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CONSIDERATIONS ABOUT THE SEVERAL INEQUALITIES IN AN INNER PRODUCT SPACE

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Abstract. The aim of this paper is to show new results concerning the Cauchy-Schwarz inequality in an inner product space. We find an improvement of Buzano's inequality and Richard's inequality, which are extensions of the Cauchy-Schwarz inequality.

1. Introduction

In a presentation Niculescu [10] makes a radiography of the inequalities that have played an important role in the Theory of Inequalities. The Cauchy-Schwarz Inequality is one of them. In 1821 Cauchy [4] shows the following identity:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)=\left(\sum_{i=1}^{n} a_i b_i\right)^2+\sum_{1\leqslant i<j\leqslant n}(a_i b_j-a_j b_i)^2. \tag{1}$$

In fact this is *Lagrange's identity*, because Lagrange in 1773 proved the identity

$$\left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right)=\left(\sum_{i=1}^{3} a_i b_i\right)^2+\sum_{1\leqslant i<j\leqslant 3}(a_i b_j-a_j b_i)^2.$$

A consequence of Lagrange's identity is the famous Cauchy-Schwarz inequality which states: if $\mathbf{a}=(a_1,\ldots,a_n)$ and $\mathbf{b}=(b_1,\ldots,b_n)$ are two $n-$tuples of real numbers, then

$$\sqrt{\left(a_1^2+\ldots+a_n^2\right)\left(b_1^2+\ldots+b_n^2\right)}\geqslant |a_1b_1+\ldots+a_nb_n|, \tag{2}$$

with equality holding if and only if $\mathbf{a}=\lambda\mathbf{b}$. This result is called the *Cauchy-Schwarz-Buniakowski inequality* or simply the *Cauchy inequality*.

Many refinements for Cauchy-Schwarz-Buniakowski inequality can be found in literature (see [2], [4], [5] and [11]). In particular, we mention one of them: Ostrowski [11], in 1952, proved the following: if $\mathbf{x}=(x_1,\ldots,x_n)$, $\mathbf{y}=(y_1,\ldots,y_n)$ and

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$\mathbf{z} = (z_1, \ldots, z_n)$ are $n-$tuples of real numbers such that $\mathbf{x}$ and $\mathbf{y}$ are not proportional and

$$\sum_{k=1}^{n} y_k z_k = 0, \quad \text{and} \quad \sum_{k=1}^{n} x_k z_k = 1, \quad \text{then} \tag{3}$$

$$\sum_{k=1}^{n} y_k^2 \Big/ \sum_{k=1}^{n} z_k^2 \leqslant \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 - \left(\sum_{k=1}^{n} x_k y_k \right)^2 .$$

For all $x, y \in X$ in an inner product space $X = (X, \langle \cdot, \cdot \rangle)$ over the field of complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$, then we have the Cauchy-Schwarz inequality, given by the following:

$$|\langle x, y \rangle| \leqslant \|x\| \cdot \|y\| . \tag{4}$$

The Cauchy-Schwarz inequality can be written, as in Aldaz [1] and Niculescu [10], in terms of the angular distance between two vectors, thus

$$\langle x, y \rangle = \|x\| \, \|y\| \left(1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \right), \tag{5}$$

for all nonzero vectors $x, y \in X$.

Buzano [3] showed an extension of the Cauchy-Schwarz inequality, given by the following:

$$|\langle a, x \rangle \langle x, b \rangle| \leqslant \frac{1}{2} \|x\|^2 \left(|\langle a, b \rangle| + \|a\| \cdot \|b\| \right). \tag{6}$$

for any $x, a, b \in X$.

It is easy to see that for $a = b$, inequality (6) becomes the Cauchy-Schwarz inequality.

Another inequality which included the Buzano inequality is mentioned by Precupanu [14] and Dragomir [6]:

$$\frac{1}{2} \|x\|^2 \left(|\langle a, b \rangle| - \|a\| \cdot \|b\| \right) \leqslant |\langle a, x \rangle \langle x, b \rangle| \leqslant \frac{1}{2} \|x\|^2 \left(|\langle a, b \rangle| + \|a\| \cdot \|b\| \right), \tag{7}$$

for any $x, a, b \in X$. In [7] Gavrea showed an extention of Buzano's inequality in inner product space. For real inner spaces, Richard [15], found the following stronger inequality

$$\left| \langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right| \leqslant \frac{1}{2} \|x\|^2 \|a\| \cdot \|b\| , \tag{8}$$

for any $x, a, b \in X$.

In [13], Popa and Raşa showed that, for any $x, a, b \in X$, the inequality

$$\left| \mathrm{Re} \left(\langle a, x \rangle \langle x, b \rangle - \frac{1}{2} \|x\|^2 \langle a, b \rangle \right) \right| \leqslant \frac{1}{2} \|x\|^2 \sqrt{\|a\|^2 \cdot \|b\|^2 - (\mathrm{Im} \, \langle a, b \rangle)^2}, \tag{9}$$

holds.

Dragomir [5] presented the following refinement of the Richard inequality:

$$\left|\langle a,b\rangle \|x\|^2 - \alpha \langle a,x\rangle \langle x,b\rangle\right| \leqslant \max\{1,|1-\alpha|\} \|a\| \cdot \|b\| \cdot \|x\|^2, \tag{10}$$

for all vectors x,a,b in an inner product space X and $\alpha \in \mathbb{C}$.

This inequality was found in another way by Khosravi *et al.* [8].

In [9], Lupu and Schwarz proved the following inequality:

$$\left|\|a\|^2 \langle b,c\rangle\right| + \left|\|b\|^2 \langle c,a\rangle\right| + \left|\|c\|^2 \langle a,b\rangle\right| \leqslant \|a\|^2 \|b\|^2 \|c\|^2 + 2\left|\langle a,b\rangle \langle b,c\rangle \langle c,a\rangle\right|, \tag{11}$$

for any vectors $x,a,b \in X$.

These inequalities are applied to the theory of Hilbert $\mathbb{C}^*-$modules over non-commutative $\mathbb{C}^*-$algebras, see Aldaz [1], Pečarić and Rajić [12] and Dragomir [5], [6].

2. Main results

For beginning we prove two lemmas:

LEMMA 1. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\|x+\alpha y\|^2 = \left|\alpha \|y\| + \frac{\langle x,y\rangle}{\|y\|}\right|^2 + \left\|x - \frac{\langle x,y\rangle}{\|y\|^2} y\right\|^2, \tag{12}$$

for all $x,y \in X$, $y \neq 0$, and for every $\alpha \in \mathbb{C}$.

Proof. By several calculations, we deduce the following:

$$\begin{aligned}\|x+ay\|^2 &= \langle x+\alpha y, x+\alpha y\rangle = \|x\|^2 + \bar{\alpha}\langle x,y\rangle + \alpha\overline{\langle x,y\rangle} + |\alpha|^2 \|y\|^2 \\ &= \left(\alpha\|y\| + \frac{\langle x,y\rangle}{\|y\|}\right)\left(\bar{\alpha}\|y\| + \frac{\overline{\langle x,y\rangle}}{\|y\|}\right) + \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} \\ &= \left|\alpha\|y\| + \frac{\langle x,y\rangle}{\|y\|}\right|^2 + \left\|x - \frac{\langle x,y\rangle}{\|y\|^2} y\right\|^2,\end{aligned}$$

because, we have $\left\|x - \frac{\langle x,y\rangle}{\|y\|^2} y\right\|^2 = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$. □

REMARK 1. Let $x,e \in X$ with $\|e\| = 1$. If we take $y = e$ and $\alpha = -\lambda$ in relation (12), then we obtain $\|x-\lambda e\|^2 = |\lambda - \langle x,e\rangle|^2 + \|x - \langle x,e\rangle e\|^2$. Consequently, we deduce $\|x - \langle x,e\rangle e\|^2 = \inf_{\lambda\in\mathbb{C}} \|x-\lambda e\|^2$ which is a result found in [9].

LEMMA 2. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\left\|\langle a,x\rangle x - \frac{1}{2}\|x\|^2 a\right\| = \frac{1}{2}\|x\|^2 \|a\|, \tag{13}$$

for all $x, a \in X$.

Proof. For $x = 0$ the equality is true. For $x \neq 0$ inequality (13) becomes $\left\| a - 2\frac{\langle a,x\rangle}{\|x\|^2} x \right\| = \|a\|$. If we take in equality (12) $\alpha = -2$, $y = \frac{\langle a,x\rangle}{\|x\|^2} x$, then by simple calculations, we deduce the following:

$$\left\| a - 2\frac{\langle a,x\rangle x}{\|x\|^2} \right\|^2 = \left| -2\frac{|\langle a,x\rangle|}{\|x\|} + \frac{|\langle a,x\rangle|}{\|x\|} \right|^2 + \left\| a - \frac{\langle a,x\rangle}{\|x\|^2} x \right\|^2 = \|a\|^2.$$

Consequently, inequality (13) is true. □

REMARK 2. A simple proof of Richard's inequality can be given by combinating the Cauchy-Schwarz inequality and relation (13), thus:

$$\begin{aligned} \left| \langle a,x\rangle \langle x,b\rangle - \frac{1}{2}\|x\|^2 \langle a,b\rangle \right| &= \left| \left\langle \langle a,x\rangle x - \frac{1}{2}\|x\|^2 a, b \right\rangle \right| \\ &\leqslant \left\| \langle a,x\rangle x - \frac{1}{2}\|x\|^2 a \right\| \|b\| \\ &= \frac{1}{2}\|x\|^2 \|a\| \|b\|. \end{aligned}$$

THEOREM 1. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$|\alpha|^2 \|y\|^2 \|z\|^2 + \|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2 \geqslant 2\mathrm{Re}\left(\bar{\alpha} \left(\langle x,y\rangle \langle y,z\rangle - \langle x,z\rangle \|y\|^2 \right) \right), \tag{14}$$

for all $x, y, z \in X$, *and for every* $\alpha \in \mathbb{C}$.

Proof. For $y = 0$, inequality (14) is true. In the situation $y \neq 0$, using Lemma 1, we obtain the following calculations:

$$\begin{aligned} &\left\| x + \alpha z - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 \\ &= \|x + \alpha z\|^2 - \frac{2\mathrm{Re}\,(\bar{\alpha}\langle x,y\rangle \langle y,z\rangle)}{\|y\|^2} - \frac{|\langle x,y\rangle|^2}{\|y\|^2} \\ &= \|x\|^2 + \bar{\alpha}\langle x,z\rangle + \alpha\overline{\langle x,z\rangle} + |\alpha|^2 \|z\|^2 - \frac{2\mathrm{Re}\,(\bar{\alpha}\langle x,y\rangle \langle y,z\rangle)}{\|y\|^2} - \frac{|\langle x,y\rangle|^2}{\|y\|^2} \\ &= |\alpha|^2 \|z\|^2 + 2\mathrm{Re}\,(\bar{\alpha}\langle x,z\rangle) - \frac{2\mathrm{Re}\,(\bar{\alpha}\langle x,y\rangle \langle y,z\rangle)}{\|y\|^2} + \frac{\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2}{\|y\|^2} \\ &= \frac{1}{\|y\|^2} \Big[|\alpha|^2 \|y\|^2 \|z\|^2 - 2\left(\mathrm{Re}\,(\bar{\alpha}\langle x,y\rangle \langle y,z\rangle) - \mathrm{Re}\,(\bar{\alpha}\langle x,z\rangle)\|y\|^2 \right) \\ &\quad + \|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2 \Big] \geqslant 0 \end{aligned}$$

Consequently, we deduce the inequality of the statement. □

COROLOLARY 1. *In an inner product space X over the field of real numbers $\mathbb{R}$, we have*

$$\frac{\|y\|^2}{\|z\|^2}\left(\frac{\langle x,y\rangle\langle y,z\rangle}{\|y\|^2}-\langle x,z\rangle\right)^2 \leqslant \|x\|^2\|y\|^2-\langle x,y\rangle^2, \tag{15}$$

for all $x,y,z\in X$, $y\neq 0$, $z\neq 0$.

Proof I. If $y\neq 0$, $z\neq 0$, then we apply Theorem 1 for $\alpha\in\mathbb{R}$, and we have

$$\|y\|^2\|z\|^2\alpha^2-2\alpha\left(\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2\right)+\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geqslant 0,$$

for all $x,y,z\in X$, and for every $\alpha\in\mathbb{R}$. Since $\|y\|^2\|z\|^2>0$, then the discriminant is negative, i.e.,

$$\Delta=\left(\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2\right)^2-\|y\|^2\|z\|^2\left(\|x\|^2\|y\|^2-\langle x,y\rangle^2\right)\leqslant 0.$$

Therefore, we prove the statement. □

Proof II. For $\alpha=\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2$ in relation (14), we have

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geqslant\left(2-\|y\|^2\|z\|^2\right)\left|\langle x,y\rangle\langle y,z\rangle-\langle x,z\rangle\|y\|^2\right|^2.$$

for $x=0$, inequality (15) is true. In the situation $x\neq 0$, $y\neq 0$, if we replace in the above relation x and y by $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$, then we deduce the statement. □

REMARK 3. If we take $\langle x,z\rangle=1$ and $\langle y,z\rangle=0$, in inequality (15), then we find the inequality of Ostrowski for inner product spaces over the field of real numbers,

$$\|y\|^2/\|z\|^2\leqslant\|x\|^2\|y\|^2-\langle x,y\rangle^2, \tag{16}$$

for all $x,y,z\in X$, $y\neq 0$, $z\neq 0$.

It is easy to see that for $x,y,z\in\mathbb{R}^n$ we obtain inequality (3).

THEOREM 2. *In an inner product space X over the field real or complex numbers, for any nonzero vectors $x,a,b\in X$, we have*

$$\frac{1}{2}\|x\|^2\|a\|\cdot\|b\|-\left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right|\geqslant\frac{A}{\|x\|^2\|a\|\cdot\|b\|}\geqslant 0, \tag{17}$$

where

$$A=\left(|\langle a,x\rangle|\left(\|x\|^2\|b\|^2-|\langle x,b\rangle|^2\right)-\frac{1}{2}\|x\|^2\left(\|a\|^2\|b\|^2-|\langle a,b\rangle|^2\right)\right)^2.$$

Proof. The equality $\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2=\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}$ implies

$$\left\|\|y\|x-\frac{\langle x,y\rangle}{\|y\|}y\right\|^2=\|x\|^2\|y\|^2-|\langle x,y\rangle|^2.$$

Hence, using the Cauchy-Schwarz inequality, we have the relation

$$\|x\|\cdot\|y\|-|\langle x,y\rangle| = \frac{\left\|\|y\|x-\frac{\langle x,y\rangle}{\|y\|}y\right\|^2}{\|x\|\cdot\|y\|+|\langle x,y\rangle|} \geqslant \frac{\left\|\|y\|x-\frac{\langle x,y\rangle}{\|y\|}y\right\|^2}{2\|x\|\cdot\|y\|}.$$

In this inequality, we make the substitutions $x \longrightarrow \langle a,x\rangle x-\frac{1}{2}\|x\|^2 a$ and $y \longrightarrow b$ in the above inequality and using the equality (13), $\left\|\langle a,x\rangle x-\frac{1}{2}\|x\|^2 a\right\| = \frac{1}{2}\|x\|^2 a$, implies

$$\frac{1}{2}\|x\|^2\|a\|\cdot\|b\| - \left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right| \geqslant \frac{\|u-v\|^2}{\|x\|^2\|a\|\cdot\|b\|},$$

where $u=\langle a,x\rangle\left(\|b\|x-\frac{\langle a,x\rangle\langle x,b\rangle}{\|b\|}b\right)$ and $v=-\frac{1}{2}\|x\|^2\left(\|b\|a-\frac{\langle a,b\rangle}{\|b\|}b\right)$.

It follows that $\|u\| = |\langle a,x\rangle|\left(\|x\|^2\|b\|^2-|\langle x,b\rangle|^2\right)$ and

$$\|u\| = \frac{1}{2}\|x\|^2\left(\|a\|^2\|b\|^2-|\langle a,b\rangle|^2\right).$$

Since $\|u-v\|^2 \geqslant |\|u\|-\|v\||^2$, $u,v\in X$, we obtain the inequality of the statement. $\square$

REMARK 4. a) For real or complex inner spaces, inequality (17) represents an improvement of Richard's inequality, given thus:

$$\left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right| \leqslant \frac{1}{2}\|x\|^2\|a\|\cdot\|b\| - \frac{A}{\|x\|^2\|a\|\cdot\|b\|},$$

where

$$A=\left(|\langle a,x\rangle|\left(\|x\|^2\|b\|^2-|\langle x,b\rangle|^2\right)-\frac{1}{2}\|x\|^2\left(\|a\|^2\|b\|^2-|\langle a,b\rangle|^2\right)\right)^2.$$

b) Also, using above inequality, and from the continuity property of the modulus, i.e., $|\alpha-\beta| \geqslant ||\alpha|-|\beta||$, α, $\beta\in\mathbb{C}$, we deduce the inequality

$$\begin{aligned}
&\frac{1}{2}\|x\|^2\left(|\langle a,b\rangle|-\|a\|\cdot\|b\|\right)+\frac{A}{\|x\|^2\|a\|\cdot\|b\|}\\
&\leqslant |\langle a,x\rangle\langle x,b\rangle|\\
&\leqslant \frac{1}{2}\|x\|^2\left(|\langle a,b\rangle|+\|a\|\cdot\|b\|\right)-\frac{A}{\|x\|^2\|a\|\cdot\|b\|}
\end{aligned}$$

which is in fact a refinement of Buzano's inequality.

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