + C_1 \sum_{i=1}^{\infty} E[X^2 I(2^{i/r} < |X| \leqslant 2^{(i+1)/r})] \sum_{m=i}^{\infty} m^2 2^{m(r-2)/r} \\
\leqslant\ & C_2 + C_3 2^{(2-r)/r} \sum_{i=1}^{\infty} i^2 2^{(i+1)(r-2)/r} E[X^2 I(2^{i/r} < |X| \leqslant 2^{(i+1)/r})] \\
\leqslant\ & C_4 + C_5 E[|X|^r \log^2(1+|X|)].
\end{aligned}$$

Next, we consider the $p=r$. By $r<2$ and (2.11), we obtain that

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(|X| \leqslant 2^{(m+1)/r})] \sum_{k=1}^{m} k^2 2^{k(1-p/r)} \\
\leqslant\ & C_1 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} E[X^2 I(|X| \leqslant 2^{(m+1)/r})] \\
=\ & C_1 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} E[X^2 I(|X| \leqslant 2^{1/r})] \\
&+ C_1 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} \sum_{i=1}^{m} E[X^2 I(2^{i/r} < |X| \leqslant 2^{(i+1)/r})] \\
\leqslant\ & C_2 + C_1 \sum_{i=1}^{\infty} E[X^2 I(2^{i/r} < |X| \leqslant 2^{(i+1)/r})] \sum_{m=i}^{\infty} m^3 2^{m(r-2)/r} \\
\leqslant\ & C_2 + C_3 2^{(2-r)/r} \sum_{i=1}^{\infty} i^3 2^{(i+1)(r-2)/r} E[X^2 I(2^{i/r} < |X| \leqslant 2^{(i+1)/r})] \\
\leqslant\ & C_4 + C_5 E[|X|^r \log^3(1+|X|)].
\end{aligned}$$

Moreover, for the case $p>r$, we check by $p<2$ that

$$\sum_{m=1}^{\infty} 2^{m(p-2)/r} E[|X|^2 I(|X| \leqslant 2^{(m+1)/r})] \leqslant CE|X|^p.$$

Consequently, it follows

$$H_{21} \leqslant \begin{cases} C_1 \sum_{m=1}^{\infty} m^2 2^{m(r-2)/r} E[X^2 I(|X| \leqslant 2^{(m+1)/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m^3 2^{m(r-2)/r} E[X^2 I(|X| \leqslant 2^{(m+1)/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(|X| \leqslant 2^{(m+1)/r})], & \text{if } p > r, \end{cases}$$

$$\leqslant \begin{cases} C_4 + C_5 E[|X|^r \log^2(1+|X|)] < \infty, & \text{if } p < r, \\ C_6 + C_7 E[|X|^r \log^3(1+|X|)] < \infty, & \text{if } p = r, \\ C_8 E|X|^p < \infty, & \text{if } p > r. \end{cases} \tag{2.12}$$

In addition, similar to the proofs of (2.8) and (2.9), we get that

$$\begin{aligned} H_{22} &\leqslant \sum_{k=1}^{\infty} k^2 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E[|X| I(|X| > s^{1/p})] ds \\ &\leqslant C_2 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X| I(|X| > 2^{m/r})] \sum_{k=1}^{m} k^2 2^{k-kp/r} \\ &\leqslant \begin{cases} C_3 \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p < r, \\ C_4 \sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p = r, \\ C_5 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X| I(|X| > 2^{m/r})], & \text{if } p > r. \end{cases} \end{aligned} \tag{2.13}$$

Similarly, we consider the case $p < r$. If $0 < r < 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \sum_{m=1}^{k} m^2 2^{m(1-1/r)} \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X| I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \\ &\leqslant C_1 E|X|. \end{aligned}$$

If $r = 1$, then

$$\begin{aligned} \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X| I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X| I(2^k < |X| \leqslant 2^{k+1})] \sum_{m=1}^{k} m^2 \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X| \log^3(1+|X|) I(2^k < |X| \leqslant 2^{k+1})] \\ &\leqslant C_1 E[|X| \log^3(1+|X|)]. \end{aligned}$$

If $r > 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \sum_{m=1}^{k} m^2 2^{m-m/r} \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X|^r \log^2(1+|X|)I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \\ &\leqslant C_1 E[|X|^r \log^2(1+|X|)].\end{aligned}$$

Second, we consider the case $p = r$. If $0 < r < 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \sum_{m=1}^{k} m^3 2^{m(1-1/r)} \\ &\leqslant C_1 \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \\ &\leqslant C_2 E|X|.\end{aligned}$$

If $r = 1$, then

$$\begin{aligned}\sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &= \sum_{k=1}^{\infty} E[|X|I(2^k < ||X \leqslant 2^{k+1})] \sum_{m=1}^{k} m^3 \\ &\leqslant C_2 \sum_{k=1}^{\infty} E[|X| \log^4(1+|X|)I(2^k < |X| \leqslant 2^{k+1})] \\ &\leqslant C_2 E[|X| \log^4(1+|X|)].\end{aligned}$$

If $r > 1$, it follows

$$\begin{aligned}\sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})] &\leqslant \sum_{k=1}^{\infty} E[|X|I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] k^3 \sum_{m=1}^{k} 2^{m-m/r} \\ &\leqslant C_1 \sum_{k=1}^{\infty} k^3 2^{k-k/r} E[|X|I(2^{k/r} < |X| \leqslant 2^{(k+1)/r})] \\ &\leqslant C_2 E[|X|^r \log^3(1+|X|)].\end{aligned}$$

Third, we consider the case $p > r$. By (2.9), (2.13) and the inequalities above, it can be seen that

$$H_{22} \leqslant \begin{cases} C_1 \sum_{m=1}^{\infty} m^2 2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m^3 2^{m-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[|X|I(|X| > 2^{m/r})], & \text{if } p > r. \end{cases}$$

$$\leqslant \begin{cases} \text{for } p<r, & \begin{cases} C_4 E|X|<\infty, \text{ if } 0<r<1, \\ C_5 E[|X|\log^3(1+|X|)]<\infty, \text{ if } r=1, \\ C_6 E[|X|^r \log^2(1+|X|)]<\infty, \text{ if } r>1, \end{cases} \\ \text{for } p=r, & \begin{cases} C_7 E|X|<\infty, \text{ if } 0<r<1, \\ C_8 E[|X|\log^4(1+|X|)]<\infty, \text{ if } r=1, \\ C_9 E[|X|^r \log^3(1+|X|)]<\infty, \text{ if } r>1, \end{cases} \\ \text{for } p>r, & \begin{cases} C_{10} E|X|<\infty, \text{ if } 0<p<1, \\ C_{11} E[|X|\log(1+|X|)]<\infty, \text{ if } p=1, \\ C_{12} E|X|^p<\infty, \text{ if } p>1. \end{cases} \end{cases} \tag{2.14}$$

Furthermore, by the independence and $EX_i=0$, it has $E(A_iX_i)=0,\ 1\leqslant i\leqslant n$. So it follows

$$\begin{aligned} &\max_{1\leqslant n\leqslant 2^k}\Big|\sum_{i=1}^{n} E(A_iX_{si})\Big| \\ &= \max_{1\leqslant n\leqslant 2^k}\Big|\sum_{i=1}^{n} E\{A_i[-s^{1/p}I(X_i<-s^{1/p})-X_iI(|X_i|>s^{1/p})+s^{1/p}I(X_i>s^{1/p})]\}\Big| \\ &\leqslant 2\sum_{i=1}^{2^k} E|A_i|E[|X_i|I(|X_i|>s^{1/p})]. \end{aligned}$$

Obviously, by the proofs of (2.8), (2.9), we can establish that

$$H_3 \leqslant C_1\sum_{k=1}^{\infty} 2^{k-kp/r}\int_{2^{kp/r}}^{\infty} s^{-1/p}E[|X|I(|X|>s^{1/p})]ds$$

$$\leqslant \begin{cases} \text{for } p<r, & \begin{cases} C_1 E|X|<\infty, \text{ if } 0<r<1, \\ C_2 E[|X|\log(1+|X|)]<\infty, \text{ if } r=1, \\ C_3 E|X|^r<\infty, \text{ if } r>1, \end{cases} \\ \text{for } p=r, & \begin{cases} C_4 E|X|<\infty, \text{ if } 0<r<1, \\ C_5 E[|X|\log^2(1+|X|)]<\infty, \text{ if } r=1, \\ C_6 E[|X|^r \log(1+|X|)]<\infty, \text{ if } r>1, \end{cases} \\ \text{for } p>r, & \begin{cases} C_7 E|X|<\infty, \text{ if } 0<p<1, \\ C_8 E[|X|\log(1+|X|)]<\infty, \text{ if } p=1, \\ C_9 E|X|^p<\infty, \text{ if } p>1. \end{cases} \end{cases} \tag{2.15}$$

Therefore, (2.3) follows from (2.1), (2.5), (2.6), (2.9)–(2.12), (2.14) and (2.15) immediately. Applying (2.3), we establish (2.4) finally. □

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