Journal of
Mathematical
Inequalities
Volume 12, Number 1 (2018), 201–217
doi:10.7153/jmi-2018-12-16

COMPLETE AND COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF $\tilde{\rho}$–MIXING RANDOM VARIABLES

FENGXIANG FENG, DINGCHENG WANG AND QUNYING WU

(*Communicated by X. Wang*)

Abstract. In this paper, we establish complete convergence results and a complete moment convergence result and prove the equivalence of them for weighted sums of $\tilde{\rho}$-mixing random variables. Our results generalize and improve the results of Baum and Katz(1965) and Peligrad and Gut (1999). As an application, we obtain the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of $\tilde{\rho}$-mixing random variables.

1. Introduction

Let $\{X_n; n \geqslant 1\}$ be a sequence of random variables on a probability space $(\Omega, \mathscr{A}, P)$. Define $\mathscr{F}_S = \sigma(X_i : i \in S \subset \mathbb{N})$. Given $\sigma-$ fields $\mathscr{F}, \mathscr{R} \subset \mathscr{A}$, let

$$\rho(\mathscr{F}, \mathscr{R}) = \sup\{|\mathrm{corr}(X,Y)|, X \in L_2(\mathscr{F}), Y \in L_2(\mathscr{R})\},$$

where $\mathrm{corr}(X,Y) = (EXY - EXEY)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$.

For $k \geqslant 0$, define the following coefficients of dependence by

$$\tilde{\rho}(k) = \sup\{\rho(\mathscr{F}_S, \mathscr{F}_T);\ S,T \subset \mathbb{N} \text{ with } \mathrm{dist}(S,T) \geqslant k\}.$$

DEFINITION 1. A sequence of random variables $\{X_n; n \geqslant 1\}$ is called $\tilde{\rho}$-mixing sequence if there exists $k \in \mathbb{N}$ such that $\tilde{\rho}(k) < 1$.

Obviously, $0 \leqslant \tilde{\rho}(k+1) \leqslant \tilde{\rho}(k) \leqslant \tilde{\rho}(0) = 1$.

Many limit results for $\tilde{\rho}$-mixing sequence of random variables have been obtained by many authors. We can refer to Bradley [4] for the central limit theorem, Bryc and Smolenski [5], and Yang [17] for moment inequalities and the strong law of large numbers, Peligrad and Gut [9], Gan [6], Wu and Jiang [16], Kuczmaszewska [8] for almost sure convergence, Utev and Peligrad [14] for maximal inequalities and the invariance principle, Peligrad and Gut [9], An and Yuan [1], Gan [6], Kuczmaszewska [7], and Sung [13] for complete convergence. Shen et al. [11] for complete convergence of weighted sums for arrays of rowwise $\tilde{\rho}$-mixing random variables. In this paper we

Mathematics subject classification (2010): 60F15.
Keywords and phrases: Complete convergence, complete moment convergence, weighted sums, $\tilde{\rho}$-mixing random variables.

© ELEMENT, Zagreb
Paper JMI-12-16

further study complete and complete moment convergence for weighted sums of $\tilde{\rho}$-mixing random variables. Our results extend and improve the results of Baum and Katz [3] and Peligrad and Gut [9].

Throughout the paper, $I(.)$ denotes the indicator function and $x^+ = xI(x \geqslant 0)$. C denotes a positive constant which may differ from one place to another.

In this article, we will consider the following condition:

Let $\{X_n, n \geqslant 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables, there exist a random variable X and a positive constant C such that for all $x \geqslant 0$

$$\sup_{i \geqslant 1} P(|X_i| > x) \leqslant CP(|X| > x). \tag{1.1}$$

2. Main results

Let's state our main results.

THEOREM 2.1. *Let $p\alpha > 1$ and $1/2 < \alpha \leqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_i = 0$ for each $i \geqslant 1$. Assume that $\{a_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \text{ for some } q > p. \tag{2.1}$$

Let $l(x) > 0$ be a nondecreasing slowly varying function.

If there exist a random variable X and a positive constant C such that (1.1) holds for all $x \geqslant 0$ and

$$E|X|^p l\left(|X|^{1/\alpha}\right) < \infty, \tag{2.2}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} l(n) P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{2.3}$$

THEOREM 2.2. *Under the conditions of Theorem 2.1, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) E\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty. \tag{2.4}$$

Furthermore, (2.3) is equivalent to (2.4).

From Theorem 2.1, we can obtain the following result:

COROLLARY 2.1. *Under the conditions of Theorem 2.1, assume that $\{a_n; n \geqslant 1\}$ is a sequence of constants satisfying*

$$\sum_{i=1}^{n} |a_i|^q = O(n) \text{ for some } q > p. \tag{2.5}$$

Then for any $\varepsilon > 0$*, we have*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} l(n) P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \tag{2.6}$$

and

$$n^{-\alpha}\sum_{i=1}^{n} a_i X_i \to 0 \ \ a.s.\ n\to\infty. \tag{2.7}$$

REMARK 1. If we take $l(x) = 1$ in Theorem 2.1, then we can get the result of Sung [13]. Theorem 2.1 extends the result of of Sung [13].

REMARK 2. If we take $l(x) = 1, a_{ni} = 1$ in Theorem 2.1, then we can get the Baum-Katz-type theorem for sequences of $\tilde{\rho}$-mixing random variables. Theorem 2.1 extends and improves the results of Baum and Katz [3] and Peligrad and Gut [9]. Theorem 2.1 extends the result of Baum and Katz [3] from the i.i.d. case to weighted sums of $\tilde{\rho}$-mixing random variables.

3. Preliminaries

In order to prove our results, we need the following lemmas.

LEMMA 3.1. (Utev and Peligrad [14]) *Let* $\{X_n; n \geqslant 1\}$ *be a sequence of* $\tilde{\rho}$*-mixing random variables with* $EX_n = 0$ *and* $E|X_n|^p < \infty$ *for some* $p \geqslant 2$ *and all* $n \geqslant 1$*. Suppose* N *is a positive integer,* $0 \leqslant r < 1$ *and* $\tilde{\rho}(N) \leqslant r$*. Then there exists a positive constant* $D = D(N, r, p)$ *such that the following statements hold:*

$$E\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leqslant D\left(\left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} + \sum_{i=1}^{n} E|X_i|^p\right),\ p \geqslant 2.$$

Lemma 3.2 can be found in Lemma 4.1.6 of Wu [15].

LEMMA 3.2. *Let* $\{X_n; n \geqslant 1\}$ *be a sequence of random variables. If there exist a random variable* X *and a positive constant* C *such that (1.1) holds for all* $x \geqslant 0$*, then for any* $\alpha > 0$ *and* $b > 0$*,*

$$E|X_n|^{\alpha} I(|X_n| \leqslant b) \leqslant C\left(E|X|^{\alpha} I(|X| \leqslant b) + b^{\alpha} P(|X| > b)\right).$$

$$E|X_n|^{\alpha} I(|X_n| > b) \leqslant CE|X|^{\alpha} I(|X| > b).$$

LEMMA 3.3. (Shen and Wu [10]) *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables and* $l(x) > 0$ $(x > 0)$ *be a nondecreasing slowly varying function as* $x \to \infty$*. Let* s *and* t *be arbitrary positive constants. If (1.1) holds, then*

$$\sup_{i\geqslant 1} E|X_i|^s l\left(|X_i|^t\right) \leqslant CE|X|^s l\left(|X|^t\right)$$

4. Proofs of Main results

Proof of Theorem 2.1. Without loss of generality, we may assume that $\sum_{i=1}^{n}|a_{ni}|^q \leqslant n$ for some $q>p$.

Firstly, we will show if $\{a_{ni}; 1\leqslant i\leqslant n, n\geqslant 1\}$ is an array of real numbers satisfying $|a_{ni}|\leqslant 1$ for $1\leqslant i\leqslant n$ and $n\geqslant 1$, then (2.3) holds.

Let $Y_{ni}=X_iI(|X_i|\leqslant n^\alpha)$ for $1\leqslant i\leqslant n, n\geqslant 1$. We can easily get

$$\begin{aligned}
&P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon n^\alpha\right)\\
&\leqslant P\left(\max_{1\leqslant i\leqslant n}|X_i|>n^\alpha\right)+P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}Y_{ni}\right|>\varepsilon n^\alpha\right)\\
&\leqslant \sum_{i=1}^{n}P(|X_i|>n^\alpha)+P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}(Y_{ni}-EY_{ni})\right|>\varepsilon n^\alpha-\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}EY_{ni}\right|\right).
\end{aligned} \tag{4.1}$$

From the proof of (2.4) of Sung [13], we have

$$n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}EY_{ni}\right|\to 0 \text{ as } n\to\infty. \tag{4.2}$$

Hence for n large enough, we have $n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}EY_{ni}\right|<\frac{\varepsilon}{2}$. It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{p\alpha-2}l(n)P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon n^\alpha\right)\\
&\leqslant \sum_{n=1}^{\infty}n^{p\alpha-2}l(n)\sum_{i=1}^{n}P(|X_i|>n^\alpha)\\
&\quad+\sum_{n=1}^{\infty}n^{p\alpha-2}l(n)P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}(Y_{ni}-EY_{ni})\right|>\varepsilon n^\alpha/2\right)\\
&=: I+CJ.
\end{aligned} \tag{4.3}$$

For I, by (1.1), (2.2) and the fact $\sum_{n=1}^{j}n^\alpha l(n)=O(j^{\alpha+1}l(j))$, $\alpha\neq -1$, we have

$$\begin{aligned}
I&=\sum_{n=1}^{\infty}n^{p\alpha-2}l(n)\sum_{i=1}^{n}P(|X_i|>n^\alpha)\\
&\leqslant C\sum_{n=1}^{\infty}n^{p\alpha-2}l(n)\sum_{i=1}^{n}P(|X|>n^\alpha)=C\sum_{n=1}^{\infty}n^{p\alpha-1}l(n)P(|X|>n^\alpha)\\
&=C\sum_{n=1}^{\infty}n^{p\alpha-1}l(n)\sum_{j=n}^{\infty}P\left(j<|X|^{1/\alpha}\leqslant j+1\right)
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{j=1}^{\infty} P\left(j<|X|^{1/\alpha}\leqslant j+1\right)\sum_{n=1}^{j} n^{p\alpha-1}l(n)\\
&\leqslant C\sum_{j=1}^{\infty} P\left(j<|X|^{1/\alpha}\leqslant j+1\right) j^{p\alpha}l(j)\leqslant CE|X|^{p}l\left(|X|^{1/\alpha}\right)<\infty. \qquad (4.4)
\end{aligned}
$$

Thus, it remains to show that $J<\infty$. we will consider the following two cases:

Case 1. $p\geqslant 2$.

By the fact $|a_{ni}|\leqslant 1$, Markov's inequality and Lemma 3.1, we have that for any $r>p$

$$
\begin{aligned}
J &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)E\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(a_{ni}Y_{ni}-a_{ni}EY_{ni})\right|^{r}\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left[\sum_{i=1}^{n}|a_{ni}|^{r}E\left|Y_{ni}-EY_{ni}\right|^{r}+\left(\sum_{i=1}^{n}a_{ni}^{2}E\left|Y_{ni}-EY_{ni}\right|^{2}\right)^{\frac{r}{2}}\right]\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\sum_{i=1}^{n}|a_{ni}|^{r}\left(E|X|^{r}I(|X|\leqslant n^{\alpha})+n^{r\alpha}P(|X|>n^{\alpha})\right)\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left[\sum_{i=1}^{n}a_{ni}^{2}\left(E|X|^{2}I(|X|\leqslant n^{\alpha})+n^{2\alpha}P(|X|>n^{\alpha})\right)\right]^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-1}l(n)E|X|^{r}I(|X|\leqslant n^{\alpha})+C\sum_{n=1}^{\infty} n^{p\alpha-1}l(n)P(|X|>n^{\alpha})\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+\frac{r}{2}}l(n)\left(E|X|^{2}I(|X|\leqslant n^{\alpha})\right)^{\frac{r}{2}}+C\sum_{n=1}^{\infty} n^{p\alpha-2+\frac{r}{2}}l(n)\left(P(|X|>n^{\alpha})\right)^{\frac{r}{2}}\\
&=: CJ_1+CJ_2+CJ_3+CJ_4. \qquad (4.5)
\end{aligned}
$$

By the fact $\sum_{n=j}^{\infty} n^{\alpha}l(n)=O(j^{\alpha+1}l(j))$, $\alpha<-1$, we have

$$
\begin{aligned}
J_1 &= \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-1}l(n)E|X|^{r}I(|X|\leqslant n^{\alpha})\\
&\leqslant \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-1}l(n)\sum_{j=1}^{n} j^{r\alpha}P\left(j-1<|X|^{1/\alpha}\leqslant j\right)\\
&= \sum_{j=1}^{\infty} j^{r\alpha}P\left(j-1<|X|^{1/\alpha}\leqslant j\right)\sum_{n=j}^{\infty} n^{p\alpha-r\alpha-1}l(n)\\
&\leqslant C\sum_{j=1}^{\infty} j^{r\alpha}P\left(j-1<|X|^{1/\alpha}\leqslant j\right) j^{\alpha(p-r)}l(j)\\
&= C\sum_{j=1}^{\infty} j^{p\alpha}l(j)P\left(j-1<|X|^{1/\alpha}\leqslant j\right)\leqslant CE|X|^{p}l\left(|X|^{1/\alpha}\right)<\infty. \qquad (4.6)
\end{aligned}
$$

From the proof of $I<\infty$, we can obtain $J_2<\infty$. For J_3, noting that $p\alpha>1$ and $p\geqslant 2$, we take $r>\max\{p,\frac{p\alpha-1}{\alpha-\frac{1}{2}}\}$, which implies that $\alpha(p-r)-2+\frac{r}{2}<-1$. Hence ,

$$\begin{aligned} J_3 &= C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+\frac{r}{2}} l(n)\left(E|X|^2 I(|X|\leqslant n^{\alpha})\right)^{\frac{r}{2}} \\ &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+\frac{r}{2}} l(n)\left(E|X|^2\right)^{\frac{r}{2}}<\infty. \end{aligned} \tag{4.7}$$

$$J_4 \leqslant \sum_{n=1}^{\infty} n^{-1-(p\alpha-1)(\frac{r}{2}-1)} l(n)\left(E|X|^p\right)^{\frac{r}{2}}<\infty. \tag{4.8}$$

Case 2. $p<2$.

We take $r=2$. Similar to the proof of $J_1<\infty$ and $J_2<\infty$, we can have

$$\begin{aligned} J &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2\alpha-2} l(n) E\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(a_{ni}Y_{ni}-a_{ni}EY_{ni})\right|^2 \\ &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2\alpha-1} l(n) E|X|^2 I(|X|\leqslant n^{\alpha})+C\sum_{n=1}^{\infty} n^{p\alpha-1} l(n) P(|X|>n^{\alpha})<\infty. \end{aligned} \tag{4.9}$$

Secondly, we will show if $\{a_{ni};1\leqslant i\leqslant n,n\geqslant 1\}$ is an array of real numbers satisfying $|a_{ni}|=0$ or $|a_{ni}|>1$ for $1\leqslant i\leqslant n$ and $n\geqslant 1$, then (2.3) holds.

Let $S'_{nj}=\sum_{i=1}^{j} a_{ni}X_i I\left(|a_{ni}X_i|\leqslant n^{\alpha}\right)$ for $1\leqslant j\leqslant n$, $n\geqslant 1$. From the proof of (2.11) of Sung [13], we have $n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|ES'_{nj}\right|\to 0$ as $n\to\infty$. Hence for n large enough, we have $n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|ES'_{nj}\right|<\frac{\varepsilon}{2}$. It follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{p\alpha-2} l(n) P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right|>\varepsilon n^{\alpha}\right) \\ &\leqslant \sum_{n=1}^{\infty} n^{p\alpha-2} l(n)\sum_{i=1}^{n} P\left(|a_{ni}X_i|>n^{\alpha}\right)+\sum_{n=1}^{\infty} n^{p\alpha-2} l(n) P\left(\max_{1\leqslant j\leqslant n}\left|S'_{nj}-ES'_{nj}\right|>\varepsilon n^{\alpha}/2\right) \\ &=: L+CM. \end{aligned} \tag{4.10}$$

For $1\leqslant j\leqslant n-1$ and $n\geqslant 1$, let

$$I_{nj}=\left\{1\leqslant i\leqslant n: n^{1/q}(j+1)^{-1/q}<|a_{ni}|\leqslant n^{1/q}j^{-1/q}\right\}.$$

Then $\{I_{nj},1\leqslant j\leqslant n-1\}$ are disjoint, $\bigcup_{j=1}^{n-1} I_{nj}=\{1\leqslant i\leqslant n: a_{ni}\neq 0\}$, and $\sum_{j=1}^{k}\sharp I_{nj}\leqslant k+1$ for $1\leqslant k\leqslant n-1$.

Let $t = 1/(\alpha - 1/q)$. Similar to the proof (2.15) of Sung [13], we have

$$
\begin{aligned}
L &= \sum_{n=2}^{\infty} n^{p\alpha-2} l(n) \sum_{j=1}^{n-1} \sum_{i \in I_{nj}} P\left(|a_{ni}X_i| > n^{\alpha}\right) \\
&= \sum_{n=}^{\infty} n^{p\alpha-2} l(n) \sum_{j=1}^{n-1} P\left(|X| > n^{\alpha-1/q} j^{1/q}\right) \sharp I_{nj} \\
&\leqslant \sum_{n=1}^{\infty} n^{p\alpha-2} l(n) \sum_{k=n}^{[n^{1+t/q}]} P\left(k < |X|^t \leqslant k+1\right) \left(\left[\left(\frac{k+1}{n}\right)^{q/t}\right] + 1\right) \\
&\quad + \sum_{n=2}^{\infty} n^{p\alpha-1} l(n) \sum_{k=[n^{1+t/q}]+1}^{\infty} P\left(k < |X|^t \leqslant k+1\right) \\
&=: L_1 + L_2.
\end{aligned} \tag{4.11}
$$

Since $p\alpha - 2 - q/t = -\alpha(q-p) - 1 < -1$, we get

$$
\begin{aligned}
L_1 &\leqslant C \sum_{n=2}^{\infty} n^{p\alpha-2-q/t} l(n) \sum_{k=n}^{[n^{1+t/q}]} P\left(k < |X|^t \leqslant k+1\right) k^{q/t} \\
&\leqslant C \sum_{k=2}^{\infty} P\left(k < |X|^t \leqslant k+1\right) k^{q/t} \sum_{n=[k^{q/(q+t)}]}^{k} n^{p\alpha-2-q/t} l(n) \\
&\leqslant C \sum_{k=2}^{\infty} P\left(k < |X|^t \leqslant k+1\right) k^{q/t} \sum_{n=[k^{q/(q+t)}]}^{\infty} n^{p\alpha-2-q/t} l(n) \\
&\leqslant C \sum_{k=2}^{\infty} P\left(k < |X|^t \leqslant k+1\right) k^{q/t - q\alpha(q-p)/(q+t)} l(k^{q/(q+t)}) \\
&\leqslant CE|X|^p l(|X|^{1/\alpha}) < \infty.
\end{aligned} \tag{4.12}
$$

we also get

$$
\begin{aligned}
L_2 &\leqslant \sum_{k=1}^{\infty} P\left(k < |X|^t \leqslant k+1\right) \sum_{n=1}^{k^{q/(q+t)}]} n^{p\alpha-1} l(n) \\
&\leqslant C \sum_{k=1}^{\infty} P\left(k < |X|^t \leqslant k+1\right) k^{p(\alpha-1/q)} l(k^{q/(q+t)}) \\
&\leqslant CE|X|^p l(|X|^{1/\alpha}) < \infty.
\end{aligned} \tag{4.13}
$$

From $L_1 < \infty$ and $L_2 < \infty$, thus it remains to show that $M < \infty$.

By Markov's inequality, Lemma 3.1 and Lemma 3.2, for any $r \geqslant 2$ we have

$$\begin{aligned}
M &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)E\max_{1\leqslant j\leqslant n}\left|S'_{nj}-ES'_{nj}\right|^{r}\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left(\sum_{i=1}^{n}E|a_{ni}X_i|^2 I\left(|a_{ni}X_i|\leqslant n^{\alpha}\right)\right)^{\frac{r}{2}}\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\sum_{i=1}^{n}E|a_{ni}X_i|^r I\left(|a_{ni}X_i|\leqslant n^{\alpha}\right)\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left(\sum_{i=1}^{n}E|a_{ni}X|^2 I\left(|a_{ni}X|\leqslant n^{\alpha}\right)+\sum_{i=1}^{n}n^{2\alpha}P\left(|a_{ni}X|> n^{\alpha}\right)\right)^{\frac{r}{2}}\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left(\sum_{i=1}^{n}E|a_{ni}X|^r I\left(|a_{ni}X|\leqslant n^{\alpha}\right)+\sum_{i=1}^{n}n^{r\alpha}P\left(|a_{ni}X|> n^{\alpha}\right)\right)\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left(\sum_{i=1}^{n}E|a_{ni}X|^2 I\left(|a_{ni}X|\leqslant n^{\alpha}\right)\right)^{\frac{r}{2}}\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\sum_{i=1}^{n}E|a_{ni}X|^r I\left(|a_{ni}X|\leqslant n^{\alpha}\right)\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-2}l(n)\sum_{i=1}^{n}P\left(|a_{ni}X|> n^{\alpha}\right)\\
&=: CM_1+CM_2+L'. \qquad (4.14)
\end{aligned}$$

From $L<\infty$, we have $L'<\infty$. For M_1 and M_2, we proceed with two cases.

(i) If $p\geqslant 2$, then we will take r large enough such that $r>\max\{\frac{p\alpha-1}{\alpha-\frac{1}{2}},q\}$. By the fact that $a_{ni}=0$ or $|a_{ni}|>1$, then we get that

$$\begin{aligned}
M_1 &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left(\sum_{i=1}^{n}|a_{ni}|^2\right)^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}l(n)\left(\sum_{i=1}^{n}|a_{ni}|^q\right)^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2}l(n)<\infty \qquad (4.15)
\end{aligned}$$

Similar to the proof of (2.21) of Sung [13], we also obtain that

$$\begin{aligned}
M_2 &= \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2}l(n)\sum_{j=1}^{n-1}\sum_{i\in I_{nj}}E|a_{ni}X|^r I\left(|a_{ni}X|\leqslant n^{\alpha}\right)\\
&\leqslant \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q}l(n)\sum_{j=1}^{n-1} j^{-r/q}\sharp I_{nj}E|X|^r I\left(|X|\leqslant n^{\alpha-1/q}(j+1)^{1/q}\right)
\end{aligned}$$

$$\leqslant \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} l(n) \sum_{j=1}^{n-1} j^{-r/q} \sharp I_{nj} \sum_{k=0}^{2n} E|X|^r I\left(k < |X|^t \leqslant k+1\right)$$
$$+ \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} l(n) \sum_{j=1}^{n-1} j^{-r/q} \sharp I_{nj} \sum_{k=2n+1}^{[n(j+1)^{t/q}]} E|X|^r I\left(k < |X|^t \leqslant k+1\right)$$
$$=: M_3 + M_4. \tag{4.16}$$

Similar to the proof of (2.22) of Sung [13], we have that

$$\begin{aligned} M_3 &\leqslant C \sum_{k=1}^{\infty} E|X|^r I\left(k < |X|^t \leqslant k+1\right) \sum_{n=[k/2]}^{\infty} n^{p\alpha-r\alpha-2+r/q} l(n) \\ &\leqslant C \sum_{k=1}^{\infty} E|X|^r I\left(k < |X|^t \leqslant k+1\right) k^{p\alpha-r\alpha-1+r/q} l(k) \\ &\leqslant C \sum_{k=1}^{\infty} P\left(k < |X|^t \leqslant k+1\right) k^{p\alpha-1} l(k) \\ &\leqslant C \sum_{k=1}^{\infty} P\left(k < |X|^t \leqslant k+1\right) k^{p/t} l(k^{1/(t\alpha)}) \\ &\leqslant CE\left(|X|^p l\left(|X|^{1/\alpha}\right)\right) < \infty. \end{aligned} \tag{4.17}$$

Since $1/t + 1/q - \alpha = 0$ and $p\alpha - 2 - q/t = -\alpha(q-p) - 1 < -1$, similar to the proof of (2.23) of Sung [13], we have that

$$\begin{aligned} M_4 &\leqslant C \sum_{k=5}^{\infty} E|X|^r I\left(k < |X|^t \leqslant k+1\right) k^{-(r-q)/t} \sum_{n=[k^{q/(q+t)}]}^{k/2} n^{p\alpha-2-q/t} l(n) \\ &\leqslant C \sum_{k=5}^{\infty} E|X|^r I\left(k < |X|^t \leqslant k+1\right) k^{-(r-q)/t-(q-p)(\alpha-1/q)} l(k^{q/(q+t)}) \\ &\leqslant CE|X|^p l(|X|^{1/\alpha}) < \infty. \end{aligned} \tag{4.18}$$

From $M_3 < \infty$ and $M_4 < \infty$, we have $M_2 < \infty$.

(ii) If $p < 2$, then we will take $r = 2$. We may assume that $p < q < 2$. Since $r > q$, as in the case $p \geqslant 2$, we have $M_1 = M_2 < \infty$.

Similar to the proof of Theorem 2.2 of Sung [13], (2.3) holds. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2.

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)^{+}$$
$$= \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_0^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha} > s\right) ds$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_0^{n^\alpha} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^\alpha > s\right) ds \\
&\quad + \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^\alpha > s\right) ds \\
&\leqslant \sum_{n=1}^{\infty} n^{p\alpha-2} l(n) P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| \geqslant \varepsilon n^\alpha\right) \\
&\quad + \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > s\right) ds.
\end{aligned}
$$

By Theorem 2.1, we only need to prove

$$
\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > s\right) ds < \infty. \tag{4.19}
$$

Firstly, we will show if $\{a_{ni}; 1\leqslant i\leqslant n, n\geqslant 1\}$ is an array of real numbers satisfying $|a_{ni}|\leqslant 1$ for $1\leqslant i\leqslant n$ and $n\geqslant 1$, then (4.19) holds.

Denote $Y_{si} = X_i I(|X_i|\leqslant s)$ for all $s > n^\alpha$, $i\geqslant 1$. Then

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > s\right) ds \\
&\quad\leqslant \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} (a_{ni}Y_{si} - a_{ni}EY_{si})\right| > s/2\right) ds \\
&\quad\quad + \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} (a_{ni}X_i I(|X_i|>s) - a_{ni}EX_i I(|X_i|>s))\right| > s/2\right) ds \\
&:= H + K.
\end{aligned}
$$

For K, by Markov inequality, Lemma 3.2, (2.2), the fact $\sum_{n=1}^{j} n^\beta l(n) = O(j^{\beta+1} l(j))$, $\beta\neq -1$ and $|a_{ni}|\leqslant 1$, we have

$$
\begin{aligned}
K &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} s^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i|>s) ds \\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha} l(n) \int_{n^\alpha}^{\infty} s^{-1} E|X| I(|X|>s) ds \\
&= C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha} l(n) \sum_{m=n}^{\infty} \int_{m^\alpha}^{(m+1)^\alpha} s^{-1} E|X| I(|X|>s) ds \\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha} l(n) \sum_{m=n}^{\infty} m^{-1} E|X| I(|X|>m^\alpha) \\
&= C\sum_{m=1}^{\infty} m^{-1} E|X| I(|X|>m^\alpha) \sum_{n=1}^{m} n^{p\alpha-1-\alpha} l(n)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\sum_{m=1}^{\infty} m^{-1}E|X|I(|X|>m^{\alpha})m^{p\alpha-\alpha}l(m)\\
&= C\sum_{m=1}^{\infty} m^{p\alpha-\alpha-1}l(m)\sum_{j=m}^{\infty} E|X|I\left(j<|X|^{\frac{1}{\alpha}}\leqslant j+1\right)\\
&= C\sum_{j=1}^{\infty} E|X|I\left(j<|X|^{\frac{1}{\alpha}}\leqslant j+1\right)\sum_{m=1}^{j} m^{p\alpha-\alpha-1}l(m)\\
&\leqslant C\sum_{j=1}^{\infty} j^{p\alpha-\alpha}l(j)E|X|I\left(j<|X|^{\frac{1}{\alpha}}\leqslant j+1\right)\leqslant CE|X|^{p}l\left(|X|^{1/\alpha}\right)<\infty. \qquad (4.20)
\end{aligned}
$$

For H, by Markov inequality and Lemma 3.1, for $r\geqslant 2$, we have

$$
\begin{aligned}
H &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha}l(n)\int_{n^{\alpha}}^{\infty} s^{-r}E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(a_{ni}Y_{si}-a_{ni}EY_{si})\right|^{r}\right)ds\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha}l(n)\int_{n^{\alpha}}^{\infty} s^{-r}\sum_{i=1}^{n}|a_{ni}|^{r}E|Y_{si}|^{r}ds\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha}l(n)\int_{n^{\alpha}}^{\infty} s^{-r}\left(\sum_{i=1}^{n}(a_{ni})^{2}E(Y_{si})^{2}\right)^{r/2}ds\\
&:= H_{1}+H_{2}. \qquad (4.21)
\end{aligned}
$$

We will discuss the following two cases:

Case 1. $p\geqslant 2$.

Taking $r>\frac{p\alpha-1}{\alpha-\frac{1}{2}}$, which implies that $\alpha(p-r)-2+\frac{r}{2}<-1$ and $\frac{p\alpha-1}{\alpha-\frac{1}{2}}\geqslant p$. By Lemma 3.2, the proof of (4.20) and the facts $\sum_{n=1}^{j} n^{\beta}l(n)=O(j^{\beta+1}l(j)),\ \beta\neq -1$, $\sum_{n=j}^{\infty} n^{\beta}l(n)=O(j^{\beta+1}l(j)),\ \beta<-1$ and $|a_{ni}|\leqslant 1$, we have

$$
\begin{aligned}
H_{1} &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha}l(n)\int_{n^{\alpha}}^{\infty} s^{-r}\sum_{i=1}^{n}|a_{ni}|^{r}(E|X|^{r}I(|X|\leqslant s)+s^{r}P(|X|>s))ds\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha}l(n)\int_{n^{\alpha}}^{\infty} s^{-r}E|X|^{r}I(|X|\leqslant s)ds\\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha}l(n)\int_{n^{\alpha}}^{\infty} s^{-1}E|X|I(|X|>s)ds\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha}l(n)\sum_{m=n}^{\infty}\int_{m^{\alpha}}^{(m+1)^{\alpha}} s^{-r}E|X|^{r}I(|X|\leqslant s)ds+C\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha}l(n)\sum_{m=n}^{\infty} m^{\alpha-1-r\alpha}E|X|^{r}I(|X|\leqslant (m+1)^{\alpha})+C\\
&= C\sum_{m=1}^{\infty} m^{\alpha-1-r\alpha}E|X|^{r}I(|X|\leqslant (m+1)^{\alpha})\sum_{n=1}^{m} n^{p\alpha-1-\alpha}l(n)+C
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\sum_{m=1}^{\infty} m^{p\alpha-1-r\alpha} l(m) E|X|^r I(|X|\leqslant (m+1)^\alpha)+C\\
&\leqslant C\sum_{m=1}^{\infty} m^{p\alpha-1-r\alpha} l(m)\sum_{k=1}^{m} E|X|^r I(k^\alpha<|X|\leqslant (k+1)^\alpha)+C\\
&= c\sum_{k=1}^{\infty} E|X|^r I(k^\alpha<|X|\leqslant (k+1)^\alpha)\sum_{m=k}^{\infty} m^{\alpha(p-r)-1} l(m)+C\\
&\leqslant C\sum_{k=1}^{\infty} k^{p\alpha} l(k) P(k<|X|^{1/\alpha}\leqslant k+1)+C\\
&\leqslant CE|X|^p l\left(|X|^{1/\alpha}\right)+C<\infty. \qquad (4.22)
\end{aligned}
$$

Note that $EX^2<\infty$ if $E|X|^p<\infty$ for $p\geqslant 2$. We have

$$
\begin{aligned}
H_2 &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n)\int_{n^\alpha}^{\infty} s^{-r}\left(\sum_{i=1}^{n}(a_{ni})^2 EX^2\right)^{r/2} ds\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha+r/2} l(n)\int_{n^\alpha}^{\infty} s^{-r} ds\\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha+r/2} l(n)<\infty.
\end{aligned}
$$

Case 2. $p<2$.

We take $r=2$. Similar to the proof of (4.21) and (4.22), we can have $H<\infty$.

Secondly, we will show if $\{a_{ni}; 1\leqslant i\leqslant n, n\geqslant 1\}$ is an array of real numbers satisfying $|a_{ni}|=0$ or $|a_{ni}|>1$ for $1\leqslant i\leqslant n$ and $n\geqslant 1$, then (4.19) holds.

Denote $U_{si}=X_i I(|a_{ni}X_i|\leqslant s)$ for all $s>n^\alpha$, $i\geqslant 1$.

By the fact $EX_i=0$, for $s>n^\alpha$, we have

$$
\begin{aligned}
s^{-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni} EU_{si}\right| &= s^{-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni} EX_i I\left(|a_{ni}X_i|>s\right)\right|\\
&\leqslant s^{-1}\sum_{i=1}^{n}|a_{ni}| E|X| I\left(|a_{ni}X|>s\right)\\
&\leqslant Cs^{-p}\sum_{i=1}^{n}|a_{ni}|^p E|X|^p\\
&\leqslant Cs^{-p}\left(\sum_{i=1}^{n}|a_{ni}|^q\right)^{p/q} n^{1-p/q} E|X|^p\\
&\leqslant Cs^{-p} nE|X|^p\leqslant Cn^{1-p\alpha}E|X|^p\to 0,\ n\to\infty.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > s\right) ds \\
&= \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} \sum_{i=1}^{n} P(|a_{ni}X_i| > s) ds \\
&\quad + \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} (a_{ni}U_{si} - Ea_{ni}U_{si})\right| > \frac{s}{2}\right) ds \\
&=: H' + K'.
\end{aligned}$$

For $1 \leqslant j \leqslant k-1$ and $k \geqslant n$, let

$$I_{kj} = \left\{1 \leqslant i \leqslant n : k^{1/q}(j+1)^{-1/q} < |a_{ni}| \leqslant k^{1/q} j^{-1/q}\right\}.$$

Then $\{I_{kj}, 1 \leqslant j \leqslant k-1\}$ are disjoint, $\bigcup_{j=1}^{k-1} I_{kj} = \{1 \leqslant i \leqslant n : a_{ni} \neq 0\}$, and $\sum_{j=1}^{m} \sharp I_{kj} \leqslant m+1$ for $1 \leqslant m \leqslant k-1$.

$$\begin{aligned}
H' &= \sum_{n=2}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_{n^\alpha}^{\infty} \sum_{i=1}^{n} P(|a_{ni}X_i| > s) ds \\
&\leqslant C \sum_{n=2}^{\infty} n^{p\alpha-2-\alpha} l(n) \sum_{k=n}^{\infty} \int_{k^\alpha}^{(k+1)^\alpha} \sum_{j=1}^{k-1} \sum_{i\in I_{kj}} P\left(|X| > s\left(\frac{j}{k}\right)^{1/q}\right) ds \\
&= C \sum_{n=2}^{\infty} n^{p\alpha-2-\alpha} l(n) \sum_{k=n}^{\infty} \int_{k^\alpha}^{(k+1)^\alpha} \sum_{j=1}^{k-1} \sharp I_{kj} P\left(|X| > k^{\alpha-\frac{1}{q}} j^{1/q}\right) ds \\
&\leqslant C \sum_{n=2}^{\infty} n^{p\alpha-2-\alpha} l(n) \sum_{k=n}^{\infty} k^{\alpha-1} \sum_{j=1}^{k-1} \sharp I_{kj} P\left(|X| > k^{\alpha-\frac{1}{q}} j^{1/q}\right) \\
&= C \sum_{k=2}^{\infty} k^{\alpha-1} \sum_{j=1}^{k-1} \sharp I_{kj} P\left(|X| > k^{\alpha-\frac{1}{q}} j^{1/q}\right) \sum_{n=2}^{k} n^{p\alpha-2-\alpha} l(n) \\
&\leqslant C \sum_{k=2}^{\infty} k^{p\alpha-2} \sum_{j=1}^{k-1} \sharp I_{kj} P\left(|X| > k^{\alpha-\frac{1}{q}} j^{1/q}\right)
\end{aligned}$$

From the proof of $L < \infty$ in Theorem 2.1, we have $H' < \infty$.

For $r \geqslant 2$, by Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
K' &\leqslant C \sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} (a_{ni}U_{si} - Ea_{ni}U_{si})\right|^r\right) ds \\
&\leqslant C \sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \leqslant s)\right)^{\frac{r}{2}} ds \\
&\quad + C \sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \sum_{i=1}^{n} E|a_{ni}X_i|^r I(|a_{ni}X_i| \leqslant s) ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \left(\sum_{i=1}^{n} E|a_{ni}X|^2 I(|a_{ni}X| \leqslant s) + \sum_{i=1}^{n} s^2 P(|a_{ni}X| > s)\right)^{\frac{r}{2}} ds \\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \left(\sum_{i=1}^{n} E|a_{ni}X|^r I(|a_{ni}X| \leqslant s) + \sum_{i=1}^{n} s^r P(|a_{ni}X| > s)\right) ds \\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \left(\sum_{i=1}^{n} E|a_{ni}X|^2 I(|a_{ni}X| \leqslant s)\right)^{\frac{r}{2}} ds \\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \sum_{i=1}^{n} E|a_{ni}X|^r I(|a_{ni}X| \leqslant s)\, ds \\
&\quad + C\sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} \sum_{i=1}^{n} P(|a_{ni}X| > s)\, ds \\
&=: CK_1' + CK_2' + CH'.
\end{aligned}
$$

Since $H' < \infty$, we only need to prove $K_1' < \infty$ and $K_2' < \infty$. For K_1' and K_2', we proceed with two cases.

(i) If $p \geqslant 2$, then we will take r large enough such that $r > \max\{\frac{p\alpha-1}{\alpha-\frac{1}{2}}, q\}$. By the fact that $a_{ni} = 0$ or $|a_{ni}| > 1$, then we get that

$$
\begin{aligned}
K_1' &\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \left(\sum_{i=1}^{n} |a_{ni}|^2\right)^{\frac{r}{2}} ds \\
&\leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} l(n) \left(\sum_{i=1}^{n} |a_{ni}|^q\right)^{\frac{r}{2}} \leqslant C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2} l(n) < \infty \\
K_2' &= C\sum_{n=2}^{\infty} n^{p\alpha-\alpha-2} l(n) \int_{n^\alpha}^{\infty} s^{-r} \sum_{i=1}^{n} E|a_{ni}X|^r I(|a_{ni}X| \leqslant s)\, ds \\
&\leqslant C\sum_{n=2}^{\infty} n^{p\alpha-\alpha-2} l(n) \sum_{k=n+1}^{\infty} \int_{(k-1)^\alpha}^{k^\alpha} s^{-r} \sum_{j=1}^{k-1} \sum_{i\in I_{kj}} \left(\frac{k}{j}\right)^{\frac{r}{q}} E|X|^r I\left(|X| \leqslant k^{\alpha-1/q}(j+1)^{1/q}\right) ds \\
&\leqslant C\sum_{n=2}^{\infty} n^{p\alpha-\alpha-2} l(n) \sum_{k=n}^{\infty} k^{r/q} \int_{(k-1)^\alpha}^{k^\alpha} s^{-r} \sum_{j=1}^{k-1} j^{-r/q} \sharp I_{kj} E|X|^r I\left(|X| \leqslant k^{\alpha-1/q}(j+1)^{1/q}\right) ds \\
&\leqslant C\sum_{n=2}^{\infty} n^{p\alpha-\alpha-2} l(n) \sum_{k=n}^{\infty} k^{\frac{r}{q}+\alpha(-r+1)-1} \sum_{j=1}^{k-1} j^{-r/q} \sharp I_{kj} E|X|^r I\left(|X| \leqslant k^{\alpha-1/q}(j+1)^{1/q}\right) \\
&\leqslant C\sum_{k=2}^{\infty} k^{\frac{r}{q}+\alpha(-r+1)-1} \sum_{n=2}^{k} n^{p\alpha-\alpha-2} l(n) \sum_{j=1}^{k-1} j^{-r/q} \sharp I_{kj} E|X|^r I\left(|X| \leqslant k^{\alpha-1/q}(j+1)^{1/q}\right) \\
&\leqslant C\sum_{k=2}^{\infty} k^{p\alpha-r\alpha-2+\frac{r}{q}} l(k) \sum_{j=1}^{k-1} j^{-r/q} \sharp I_{kj} E|X|^r I\left(|X| \leqslant k^{\alpha-1/q}(j+1)^{1/q}\right)
\end{aligned}
$$

From the proof of $M_2 < \infty$ in Theorem 2.1, we have $K_2' < \infty$.

(ii) If $p < 2$, then we will take $r = 2$. We may assume that $p < q < 2$. Since $r > q$, as in the case $p \geqslant 2$, we have $K_1' = K_2' < \infty$.

Similar to the proof of Theorem 2.2 of Sung [13], (4.19) holds. Hence (2.4) holds.

We will prove the equivalence of (2.3) and (2.4). Under the conditions of Theorem 2.1, obviously, (2.3) implies (2.4).

Next, we prove that (2.4) implies (2.3). We can easily get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)^{+} \\
&= \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_0^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon n^{\alpha} > t\right) dt \\
&\geqslant \sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} l(n) \int_0^{\varepsilon n^{\alpha}} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha} + t\right) dt \\
&\geqslant \varepsilon \sum_{n=1}^{\infty} n^{p\alpha-2} l(n) P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > 2\varepsilon n^{\alpha}\right).
\end{aligned}$$

This completes the proof of Theorem 2.2. □

Proof of Corollary 2.1. Similar to the proof of Theorem 2.2, we get (2.6) holds. Let $l(n) = 1$, For any $\varepsilon > 0$, we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon n^{\alpha}\right) \\
&= \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{p\alpha-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon n^{\alpha}\right) \\
&\geqslant \begin{cases} \sum_{k=0}^{\infty} (2^k)^{p\alpha-2} 2^k P\left(\max_{1\leqslant j\leqslant 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right), & \text{if } p\alpha \geqslant 2, \\ \sum_{k=0}^{\infty} (2^{k+1})^{p\alpha-2} 2^k P\left(\max_{1\leqslant j\leqslant 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right), & \text{if } 1 < p\alpha < 2, \end{cases} \\
&\geqslant \begin{cases} \sum_{k=0}^{\infty} P\left(\max_{1\leqslant j\leqslant 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right), & \text{if } p\alpha \geqslant 2, \\ \frac{1}{2}\sum_{k=0}^{\infty} P\left(\max_{1\leqslant j\leqslant 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon 2^{(k+1)\alpha}\right), & \text{if } 1 < p\alpha < 2. \end{cases}
\end{aligned} \tag{4.22}$$

By Borel-Cantelli Lemma, we have

$$\frac{\max_{1\leqslant j\leqslant 2^k}\left|\sum_{i=1}^{j} a_i X_i\right|}{2^{(k+1)\alpha}} \to 0 \text{ a.s., } k \to \infty.$$

For any positive integer n, there exists some positive integer k such that $2^{k-1} < n \leqslant 2^k$.

By (4.22), we get

$$n^{-\alpha}\left|\sum_{i=1}^{n} a_i X_i\right| \leqslant \max_{2^{k-1}\leqslant n<2^k} n^{-\alpha}\left|\sum_{i=1}^{j} a_i X_i\right| \leqslant \frac{1}{2^{(k-1)\alpha}} \max_{1\leqslant j\leqslant 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| \to 0 \text{ a.s., } k\to\infty.$$

This completes the proof of Corollary 2.1. □

Acknowledgements. The authors are deeply grateful to the anonymous referee and the Editor and the Associate Editor of JMI for the careful reading, valuable comments and correcting some errors, which have greatly improved the quality of the paper. This research is supported by the National Natural Science Foundation of China (71271042 and 11361019).

REFERENCES

[1] J. AN, D. YUAN, *Complete convergence of weighted sums for ρ̃-mixing sequence of random variables*, Statist. Probab. Lett. **78**, 1466–1472 (2008).

[2] Z. D. BAI, C. SU, *The complete convergence for partial sums of i.i.d. random variables*, Sci. Sinica Ser. A. **28**, 1261–1277 (1985).

[3] L. E. BAUM, M. KATZ, *Convergence rates in the law of large numbers*, Trans. Am. Math. Soc. **120**, 108–123 (1965).

[4] R. C. BRADLEY, *On the spectral density and asymptotic normality of weakly dependent random fields*, J. Theoret. Probab. **5**, 355–373 (1992).

[5] W. BRYC, W. SMOLENSKI, *Moment conditions for almost sure convergence of weakly correlated random variables*, Proc. Amer. Math. Soc. **199**, 629–635 (1993).

[6] S. X. GAN, *Almost sure convergence for ρ̃-mixing random variable sequences*, Statist. Probab. Lett. **67**, 289–298 (2004).

[7] A. KUCZMASZEWSKA, *On complete convergence for arrays of rowwise dependent random variables*, Statist. Probab. Lett. **77**, 1050–1060 (2007).

[8] A. KUCZMASZEWSKA, *On Chung-Teicher type strong law of large numbers for ρ̃-mixing random variables*, Discrete. Dyn. Nat. Soc. **2008**, Article ID 140548, 10 pages (2008).

[9] M. PELIGRAD, A. GUT, *Almost sure results for a class of dependent random variables*, J. Theoret. Probab. **12**, 87–104 (1999).

[10] A. SHEN, R. WU, *Strong convergence for sequences of asymptotically almost negatively associated random variables*, Stochastics An International Journal of Probability and Stochastic Processes: formerly Stochastics and Stochastics Reports, **86**, 291–303 (2014), doi:10.1080/17442508.2013.775289.

[11] A. SHEN, R. WU, X. H. WANG, Y. SHEN, *Complete convergence for weighted sums of arrays of rowwise ρ̃-mixing random variables*, J. Inequal. Appl. **2013**, 356, doi:10.1186/1029-242X-2013-356 (2013).

[12] S. H. SUNG, *Inequalities and complete moment convergence*, J. Inequal. Appl. **2009**, Article ID 271265 (2009).

[13] S. H. SUNG, *Complete convergence for weighted sums of ρ̃-mixing random variables*, Discrete. Dyn. Nat. Soc. **2010**, Article ID 630608, 13 pages (2010).

[14] S. UTEV, M. PELIGRAD, *Maximal inequalities and an invariance principle for a class of weakly dependent random variables*, J. Theoret. Probab. **16**, 101–115 (2003).

[15] Q. Y. WU, *Limit Theorems of Probability Theory for Mixing Sequences*, Science Press, Beijing, 2006.
[16] Q. Y. WU, Y. Y. JIANG, *Some strong limit theorems for $\tilde{\rho}$-mixing sequences of random variables*, Statist. Probab. Lett, **78**, 1017–1023 (2008).
[17] S. C. YANG, *Some moment inequalities for partial sums of random variables and their applications*, Chinese Sci. Bull. **43**, 1823–1827 (1998).

(Received March 16, 2016)

Fengxiang Feng
School of Mathematical Science
University of Electronic Science and Technology of China
Chengdu, Sichuan 611731, PR China
and
College of Science
Guilin University of Technology
Guilin, Guangxi 541004, PR China
e-mail: fengfengxiang2013@163.com

Qunying Wu
College of Science
Guilin University of Technology
Guilin, Guangxi 541004, PR China
e-mail: wqy666@glut.edu.cn

Dingcheng Wang
School of Mathematical Science
University of Electronic Science and Technology of China
Chengdu, Sichuan 611731, PR China
e-mail: wangdc@uestc.edu.cn