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NONLINEAR RETARDED INTEGRAL INEQUALITIES ON TIME SCALES AND THEIR APPLICATIONS

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Abstract. In this paper, some new nonlinear retarded integral inequalities on time scales are established, which provide a handy tool in the study of some retarded integral equations and dynamic equations on time scales. The results unify and extend some continuous inequalities and their corresponding discrete analogues. Some applications are also presented to illustrate the usefulness of some of our results.

1. Introduction

The calculus on time scales, which was initiated by Hilger in 1990 [17] has received considerable attention in recent years due to its broad applications in economics, population's models, quantum physics and other science fields (see [4, 8, 18] and the references therein). Many authors have expounded on various aspects of the theory of dynamic equations and integral equations on time scales (see [1, 5–7, 9–12, 28] and the references therein), and the interest in the subject remains growing.

Recently, there has been much research activity concerning the integral inequalities on time scales, and a lot of integral inequalities on time scales have been established, for example, see [2, 3, 13–15, 18–27, 29–32] and the references cited therein, which have been designed in order to unify continuous and discrete analysis.

In this paper, we establish some new nonlinear retarded integral inequalities on time scales. Our results generalize some of the presented inequalities in [14, 20, 24, 29] and can be used as important tools in the qualitative theory of certain retarded integral equations and dynamic equations on time scales. As an application, we study the qualitative property of some retarded integral equations on time scales.

2. Preliminaries

Throughout this paper, we always assume that $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}_+ = [0,+\infty)$, $\mathbf{Z}$ denotes the set of integers, $\mathbf{T}$ is an arbitrary time scale (nonempty closed subset of the real numbers $\mathbf{R}$), $t_0 \in \mathbf{T}^\kappa$ is a fixed number ($\mathbf{T}^\kappa$ is defined to be $\mathbf{T}-\{m\}$ if $\mathbf{T}$ has a maximum m which is left-scattered, otherwise, $\mathbf{T}^\kappa = \mathbf{T}$), $[a,b]_{\mathbf{T}^\kappa} =$

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$[a,b]\cap\mathbf{T}^{\kappa}$, and $f_t^{\Delta}(t,s)$ denotes for each fixed s the derivative of $f(t,s)$ with respect to t.

Next, we present some basic concepts and results concerning time scales which will be essential to prove our main results. It is assumed that the readers are familiar with the time scale calculus. For more details, the readers may want to consult [6] and [7].

LEMMA 1. ([6, Theorem 1.90]) *Let $f:\mathbf{R}\to\mathbf{R}$ be continuously differentiable and suppose $g:\mathbf{T}\to\mathbf{R}$ is delta differentiable. Then $f\circ g:\mathbf{T}\to\mathbf{R}$ is delta differentiable and the formula*

$$(f\circ g)^{\Delta}(t)=\Big\{\int_0^1 f'\Big(g(t)+h\mu(t)g^{\Delta}(t)\Big)\mathrm{d}h\Big\}g^{\Delta}(t)$$

holds.

LEMMA 2. ([6, Theorem 1.98]) *Assume that $\nu:\mathbf{T}\to\mathbf{R}$ is a strictly increasing function and $\widetilde{\mathbf{T}}:=\nu(\mathbf{T})$ is a time scale. If $f:\mathbf{T}\to\mathbf{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a,b\in\mathbf{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t=\int_{\nu(a)}^{\nu(b)}(f\circ\nu^{-1})(s)\widetilde{\Delta}s.$$

LEMMA 3. ([6, Theorem 1.117]) *Let $a\in\mathbf{T}^{\kappa}$ and assume $f:\mathbf{T}\times\mathbf{T}^{\kappa}\to\mathbf{R}$ is continuous at (t,t), where $t\in\mathbf{T}^{\kappa}$ with $t>a$. Also assume that for each $t\in\mathbf{T}^{\kappa}$, $f_t^{\Delta}(t,\cdot)$ is rd-continuous on $[a,\sigma(t)]$ and for each $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[a,\sigma(t)]$, such that*

$$|f(\sigma(t),\tau)-f(s,\tau)-f_t^{\Delta}(t,\tau)(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|,\ \textit{for all } s\in U,$$

then

$$g(t):=\int_a^t f(t,\tau)\Delta\tau,\quad t\in\mathbf{T}^{\kappa},$$

implies

$$g^{\Delta}(t)=\int_a^t f_t^{\Delta}(t,\tau)\Delta\tau+f(\sigma(t),t),\quad t\in\mathbf{T}^{\kappa}.$$

LEMMA 4. *Let $\alpha:\mathbf{T}^{\kappa}\to\mathbf{T}^{\kappa}$ be a continuous and strictly increasing function such that $\alpha(t)\leqslant t$ and α^{Δ} is continuous at $t\in\mathbf{T}^{\kappa}$. Assume that $f:\mathbf{T}\times\mathbf{T}^{\kappa}\to\mathbf{R}$ is a function such that f is continuous at $(t,\alpha(t))$, where $t\in\mathbf{T}^{\kappa}$ with $t>t_0$, and for each $t\in\mathbf{T}^{\kappa}$, $f(t,\alpha(\cdot))$, $f_t^{\Delta}(t,\alpha(\cdot))$ are rd-continuous on $[t_0,\sigma(t)]$. Suppose that for each $t\in\mathbf{T}^{\kappa}$, and for any $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[t_0,\sigma(t)]$, such that*

$$|f(\sigma(t),\alpha(\tau))-f(s,\alpha(\tau))-f_t^{\Delta}(t,\alpha(\tau))(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|,\tag{2.1}$$

for all $s \in U$, *then*

$$g(t) := \int_{\alpha(t_0)}^{\alpha(t)} f(t,\tau)\Delta\tau, \quad t \in \mathbf{T}^\kappa, \tag{2.2}$$

implies

$$g^\Delta(t) = \int_{\alpha(t_0)}^{\alpha(t)} f_t^\Delta(t,\tau)\Delta\tau + f(\sigma(t),\alpha(t))\alpha^\Delta(t), \quad t \in \mathbf{T}^\kappa. \tag{2.3}$$

Proof. From (2.2), we get for any $t \in \mathbf{T}^\kappa$,

$$g(t) = \int_{\alpha(t_0)}^{\alpha(t)} f(t,\tau)\Delta\tau = \int_{\alpha(t_0)}^{\alpha(t)} f(t,\alpha(\alpha^{-1}(\tau)))\Delta\tau.$$

By Lemma 2, we obtain

$$g(t) = \int_{t_0}^{t} f(t,\alpha(\tau))\alpha^\Delta(\tau)\Delta\tau, \quad t \in \mathbf{T}^\kappa.$$

Since α^Δ is continuous at $t \in \mathbf{T}^\kappa$, there exists a positive constant M such that $|\alpha^\Delta(\tau)| \leqslant M$ for any $\tau \in [t_0,\sigma(t)]$. Then from (2.1), we get for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0,\sigma(t)]$, such that

$$\begin{aligned} &|f(\sigma(t),\alpha(\tau))\alpha^\Delta(\tau) - f(s,\alpha(\tau))\alpha^\Delta(\tau) - f_t^\Delta(t,\alpha(\tau))\alpha^\Delta(\tau)(\sigma(t)-s)| \\ &\leqslant M|f(\sigma(t),\alpha(\tau)) - f(s,\alpha(\tau)) - f_t^\Delta(t,\alpha(\tau))(\sigma(t)-s)| \\ &\leqslant \varepsilon|\sigma(t)-s|. \end{aligned}$$

Applying Lemma 2 and Lemma 3, we get

$$\begin{aligned} g^\Delta(t) &= \int_{t_0}^{t} f_t^\Delta(t,\alpha(\tau))\alpha^\Delta(\tau)\Delta\tau + f(\sigma(t),\alpha(t))\alpha^\Delta(t) \\ &= \int_{\alpha(t_0)}^{\alpha(t)} f_t^\Delta(t,\tau)\Delta\tau + f(\sigma(t),\alpha(t))\alpha^\Delta(t), \qquad t \in \mathbf{T}^\kappa. \quad \Box \end{aligned}$$

In what follows, we always assume that:

(H_1) $\alpha : \mathbf{T}^\kappa \to \mathbf{T}^\kappa$ is a continuous and strictly increasing function such that $\alpha(t) \leqslant t$ and α^Δ is continuous at $t \in \mathbf{T}^\kappa$.

(H_2) $\beta : \mathbf{T}^\kappa \to \mathbf{T}^\kappa$ is a continuous and strictly increasing function such that $\beta(t) \leqslant t$ and β^Δ is continuous at $t \in \mathbf{T}^\kappa$.

(H_3) $k : \mathbf{T}^\kappa \to (0,+\infty)$ is a nondecreasing and rd-continuous function, and $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ is a nondecreasing and continuous function with $\omega(u) > 0$ for $u > 0$.

(H_4) $f : \mathbf{T} \times \mathbf{T}^\kappa \to \mathbf{R}_+$ is a function such that f is continuous at $(t,\alpha(t))$, where $t \in \mathbf{T}^\kappa$ with $t > t_0$, $f_t^\Delta(t,s) \geqslant 0$ for $t \geqslant s$, and for each $t \in \mathbf{T}^\kappa$, $f(t,\alpha(\cdot))$, $f_t^\Delta(t,\alpha(\cdot))$ are rd-continuous on $[t_0,\sigma(t)]$.

(H_5) $g : \mathbf{T} \times \mathbf{T}^\kappa \to \mathbf{R}_+$ is a function such that g is continuous at $(t,\beta(t))$, where $t \in \mathbf{T}^\kappa$ with $t > t_0$, $g_t^\Delta(t,s) \geqslant 0$ for $t \geqslant s$, and for each $t \in \mathbf{T}^\kappa$, $g(t,\beta(\cdot))$, $g_t^\Delta(t,\beta(\cdot))$ are rd-continuous on $[t_0,\sigma(t)]$.

LEMMA 5. *Assume that (H_1)–(H_5) hold. Let $u:\mathbf{T}^\kappa\to\mathbf{R}_+$ be a rd-continuous function and $p>0$ be a constant. Suppose that for each $t\in\mathbf{T}^\kappa$, and for any $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[t_0,\sigma(t)]$, such that*

$$|f(\sigma(t),\alpha(\tau))-f(s,\alpha(\tau))-f_t^\Delta(t,\alpha(\tau))(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|,\ \textit{for all}\ s\in U,\quad(2.4)$$

and

$$|g(\sigma(t),\beta(\tau))-g(s,\beta(\tau))-g_t^\Delta(t,\beta(\tau))(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|,\ \textit{for all}\ s\in U.\quad(2.5)$$

Let

$$G(x)=\int_1^x\frac{1}{\omega(s^{\frac{1}{p}})}\mathrm{d}s,\quad x>0,\quad \textit{with}\quad \lim_{x\to+\infty}G(x)=+\infty.\quad(2.6)$$

If u satisfies

$$u^p(t)\leqslant k(t)+\int_{\alpha(t_0)}^{\alpha(t)}f(t,s)\omega(u(s))\Delta s+\int_{\beta(t_0)}^{\beta(t)}g(t,s)\omega(u(s))\Delta s,\quad t\in\mathbf{T}^\kappa,\quad(2.7)$$

then

$$u(t)\leqslant\left\{G^{-1}\left[G(k(t))+\int_{\alpha(t_0)}^{\alpha(t)}f(t,s)\Delta s+\int_{\beta(t_0)}^{\beta(t)}g(t,s)\Delta s\right]\right\}^{\frac{1}{p}},\quad t\in\mathbf{T}^\kappa,\quad(2.8)$$

where G^{-1} is the inverse of G.

Proof. Let $T\in\mathbf{T}^\kappa$ be fixed, and denote

$$z(t)=k(T)+\int_{\alpha(t_0)}^{\alpha(t)}f(t,s)\omega(u(s))\Delta s+\int_{\beta(t_0)}^{\beta(t)}g(t,s)\omega(u(s))\Delta s,\qquad t\in[t_0,T]_{\mathbf{T}^\kappa}.$$

Then from (2.7), we obtain

$$u(t)\leqslant z^{\frac{1}{p}}(t),\ \ t\in[t_0,T]_{\mathbf{T}^\kappa},\quad(2.9)$$

From the assumptions on ω,α,β and u, for each $t\in\mathbf{T}^\kappa$, we have there exists a constant M, such that for $\tau\in[t_0,\sigma(t)]$, $|\omega(u(\alpha(\tau)))|\leqslant M$, and $|\omega(u(\beta(\tau)))|\leqslant M$. So from (2.4) and (2.5), for any $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[t_0,\sigma(t)]$, such that

$$\begin{aligned}&|f(\sigma(t),\alpha(\tau))\omega(u(\alpha(\tau)))-f(s,\alpha(\tau))\omega(u(\alpha(\tau)))\\&\quad-f_t^\Delta(t,\alpha(\tau))\omega(u(\alpha(\tau)))(\sigma(t)-s)|\\&\leqslant M|f(\sigma(t),\alpha(\tau))-f(s,\alpha(\tau))-f_t^\Delta(t,\alpha(\tau))(\sigma(t)-s)|\\&\leqslant\varepsilon|\sigma(t)-s|,\ \ \text{for all}\ s\in U,\end{aligned}$$

and

$$\begin{aligned}&|g(\sigma(t),\beta(\tau))\omega(u(\beta(\tau)))-g(s,\beta(\tau))\omega(u(\beta(\tau)))\\&\quad-g_t^\Delta(t,\beta(\tau))\omega(u(\beta(\tau)))(\sigma(t)-s)|\\&\leqslant M|g(\sigma(t),\beta(\tau))-g(s,\beta(\tau))-g_t^\Delta(t,\beta(\tau))(\sigma(t)-s)|\\&\leqslant\varepsilon|\sigma(t)-s|,\ \ \text{for all}\ s\in U.\end{aligned}$$

Then from Lemma 4, we get

$$\begin{aligned} z^{\Delta}(t) = & \int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)\omega(u(s))\Delta s + f(\sigma(t),\alpha(t))\omega(u(\alpha(t)))\alpha^{\Delta}(t) \\ & + \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)\omega(u(s))\Delta s + g(\sigma(t),\beta(t))\omega(u(\beta(t)))\beta^{\Delta}(t). \end{aligned} \tag{2.10}$$

Our assumptions on α,β,f,g and ω imply that z is nondecreasing on $[t_0,T]_{\mathbf{T}^\kappa}$. Hence, for $t \in [t_0,T]_{\mathbf{T}^\kappa}$, from Lemma 4, (2.9) and (2.10), we have

$$\begin{aligned} z^{\Delta}(t) \leqslant & \int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + f(\sigma(t),\alpha(t))\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha^{\Delta}(t) \\ & + \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + g(\sigma(t),\beta(t))\omega(z^{\frac{1}{p}}(\beta(t)))\beta^{\Delta}(t) \\ \leqslant & \; \omega(z^{\frac{1}{p}}(t))\Big(\int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)\Delta s + f(\sigma(t),\alpha(t))\alpha^{\Delta}(t) \\ & + \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)\Delta s + g(\sigma(t),\beta(t))\beta^{\Delta}(t)\Big) \\ = & \; \omega(z^{\frac{1}{p}}(t))\Big(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s\Big)^{\Delta}, \end{aligned}$$

which yields

$$\frac{z^{\Delta}(t)}{\omega(z^{\frac{1}{p}}(t))} \leqslant \Big(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s\Big)^{\Delta}. \tag{2.11}$$

Furthermore, for $t \in [t_0,T]_{\mathbf{T}^\kappa}$, if $\sigma(t) > t$,

$$\begin{aligned} [G(z(t))]^{\Delta} = & \frac{G(z(\sigma(t))) - G(z(t))}{\sigma(t)-t} = \frac{1}{\sigma(t)-t}\int_{z(t)}^{z(\sigma(t))} \frac{1}{\omega(s^{\frac{1}{p}})}\mathrm{d}s \\ \leqslant & \frac{z(\sigma(t))-z(t)}{\sigma(t)-t}\frac{1}{\omega(z^{\frac{1}{p}}(t))} = \frac{z^{\Delta}(t)}{\omega(z^{\frac{1}{p}}(t))}. \end{aligned} \tag{2.12}$$

If $\sigma(t) = t$,

$$\begin{aligned} [G(z(t))]^{\Delta} = & \lim_{s\to t}\frac{G(z(t)) - G(z(s))}{t-s} = \lim_{s\to t}\frac{1}{t-s}\int_{z(s)}^{z(t)} \frac{1}{\omega(s^{\frac{1}{p}})}\mathrm{d}s \\ = & \lim_{s\to t}\frac{z(t)-z(s)}{t-s}\frac{1}{\omega(\xi^{\frac{1}{p}})} = \frac{z^{\Delta}(t)}{\omega(z^{\frac{1}{p}}(t))}, \end{aligned} \tag{2.13}$$

where ξ lies between $z(s)$ and $z(t)$. So from (2.12) and (2.13) we always have

$$[G(z(t))]^{\Delta} \leqslant \frac{z^{\Delta}(t)}{\omega(z^{\frac{1}{p}}(t))}. \tag{2.14}$$

Combining (2.11) and (2.14), we deduce

$$[G(z(t))]^{\Delta} \leqslant \Big(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big)^{\Delta}. \tag{2.15}$$

Setting $t=\tau$ in (2.15), an integration with respect to τ from t_0 to t yields

$$G(z(t)) - G(z(t_0)) \leqslant \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s, \quad t \in [t_0, T]_{\mathbf{T}^{\kappa}},$$

where G is defined in (2.6). On the basis of $z(t_0) = k(T)$, and G is increasing, one has

$$z(t) \leqslant G^{-1}\Big[G(k(T)) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big], \quad t \in [t_0, T]_{\mathbf{T}^{\kappa}}.$$

Letting $t=T$ in the above relation, we obtain

$$z(T) \leqslant G^{-1}\Big[G(k(T)) + \int_{\alpha(t_0)}^{\alpha(T)} f(T,s)\Delta s + \int_{\beta(t_0)}^{\beta(T)} g(T,s)\Delta s \Big].$$

Combine the above inequality with (2.9), we obtain

$$u(T) \leqslant \Big\{ G^{-1}\Big[G(k(T)) + \int_{\alpha(t_0)}^{\alpha(T)} f(T,s)\Delta s + \int_{\beta(t_0)}^{\beta(T)} g(T,s)\Delta s \Big] \Big\}^{\frac{1}{p}}. \tag{2.16}$$

Since $T \in \mathbf{T}^{\kappa}$ is arbitrary, then after substituting T with t in (2.16), we obtain the desired inequality (2.8). $\square$

3. Main results

THEOREM 1. *Assume that (H_1)–(H_5) hold. Let $u: \mathbf{T}^{\kappa} \to \mathbf{R}_+$ be a rd-continuous function and $p > q > 0$ be constants. Suppose that for each $t \in \mathbf{T}^{\kappa}$, and for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that (2.4) and (2.5) hold. Let*

$$G(x) = \int_1^x \frac{1}{\omega(s^{\frac{1}{p-q}})} \mathrm{d}s, \quad x > 0, \quad \textit{with} \quad \lim_{x \to +\infty} G(x) = +\infty. \tag{3.1}$$

If u satisfies

$$u^p(t) \leqslant k^{\frac{p}{p-q}}(t) + \frac{p}{p-q} \int_{\alpha(t_0)}^{\alpha(t)} f(t,s) u^q(s) \omega(u(s)) \Delta s + \frac{p}{p-q} \int_{\beta(t_0)}^{\beta(t)} g(t,s) u^q(s) \Delta s, \ t \in \mathbf{T}^{\kappa}, \tag{3.2}$$

then

$$u(t) \leqslant \Big\{ G^{-1}\Big[G\Big(k(t) + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s \Big] \Big\}^{\frac{1}{p-q}}, \ t \in \mathbf{T}^{\kappa}, \tag{3.3}$$

where G^{-1} is the inverse of G.

Proof. Let $T \in \mathbf{T}^{\kappa}$ be fixed. For $t \in [t_0, T]_{\mathbf{T}^{\kappa}}$, denote

$$z(t) = k^{\frac{p}{p-q}}(T) + \frac{p}{p-q}\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)u^q(s)\omega(u(s))\Delta s + \frac{p}{p-q}\int_{\beta(t_0)}^{\beta(t)} g(t,s)u^q(s)\Delta s.$$

Then from (3.3), we have

$$u(t) \leqslant z^{\frac{1}{p}}(t), \qquad t \in [t_0, T]_{\mathbf{T}^{\kappa}}. \tag{3.4}$$

From the assumptions on ω, α, β and u, for each $t \in \mathbf{T}^{\kappa}$, we have there exists a constant M, such that for $\tau \in [t_0, \sigma(t)]$, $|u^q(\alpha(\tau))\omega(u(\alpha(\tau)))| \leqslant M$, and $|u^q(\beta(\tau))| \leqslant M$. So from (2.4) and (2.5), for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that

$$\begin{aligned}
&\big|f(\sigma(t),\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau))) - f(s,\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau)))\\
&\quad - f_t^{\Delta}(t,\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau)))(\sigma(t)-s)\big|\\
&\leqslant M|f(\sigma(t),\alpha(\tau)) - f(s,\alpha(\tau)) - f_t^{\Delta}(t,\alpha(\tau))(\sigma(t)-s)|\\
&\leqslant \varepsilon|\sigma(t)-s|, \ \text{ for all } s \in U,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\big|g(\sigma(t),\beta(\tau))u^q(\beta(\tau)) - g(s,\beta(\tau))u^q(\beta(\tau)) - g_t^{\Delta}(t,\beta(\tau))u^q(\beta(\tau))(\sigma(t)-s)\big|\\
&\leqslant M|g(\sigma(t),\beta(\tau)) - g(s,\beta(\tau)) - g_t^{\Delta}(t,\beta(\tau))(\sigma(t)-s)|\\
&\leqslant \varepsilon|\sigma(t)-s|, \ \text{ for all } s \in U.
\end{aligned} \tag{3.6}$$

By (H_1)–(H_5), (3.5) and (3.6), using Lemma 4, we have

$$\begin{aligned}
z^{\Delta}(t) = \frac{p}{p-q}\bigg\{&\int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)u^q(s)\omega(u(s))\Delta s\\
&+ f(\sigma(t),\alpha(t))u^q(\alpha(t))\omega(u(\alpha(t)))\alpha^{\Delta}(t)\\
&+ \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)u^q(s)\Delta s + g(\sigma(t),\beta(t))u^q(\beta(t))\beta^{\Delta}(t)\bigg\}.
\end{aligned}$$

The assumptions on $\alpha, \beta, f, g, \omega$ and u imply that z is nondecreasing on $\mathbf{T}^{\kappa}$. Hence, for $t \in [t_0, T]_{\mathbf{T}^{\kappa}}$, from Lemma 4 and (3.4), we have

$$\begin{aligned}
z^{\Delta}(t) \leqslant \frac{p}{p-q}\bigg\{&\int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)z^{\frac{q}{p}}(s)\omega(z^{\frac{1}{p}}(s))\Delta s\\
&+ f(\sigma(t),\alpha(t))z^{\frac{q}{p}}(\alpha(t))\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha^{\Delta}(t)\\
&+ \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)z^{\frac{q}{p}}(s)\Delta s + g(\sigma(t),\beta(t))z^{\frac{q}{p}}(\beta(t))\beta^{\Delta}(t)\bigg\}
\end{aligned}$$

$$\leqslant \frac{p}{p-q} z^{\frac{q}{p}}(t) \Big\{ \int_{\alpha(t_0)}^{\alpha(t)} f_t^\Delta(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + f(\sigma(t),\alpha(t))\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha^\Delta(t) + \int_{\beta(t_0)}^{\beta(t)} g_t^\Delta(t,s)\Delta s + g(\sigma(t),\beta(t))\beta^\Delta(t) \Big\}$$

$$= \frac{p}{p-q} z^{\frac{q}{p}}(t) \Big(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big)^\Delta,$$

that is,

$$\frac{z^\Delta(t)}{z^{\frac{q}{p}}(t)} \leqslant \frac{p}{p-q} \Big(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big)^\Delta. \tag{3.7}$$

According to Lemma 1, considering $z^\Delta(t) \geqslant 0$, we have

$$\begin{aligned}\Big(\frac{p}{p-q} z^{\frac{p-q}{p}}(t) \Big)^\Delta &= z^\Delta(t) \int_0^1 [z(t) + h\mu(t) z^\Delta(t)]^{-\frac{q}{p}} \mathrm{d}h \\ &= \frac{z^\Delta(t)}{z^{\frac{q}{p}}(t)} \int_0^1 [1 + h\mu(t)\frac{z^\Delta(t)}{z(t)}]^{-\frac{q}{p}} \mathrm{d}h \\ &\leqslant \frac{z^\Delta(t)}{z^{\frac{q}{p}}(t)}. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\Big(z^{\frac{p-q}{p}}(t) \Big)^\Delta \leqslant \Big(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big)^\Delta. \tag{3.9}$$

Setting $t = \tau$ in (3.9), an integration with respect to τ from t_0 to t yields

$$z^{\frac{p-q}{p}}(t) - z^{\frac{p-q}{p}}(t_0) \leqslant \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s. \tag{3.10}$$

Since $z(t_0) = k^{\frac{p}{p-q}}(T)$, then (3.10) implies

$$\begin{aligned} z^{\frac{p-q}{p}}(t) &\leqslant k(T) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \\ &= k(T) + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s, \quad t \in [t_0, T]_{\mathbf{T}^\kappa}. \end{aligned}$$

By Lemma 4, and the assumptions on β, g, we get

$$\Big(k(T) + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big)^\Delta = \int_{\beta(t_0)}^{\beta(t)} g_t^\Delta(t,s)\Delta s + g(\sigma(t),\beta(t))\beta^\Delta(t) \geqslant 0,$$

and then $k(T) + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s$ is nondecreasing and rd-continuous. According to Lemma 5, we obtain for $t \in [t_0, T]_{\mathbf{T}^\kappa}$,

$$z^{\frac{1}{p}}(t) \leqslant \Big\{ G^{-1} \Big[G\Big(k(T) + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s \Big] \Big\}^{\frac{1}{p-q}}. \tag{3.11}$$

Combining (3.4) and (3.11), we have

$$u(t) \leqslant \left\{ G^{-1}\left[G\Big(k(T) + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s \right] \right\}^{\frac{1}{p-q}}. \tag{3.12}$$

Let $t = T$ in the above inequality, and since $T \in \mathbf{T}^{\kappa}$ was arbitrarily chosen, after substituting T with t, we obtain the desired inequality (3.3). □

If we let $\beta(t) = \alpha(t)$ in Theorem 1, then we have the following corollary.

COROLLARY 1. *Let $u : \mathbf{T}^{\kappa} \to \mathbf{R}_{+}$ be a rd-continuous function and $p > q > 0$ be constants. Assume that (H_1), (H_3)–(H_5), (2.4) and (2.5) hold and G is defined as in (3.1). If u satisfies*

$$u^p(t) \leqslant k^{\frac{p}{p-q}}(t) + \frac{p}{p-q}\int_{\alpha(t_0)}^{\alpha(t)} \Big[f(t,s)u^q(s)\omega(u(s)) + g(t,s)u^q(s) \Big]\Delta s, \qquad t \in \mathbf{T}^{\kappa},$$

then

$$u(t) \leqslant \left\{ G^{-1}\left[G\Big(k(t) + \int_{\alpha(t_0)}^{\alpha(t)} g(t,s)\Delta s \Big) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s \right] \right\}^{\frac{1}{p-q}}, \quad t \in \mathbf{T}^{\kappa}.$$

If we let $p > 1$, $q = p-1$, $\omega(u) = u$ in Theorem 1, in this case, we have $G(x) = \ln x$, then we obtain the following corollary.

COROLLARY 2. *Let $u : \mathbf{T}^{\kappa} \to \mathbf{R}_{+}$ be a rd-continuous function and $p > 1$ be a constant. Assume that (H_1)–(H_5), (2.4) and (2.5) hold. If u satisfies*

$$u^p(t) \leqslant k^p(t) + p\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)u^p(s)\Delta s + p\int_{\beta(t_0)}^{\beta(t)} g(t,s)u^{p-1}(s)\Delta s, \qquad t \in \mathbf{T}^{\kappa},$$

then

$$u(t) \leqslant \Big(k(t) + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \Big) \exp\Big(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s \Big), \quad t \in \mathbf{T}^{\kappa}.$$

THEOREM 2. *Assume that (H_1)–(H_5) hold and G is defined as in (3.1). Let $u : \mathbf{T}^{\kappa} \to \mathbf{R}_{+}$ be a rd-continuous function and $p > q \geqslant 0$ be constants. Suppose that for each $t \in \mathbf{T}^{\kappa}$, and for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that (2.4) and (2.5) hold. If u satisfies*

$$\begin{aligned} u^p(t) \leqslant k^{\frac{p}{p-q}}(t) &+ \frac{p}{p-q}\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)u^q(s)\omega(u(s))\Delta s \\ &+ \frac{p}{p-q}\int_{\beta(t_0)}^{\beta(t)} g(t,s)u^q(s)\omega(u(s))\Delta s, \quad t \in \mathbf{T}^{\kappa}, \end{aligned} \tag{3.13}$$

then

$$u(t) \leqslant \left\{ G^{-1}\left[G\big(k(t)\big) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s \right] \right\}^{\frac{1}{p-q}}, \quad t \in \mathbf{T}^{\kappa}. \tag{3.14}$$

Proof. Let $T \in \mathbf{T}^{\kappa}$ be fixed. For $t \in [t_0, T]_{\mathbf{T}^{\kappa}}$, denote

$$z(t) = k^{\frac{p}{p-q}}(T) + \frac{p}{p-q}\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)u^q(s)\omega(u(s))\Delta s + \frac{p}{p-q}\int_{\beta(t_0)}^{\beta(t)} g(t,s)u^q(s)\omega(u(s))\Delta s.$$

Then from (3.13), we have

$$u(t) \leqslant z^{\frac{1}{p}}(t), \qquad t \in [t_0, T]_{\mathbf{T}^{\kappa}}. \tag{3.15}$$

Similar to the proof of Theorem 1, from (2.4) and (2.5) we get

$$\begin{aligned} &\big| f(\sigma(t),\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau))) - f(s,\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau))) \\ &\quad - f_t^{\Delta}(t,\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau)))(\sigma(t)-s)\big| \leqslant \varepsilon|\sigma(t)-s|, \text{ for all } s \in U, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &|g(\sigma(t),\beta(\tau))u^q(\beta(\tau))\omega(u(\beta(\tau))) - g(s,\beta(\tau))u^q(\beta(\tau))\omega(u(\beta(\tau))) \\ &\quad - g_t^{\Delta}(t,\beta(\tau))u^q(\beta(\tau))\omega(u(\beta(\tau)))(\sigma(t)-s)| \leqslant \varepsilon|\sigma(t)-s|, \text{ for all } s \in U. \end{aligned} \tag{3.17}$$

By (H_1)–(H_5), (3.16) and (3.17), using Lemma 4, we obtain

$$\begin{aligned} z^{\Delta}(t) = \frac{p}{p-q}\Big\{ &\int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)u^q(s)\omega(u(s))\Delta s + f(\sigma(t),\alpha(t))u^q(\alpha(t))\omega(u(\alpha(t)))\alpha^{\Delta}(t) \\ &+ \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)u^q(s)\omega(u(s))\Delta s + g(\sigma(t),\beta(t))u^q(\beta(t))\omega(u(\beta(t)))\beta^{\Delta}(t)\Big\}. \end{aligned} \tag{3.18}$$

The assumptions on $\alpha, \beta, f, g, \omega$ and u imply that z is nondecreasing on $\mathbf{T}^{\kappa}$. Hence, for $t \in [t_0, T]_{\mathbf{T}^{\kappa}}$, from Lemma 4, (3.15) and (3.18), we have

$$\begin{aligned} z^{\Delta}(t) &\leqslant \frac{p}{p-q}\Big\{ \int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)z^{\frac{q}{p}}(s)\omega(z^{\frac{1}{p}}(s))\Delta s \\ &\quad + f(\sigma(t),\alpha(t))z^{\frac{q}{p}}(\alpha(t))\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha^{\Delta}(t) \\ &\quad + \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)z^{\frac{q}{p}}(s)\omega(z^{\frac{1}{p}}(s))\Delta s + g(\sigma(t),\beta(t))z^{\frac{q}{p}}(\beta(t))\omega(z^{\frac{1}{p}}(\beta(t)))\beta^{\Delta}(t)\Big\} \\ &\leqslant \frac{p}{p-q}z^{\frac{q}{p}}(t)\Big\{ \int_{\alpha(t_0)}^{\alpha(t)} f_t^{\Delta}(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + f(\sigma(t),\alpha(t))\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha^{\Delta}(t) \\ &\quad + \int_{\beta(t_0)}^{\beta(t)} g_t^{\Delta}(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + g(\sigma(t),\beta(t))\omega(z^{\frac{1}{p}}(\beta(t)))\beta^{\Delta}(t)\Big\} \\ &= \frac{p}{p-q}z^{\frac{q}{p}}(t)\left(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s \right)^{\Delta}, \end{aligned}$$

that is,

$$\frac{z^{\Delta}(t)}{z^{\frac{q}{p}}(t)} \leqslant \frac{p}{p-q}\left(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s\right)^{\Delta}. \tag{3.19}$$

Combining (3.8) and (3.19), we get

$$\left(z^{\frac{p-q}{p}}(t)\right)^{\Delta} \leqslant \left(\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s\right)^{\Delta}. \tag{3.20}$$

Setting $t=\tau$ in (3.20), an integration with respect to τ from t_0 to t yields

$$z^{\frac{p-q}{p}}(t) - z^{\frac{p-q}{p}}(t_0) \leqslant \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s. \tag{3.21}$$

Since $z(t_0) = k^{\frac{p}{p-q}}(T)$, then (3.21) implies

$$z^{\frac{p-q}{p}}(t) \leqslant k(T) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s.$$

According to Lemma 5, we get for $t \in [t_0,T]_{\mathbf{T}^{\kappa}}$,

$$z^{\frac{1}{p}}(t) \leqslant \left\{G^{-1}\left[G\big(k(T)\big) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s\right]\right\}^{\frac{1}{p-q}}. \tag{3.22}$$

Combining (3.15) and (3.22), we have

$$u(t) \leqslant \left\{G^{-1}\left[G\big(k(T)\big) + \int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s + \int_{\beta(t_0)}^{\beta(t)} g(t,s)\Delta s\right]\right\}^{\frac{1}{p-q}}. \tag{3.23}$$

Let $t=T$ in the above inequality, and since $T \in \mathbf{T}^{\kappa}$ was arbitrarily chosen, after substituting T with t, we obtain the desired inequality (3.14). □

If we let $\beta(t) = \alpha(t)$ in Theorem 2, then we have the following corollary.

COROLLARY 3. *Let $u : \mathbf{T}^{\kappa} \to \mathbf{R}_+$ be a rd-continuous function and $p > q > 0$ be constants. Assume that (H_1), (H_3)–(H_5), (2.4) and (2.5) hold and G is defined as in (3.1). If u satisfies*

$$u^p(t) \leqslant k^{\frac{p}{p-q}}(t) + \frac{p}{p-q}\int_{\alpha(t_0)}^{\alpha(t)} \Big[f(t,s)+g(t,s)\Big]u^q(s)\omega(u(s))\Delta s, \quad t \in \mathbf{T}^{\kappa}.$$

then

$$u(t) \leqslant \left\{G^{-1}\left[G\big(k(t)\big) + \int_{\alpha(t_0)}^{\alpha(t)} \Big[f(t,s)+g(t,s)\Big]\Delta s\right]\right\}^{\frac{1}{p-q}}, \quad t \in \mathbf{T}^{\kappa}.$$

THEOREM 3. *Assume that $(H_1),(H_3),(H_4)$ hold, G is defined as in (3.1), and $h:\mathbf{T}^\kappa\to\mathbf{R}_+$ is continuous. Let $u:\mathbf{T}^\kappa\to\mathbf{R}_+$ be a continuous function and $p>q\geqslant 0$ be constants. Suppose that for each $t\in\mathbf{T}^\kappa$, and for any $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[t_0,\sigma(t)]$, such that (2.4) holds. If u satisfies*

$$u^p(t)\leqslant k^{\frac{p}{p-q}}(t)+\frac{p}{p-q}\int_{\alpha(t_0)}^{\alpha(t)}\left[f(t,s)u^q(s)\omega(u(s))+\int_{\alpha(t_0)}^{s}h(\tau)u^q(\tau)\Delta\tau\right]\Delta s,\ t\in\mathbf{T}^\kappa, \tag{3.24}$$

then

$$u(t)\leqslant\left\{G^{-1}\left[G\Big(k(t)+\int_{\alpha(t_0)}^{\alpha(t)}\big[\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau+f(t,s)\big]\Delta s\Big)\right]\right\}^{\frac{1}{p-q}},\quad t\in\mathbf{T}^\kappa. \tag{3.25}$$

Proof. Let $T\in\mathbf{T}^\kappa$ be fixed. For $t\in[t_0,T]_{\mathbf{T}^\kappa}$, denote

$$z(t)=k^{\frac{p}{p-q}}(T)+\frac{p}{p-q}\int_{\alpha(t_0)}^{\alpha(t)}\left[f(t,s)u^q(s)\omega(u(s))+\int_{\alpha(t_0)}^{s}h(\tau)u^q(\tau)\Delta\tau\right]\Delta s.$$

Then from (3.24), we have

$$u(t)\leqslant z^{\frac{1}{p}}(t),\qquad t\in[t_0,T]_{\mathbf{T}^\kappa}. \tag{3.26}$$

Similar to the proof of Theorem 1, from (2.4) we get

$$\begin{aligned}&\big|f(\sigma(t),\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau)))-f(s,\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau)))\\&\quad-f_t^\Delta(t,\alpha(\tau))u^q(\alpha(\tau))\omega(u(\alpha(\tau)))(\sigma(t)-s)\big|\leqslant\varepsilon|\sigma(t)-s|,\ \text{ for all } s\in U.\end{aligned} \tag{3.27}$$

Applying Lemma 4, we obtain

$$\begin{aligned}z^\Delta(t)=\frac{p}{p-q}\Big\{&\int_{\alpha(t_0)}^{\alpha(t)}f_t^\Delta(t,s)u^q(s)\omega(u(s))\Delta s\\&+f(\sigma(t),\alpha(t))u^q(\alpha(t))\omega(u(\alpha(t)))\alpha^\Delta(t)+\alpha^\Delta(t)\int_{\alpha(t_0)}^{\alpha(t)}h(\tau)u^q(\tau)\Delta\tau\Big\}.\end{aligned} \tag{3.28}$$

The assumptions on α,f,h,ω and u imply that z is nondecreasing on $\mathbf{T}^\kappa$. Hence, for $t\in[t_0,T]_{\mathbf{T}^\kappa}$, from Lemma 4, (3.26) and (3.28), we have

$$\begin{aligned}z^\Delta(t)&\leqslant\frac{p}{p-q}\Big\{\int_{\alpha(t_0)}^{\alpha(t)}f_t^\Delta(t,s)z^{\frac{q}{p}}(s)\omega(z^{\frac{1}{p}}(s))\Delta s\\&\qquad+f(\sigma(t),\alpha(t))z^{\frac{q}{p}}(\alpha(t))\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha^\Delta(t)+\alpha^\Delta(t)\int_{\alpha(t_0)}^{\alpha(t)}h(\tau)z^{\frac{q}{p}}(\tau)\Delta\tau\Big\}\\&\leqslant\frac{p}{p-q}z^{\frac{q}{p}}(t)\Big\{\int_{\alpha(t_0)}^{\alpha(t)}f_t^\Delta(t,s)\omega(z^{\frac{1}{p}}(s))\\&\qquad+f(\sigma(t),\alpha(t))\omega(z^{\frac{1}{p}}(\alpha(t)))\alpha^\Delta(t)+\alpha^\Delta(t)\int_{\alpha(t_0)}^{\alpha(t)}h(\tau)\Delta\tau\Big\}\\&=\frac{p}{p-q}z^{\frac{q}{p}}(t)\bigg(\int_{\alpha(t_0)}^{\alpha(t)}\Big[f(t,s)\omega(z^{\frac{1}{p}}(s))+\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\Big]\Delta s\bigg)^\Delta,\end{aligned}$$

that is,

$$\frac{z^{\Delta}(t)}{z^{\frac{q}{p}}(t)} \leqslant \frac{p}{p-q}\left(\int_{\alpha(t_0)}^{\alpha(t)}\left[f(t,s)\omega(z^{\frac{1}{p}}(s))+\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\right]\Delta s\right)^{\Delta}. \tag{3.29}$$

Combining (3.8) and (3.29), we obtain

$$\left(z^{\frac{p-q}{p}}(t)\right)^{\Delta} \leqslant \left(\int_{\alpha(t_0)}^{\alpha(t)}\left[f(t,s)\omega(z^{\frac{1}{p}}(s))+\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\right]\Delta s\right)^{\Delta}. \tag{3.30}$$

Setting $t=\tau$ in (3.30), an integration with respect to τ from t_0 to t yields

$$z^{\frac{p-q}{p}}(t)-z^{\frac{p-q}{p}}(t_0) \leqslant \int_{\alpha(t_0)}^{\alpha(t)}\left[f(t,s)\omega(z^{\frac{1}{p}}(s))+\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\Delta\tau\right]\Delta s. \tag{3.31}$$

Since $z(t_0)=k^{\frac{p}{p-q}}(T)$, then (3.31) implies

$$\begin{aligned} z^{\frac{p-q}{p}}(t) &\leqslant k(T)+\int_{\alpha(t_0)}^{\alpha(t)}\left[f(t,s)\omega(z^{\frac{1}{p}}(s))+\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\right]\Delta s \\ &= k(T)+\int_{\alpha(t_0)}^{\alpha(t)}\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\Delta s+\int_{\alpha(t_0)}^{\alpha(t)}f(t,s)\omega(z^{\frac{1}{p}}(s))\Delta s. \end{aligned}$$

By Lemma 4, and the assumptions on α and h, we get

$$\left(k(T)+\int_{\alpha(t_0)}^{\alpha(t)}\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\Delta s\right)^{\Delta}=\alpha^{\Delta}(t)\int_{\alpha(t_0)}^{\alpha(t)}h(\tau)\Delta\tau \geqslant 0,$$

and then $k(T)+\int_{\alpha(t_0)}^{\alpha(t)}\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\Delta s$ is nondecreasing and rd-continuous. According to Lemma 5, we get for $t\in[t_0,T]_{\mathbf{T}^{\kappa}}$,

$$z^{\frac{1}{p}}(t) \leqslant \left\{G^{-1}\left[G\left(k(T)+\int_{\alpha(t_0)}^{\alpha(t)}\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\Delta s\right)+\int_{\alpha(t_0)}^{\alpha(t)}f(t,s)\Delta s\right]\right\}^{\frac{1}{p-q}}. \tag{3.32}$$

Combining (3.26) and (3.32), we have

$$u(t) \leqslant \left\{G^{-1}\left[G\left(k(T)+\int_{\alpha(t_0)}^{\alpha(t)}\int_{\alpha(t_0)}^{s}h(\tau)\Delta\tau\Delta s\right)+\int_{\alpha(t_0)}^{\alpha(t)}f(t,s)\Delta s\right]\right\}^{\frac{1}{p-q}}. \tag{3.33}$$

Let $t=T$ in the above inequality, and since $T\in\mathbf{T}^{\kappa}$ was arbitrarily chosen, after substituting T with t, we obtain the desired inequality (3.25). □

REMARK 1. If we take $\mathbf{T}=\mathbf{R}$, $t_0=0$, $k(t)\equiv c$, $\beta(t)=t$, $f(t,s)=f(s)$, $g(t,s)=g(s)$, then Theorem 2 reduces to [29, Theorem 2.2]. If we take $\mathbf{T}=\mathbf{R}$, $t_0=0$, $k(t)\equiv c$, $\beta(t)=t$, $f(t,s)=f(s)$, $g(t,s)=g(s)$, $p=2$, $q=1$, then Theorem 2 reduces to [20, Theorem 2]. If we take $\mathbf{T}=\mathbf{Z}$, $t_0=0$, $k(t)\equiv c$, $\alpha(t)=\beta(t)=t$, $f(t,s)=f(s)$, $g(t,s)=g(s)$, $p=2$, $q=1$, then Theorem 1 reduces to [24, Theorem 6 (b6)]. If we take $k(t)\equiv c$, $f(t,s)=f(s)$, $g(t,s)=g(s)$, then Theorem 1 reduces to [14, Theorem 3.2].

4. Applications

In this section, we will apply the results established above to derive explicit bounds for solutions of certain retarded integral equations on time scales.

Consider the following retarded integral equation on time scales:

$$u^p(t) = a(t) + \int_{\alpha(t_0)}^{\alpha(t)} \Phi(t,s,u(s))\Delta s, \quad t \in \mathbf{T}^\kappa, \tag{4.1}$$

where $a : \mathbf{T}^\kappa \to \mathbf{R}$ is rd-continuous, α is defined as in Theorem 1, $\Phi : \mathbf{T}^\kappa \times \mathbf{T}^\kappa \times \mathbf{R} \to \mathbf{R}$ is continuous, and p is a constant with $p > 1$.

THEOREM 4. *Suppose that*

$$|\Phi(t,s,u)| \leqslant f(t,s)|u|^p + g(t,s)|u|^{p-1}, \quad t,s \in \mathbf{T}^\kappa,\ u \in \mathbf{R}, \tag{4.2}$$

where f, g satisfy (H_4) and (H_5). If u is a solution of Eq.(4.1), then

$$u(t) \leqslant \left(|a(t)|^{\frac{1}{p}} + \frac{1}{p}\int_{\alpha(t_0)}^{\alpha(t)} g(t,s)\Delta s\right) \exp\left(\frac{1}{p}\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s\right), \ t \in \mathbf{T}^\kappa. \tag{4.3}$$

Proof. Using (4.1) and (4.2), we have

$$\begin{aligned} |u^p(t)| &\leqslant |a(t)| + \int_{\alpha(t_0)}^{\alpha(t)} |\Phi(t,s,u(s))|\Delta s \\ &\leqslant |a(t)| + \int_{\alpha(t_0)}^{\alpha(t)} \left[f(t,s)|u(s)|^p + g(t,s)|u(s)|^{p-1}\right]\Delta s, \quad t \in \mathbf{T}^\kappa. \end{aligned}$$

By Corollary 2, we obtain the desired inequality (4.3). □

Next, we consider the following retarded integral equation on time scales:

$$u^p(t) = a(t) + \int_{\alpha(t_0)}^{\alpha(t)} F\left(t,s,u(s), \int_{\alpha(t_0)}^{s} \Psi(\tau,u(\tau))\Delta\tau\right)\Delta s, \quad t \in \mathbf{T}^\kappa, \tag{4.4}$$

where $a : \mathbf{T}^\kappa \to \mathbf{R}$ is rd-continuous, α is defined as in Theorem 1, $F : \mathbf{T}^\kappa \times \mathbf{T}^\kappa \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\Psi : \mathbf{T}^\kappa \times \mathbf{R} \to \mathbf{R}$ are continuous, and p is a constant with $p > 1$.

THEOREM 5. *Suppose that*

$$|F(t,s,u,v)| \leqslant f(t,s)|u|^p + |v|, \quad t,s \in \mathbf{T}^\kappa, u,v \in \mathbf{R}, \tag{4.5}$$

$$|\Psi(t,u)| \leqslant h(t)|u|^{p-1}, \quad t,s \in \mathbf{T}^\kappa, u \in \mathbf{R}, \tag{4.6}$$

where f, and h are defined as in Theorem 3. If u is a solution of Eq.(4.4), then

$$u(t) \leqslant \left(|a(t)|^{\frac{1}{p}} + \frac{1}{p}\int_{\alpha(t_0)}^{\alpha(t)}\int_{\alpha(t_0)}^{s} h(\tau)\Delta\tau\Delta s\right) \exp\left(\frac{1}{p}\int_{\alpha(t_0)}^{\alpha(t)} f(t,s)\Delta s\right), \ t \in \mathbf{T}^\kappa. \tag{4.7}$$

Proof. By (4.4) – (4.6), we get

$$|u^p(t)| \leqslant |a(t)| + \int_{\alpha(t_0)}^{\alpha(t)} |F\Big(t,s,u(s),\int_{\alpha(t_0)}^{s} \Psi(\tau,u(\tau))\Delta\tau\Big)|\Delta s$$
$$\leqslant |a(t)| + \int_{\alpha(t_0)}^{\alpha(t)} \Big[f(t,s)|u(s)|^p + \int_{\alpha(t_0)}^{s} h(\tau)|u(\tau)|^{p-1}\Delta\tau\Big]\Delta s, \ \ t \in \mathbf{T}^{\kappa}.$$

Using Theorem 3, we obtain the desired inequality (4.7). □

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