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EXPLICIT BOUNDS OF UNKNOWN FUNCTION OF SOME NEW WEAKLY SINGULAR RETARDED INTEGRAL INEQUALITIES WITH APPLICATIONS

RICAI LUO, WU-SHENG WANG AND ZONGYI HOU

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Abstract. Some new retarded weakly singular integral inequalities of Gronwall-Bellman type are established, which generalized some known weakly singular inequalities and can be used in the analysis of various problems in the theory of certain classes of singular differential equations, singular integral equations and singular evolution equations. Using the modification of Medveď's method, the explicit estimations of unknown function in the inequalities are obtained. Finally, we give a examples to illustrate applications of our results.

1. Introduction

Gronwall [1] and Bellman [2] established the integral inequality

$$u(t) \leqslant c + \int_a^t f(s)u(s)ds, \quad t \in [a,b],$$

for some constant $c \geqslant 0$, obtained the estimation of unknown function

$$u(t) \leqslant c\exp\left(\int_a^t f(s)ds\right), \quad t \in [a,b]. \tag{1}$$

In 2016, Abdeldaim[3] discussed the following nonlinear integral inequality

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\left[u^{2-p}(s) + \int_0^s g(\tau)u^q(\tau)d\tau\right]^p ds, \quad p \in [0,1), \tag{2}$$

$$u(t) \leqslant n(t) + \int_0^{\alpha(t)} f(s)\left[u(s) + \int_0^s g(\tau)u(\tau)d\tau\right]^p ds, \quad p \in [0,1). \tag{3}$$

Being an important tool in the study of qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall-Bellman integral

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The corresponding author: Wu-Sheng Wang.

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inequality and their applications have attracted great interests of many mathematicians (such as [3–12]). Usually, this type integral inequalities have regular or continuous integral kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, to prove a global existence and an exponential decay result for a parabolic Cauchy problem; Henry [13] investigated the following linear singular integral inequality

$$u(t) \leqslant a + b\int_0^t (t-s)^{\beta-1}u(s)ds.$$

Sano and Kunimatsu[14] generalized Henry's type inequality to

$$0 \leqslant u(t) \leqslant c_1 + c_2t^{\alpha-1} + c_3\int_0^t u(s)ds + c_4\int_0^t (t-s)^{\beta-1}u(s)ds,$$

and gave a sufficient condition for stabilization of semilinear parabolic distributed systems. Ye at el. [15] discussed the linear singular integral inequality

$$u(t) \leqslant a(t) + b(t)\int_0^t (t-s)^{\beta-1}u(s)ds,$$

and used it to study the dependence of the solution on the order and the initial condition to a certain fractional differential equation with Riemann-Liouville fractional derivatives. All this type inequalities are proved by an iteration argument and the estimation formulas are expressed by a complicated power series which are sometimes not very convenient for applications. To avoid the weakness, Medveď [16] presented a new method to solve integral inequalities of Henry-Gronwall type, then he got the explicit bounds with a quite simple formula, similar to the classic Gronwall-Bellman inequalities. Furthermore, he also obtained global solutions of the semilinear evolutions in [17]. In 2008, Ma and Pečarić [18] used the modification of Medveď's method to study a new weakly singular integral inequality

$$u^p(t) \leqslant a(t) + b(t)\int_0^t (t^\beta - s^\beta)^{\gamma-1}s^{\xi-1}f(s)u^q(s)ds, \quad t \in [0,+\infty). \tag{4}$$

Besides the results mentioned above, various investigators have discovered many useful and new weakly singular integral inequalities, mainly inspired by their applications in various branches of fractional differential equations; see [19-28] and the references cited therein.

However, only a few papers studied the delay weakly singular integral inequalities, as far as we know. In order to achieve a diversity of desired goals, in this paper, based on the works of [3, 16, 18], we discuss a class of retarded integral inequalities with weak singularity

$$u(t) \leqslant a(t)+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}A_1(s)u(s)ds \\ +b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}A_2(s)\int_0^s A_3(\tau)u(\tau)d\tau ds, \quad t\in\mathbf{R}_+, \tag{5}$$

$$u^p(t) \leqslant d(t)+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}f(s)\left[u^m(s)+\int_0^s g(\tau)u^n(\tau)d\tau\right]^q ds,\ t\in\mathbf{R}_+, \tag{6}$$

which generalize the inequality (2) in [3] to the weakly singular integral inequality, and (4) in [18] to the retarded inequality. We use the modification of Medveď's method to obtain the explicit estimations of unknown function in the inequality (6). Finally, we give an example to illustrate applications of our results.

2. Main result

LEMMA 1. (Hölders inequality [16]) *Suppose that $f(x)$ and $g(x)$ are nonnegative and continuous functions on $[c,d]$. Let $p>1$, $\frac{1}{q}+\frac{1}{p}=1$. Then*

$$\int_c^d f(s)g(s)ds \leqslant \left(\int_c^d f^p(s)ds\right)^{1/p}\left(\int_c^d g^q(s)ds\right)^{1/q}. \tag{7}$$

Let $\alpha(t)$ be continuous, differentiable and increasing functions on $[t_0,+\infty)$ with $\alpha(t)\leqslant t$, $\alpha(t_0)=t_0$, then

$$\int_{\alpha(t_0)}^{\alpha(t)} f(s)g(s)ds \leqslant \left(\int_{\alpha(t_0)}^{\alpha(t)} f^p(s)ds\right)^{1/p}\left(\int_{\alpha(t_0)}^{\alpha(t)} g^q(s)ds\right)^{1/q}. \tag{8}$$

Proof. We prove the inequality (8). Using the inequality (7), we obtain

$$\begin{aligned}\int_{\alpha(t_0)}^{\alpha(t)} f(s)g(s)ds &= \int_{t_0}^t f(\alpha(s))g(\alpha(s))\alpha'(s)ds \\ &= \int_{t_0}^t f(\alpha(s))(\alpha'(s))^{1/p}g(\alpha(s))(\alpha'(s))^{1/q}ds \\ &\leqslant \left(\int_{t_0}^t f^p(\alpha(s))\alpha'(s)ds\right)^{1/p}\left(\int_{t_0}^t g^q(\alpha(s))\alpha'(s)ds\right)^{1/q} \\ &= \left(\int_{\alpha(t_0)}^{\alpha(t)} f^p(s)ds\right)^{1/p}\left(\int_{\alpha(t_0)}^{\alpha(t)} g^q(s)ds\right)^{1/q}. \quad \square\end{aligned} \tag{9}$$

LEMMA 2. (Discrete Jensen inequality [29]) *Let $A_1,A_2,\cdots,A_n$ be nonnegative real numbers, $l>1$ is a real number, and n is a natural number. Then*

$$(A_1+A_2+\cdots+A_n)^l \leqslant n^{l-1}(A_1^l+A_2^l+\cdots+A_n^l). \tag{10}$$

LEMMA 3. (see [30, 18]) *Let $a \geqslant 0$, $p \geqslant q \geqslant 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}} \tag{11}$$

for any $K > 0$.

LEMMA 4. (see [20, 18]) *Let β, γ, ξ and p be positive constants. Then*

$$\int_0^t (t^\beta - s^\beta)^{p(\gamma-1)} s^{p(\xi-1)} ds = \frac{t^\theta}{\beta} B\Big[\frac{p(\xi-1)+1}{\beta}, p(\gamma-1)+1\Big], \quad t \in [0,+\infty). \tag{12}$$

Let $\alpha(t)$ be continuous, differentiable and increasing functions on $[t_0,+\infty)$ with $\alpha(t) \leqslant t$, $\alpha(t_0) = t_0$, then

$$\int_{\alpha(t_0)}^{\alpha(t)} (\alpha^\beta(t) - s^\beta)^{p(\gamma-1)} s^{p(\xi-1)} ds \leqslant \frac{\alpha^\theta(t)}{\beta} B\Big[\frac{p(\xi-1)+1}{\beta}, p(\gamma-1)+1\Big], \quad t \in [0,+\infty), \tag{13}$$

where $B[x,y] = \int_0^1 s^{x-1}(1-s)^{y-1} ds$ $(x > 0,\ y > 0)$ is the well-known B-function and $\theta = p[\beta(\gamma-1)+\xi-1]+1$.

LEMMA 5. (see [20, 18]) *Suppose that the positive constants β, γ, ξ, p_1 and p_2 satisfy conditions:*

(1) if $\beta \in (0,1], \gamma \in (1/2,1)$ and $\xi \geqslant 3/2-\gamma$, $p_1 = 1/\gamma$;

(2) if $\beta \in (0,1], \gamma \in (0,1/2]$ and $\xi > (1-2\gamma^2)/(1-\gamma^2)$, $p_2 = (1+4\gamma)/(1+3\gamma)$, then

$$B\Big[\frac{p_i(\xi-1)+1}{\beta}, p_i(\gamma-1)+1\Big] \in [0,+\infty), \tag{14}$$

and $\theta_i = p_i[\beta(\gamma-1)+\xi-1]+1 \geqslant 0$ are valid for $i = 1,2$.

LEMMA 6. *Let $u(t)$, $a(t)$, $b(t)$ and $h(t)$ be nonnegative continuous functions on $\mathbf{R}_+$, and let $\alpha(t)$ be continuous, differentiable and increasing functions on $\mathbf{R}_+$ with $\alpha(t) \leqslant t$, $\alpha(0) = 0$. If $u(t)$ satisfies the following inequality*

$$u(t) \leqslant a(t) + b(t) \int_0^{\alpha(t)} h(s)u(s)ds, \quad t \in \mathbf{R}_+. \tag{15}$$

Then

$$u(t) \leqslant a(t) + \frac{b(t)}{e(\alpha(t))} \int_0^{\alpha(t)} h(s)a(s)e(s)ds, \quad t \in \mathbf{R}_+, \tag{16}$$

where

$$e(t) = \exp\Big(-\int_0^t h(s)b(s)ds\Big). \tag{17}$$

Proof. Define a function $v(t)$ on $\mathbf{R}_+$ by

$$v(t) = e(\alpha(t)) \int_0^{\alpha(t)} h(s)u(s)ds, \tag{18}$$

we have $v(0) = 0$. Differentiating $v(t)$ with respect to t and using (15) and (17), we have

$$\begin{aligned} v'(t) &= \alpha'(t)h(\alpha(t))u(\alpha(t))e(\alpha(t)) - \alpha'(t)h(\alpha(t))b(\alpha(t))e(\alpha(t)) \int_0^{\alpha(t)} h(s)u(s)ds \\ &\leqslant \alpha'(t)h(\alpha(t))a(\alpha(t))e(\alpha(t)) + \alpha'(t)h(\alpha(t))e(\alpha(t))b(\alpha(t)) \int_0^{\alpha(t)} h(s)u(s)ds \\ &\quad -\alpha'(t)h(\alpha(t))b(\alpha(t))e(\alpha(t)) \int_0^{\alpha(t)} h(s)u(s)ds \\ &\leqslant \alpha'(t)h(\alpha(t))a(\alpha(t))e(\alpha(t)). \end{aligned} \tag{19}$$

Integrating both sides of the inequality (19) from 0 to t, since $v(0) = 0$ we get

$$v(t) \leqslant \int_0^t \alpha'(s)h(\alpha(s))a(\alpha(s))e(\alpha(s))ds = \int_0^{\alpha(t)} h(s)a(s)e(s)ds. \tag{20}$$

From (18) and (20), we obtain

$$\int_0^{\alpha(t)} h(s)u(s)ds \leqslant \frac{1}{e(\alpha(t))} \int_0^{\alpha(t)} h(s)a(s)e(s)ds. \tag{21}$$

Substituting the inequality (21) into (15) we get the required estimation (16). The proof is completed. □

THEOREM 1. *Let $a(t)$, $b(t)$, $A_1(t)$, $A_2(t)$ and $A_3(t)$ be nonnegative continuous functions on $\mathbf{R}_+$, and both $a(t)$ and $b(t)$ are nondecreasing functions, and let $\alpha(t)$ be continuous, differentiable and increasing functions on $\mathbf{R}_+$ with $\alpha(t) \leqslant t$, $\alpha(0) = 0$. Let β, γ, ξ be positive constants. Suppose that $u(t)$ satisfies the inequality (5).*

(1) If $\beta \in (0,1]$, $\gamma \in (1/2,1)$ and $\xi \geqslant 3/2-\gamma$; then for any $K > 0$, we have

$$u(t) \leqslant \left(\tilde{a}_1(t) + \frac{\tilde{b}_1(t)}{\tilde{e}_1(\alpha(t))} \int_0^{\alpha(t)} \tilde{h}_1(s)\tilde{a}_1(s)\tilde{e}_1(s)ds\right)^{1-\gamma}, \quad t \in \mathbf{R}_+, \tag{22}$$

where

$$\tilde{a}_1(t) = \left(3M_1\alpha^{\theta_1}(t)\right)^{\frac{\gamma}{1-\gamma}} a^{\frac{1}{1-\gamma}}(t), \tag{23}$$

$$\tilde{b}_1(t) = \left(3M_1\alpha^{\theta_1}(t)\right)^{\frac{\gamma}{1-\gamma}} b^{\frac{1}{1-\gamma}}(t), \tag{24}$$

$$\tilde{h}_1(t) = A_1^{\frac{1}{1-\gamma}}(t) + \left(A_2(t)\int_0^t A_3(\tau)d\tau\right)^{\frac{1}{1-\gamma}}, \tag{25}$$

$$\tilde{e}_1(t) = \exp\left(-\int_0^t \tilde{h}_1(s)\tilde{b}_1(s)ds\right), \tag{26}$$

$$M_1 = \frac{1}{\beta}B\left[\frac{\gamma+\xi-1}{\beta\gamma}, \frac{2\gamma-1}{\gamma}\right], \tag{27}$$

$$\theta_1 = \frac{1}{\gamma}[\beta(\gamma-1)+\xi-1]+1. \tag{28}$$

(2) If $\beta \in (0,1]$, $\gamma \in (0,1/2]$ and $\xi > (1-2\gamma^2)/(1-\gamma^2)$, then for any $K>0$, we have

$$w(t) \leqslant \left(\tilde{a}_2(t) + \frac{\tilde{b}_2(t)}{\tilde{e}_2(\alpha(t))}\int_0^{\alpha(t)} \tilde{h}_2(s)\tilde{a}_2(s)\tilde{e}_2(s)ds\right)^{\frac{\gamma}{1+4\gamma}}, \quad t \in \mathbf{R}_+, \tag{29}$$

where

$$\tilde{a}_2(t) = \left(3M_2\alpha^{\theta_2}(t)\right)^{\frac{1+3\gamma}{\gamma}} a^{\frac{1+4\gamma}{\gamma}}(t), \tag{30}$$

$$\tilde{b}_2(t) = \left(3M_2\alpha^{\theta_2}(t)\right)^{\frac{1+3\gamma}{\gamma}} b^{\frac{1+4\gamma}{\gamma}}(t), \tag{31}$$

$$\tilde{h}_2(t) = A_1^{\frac{1+4\gamma}{\gamma}}(s) + \left(A_2(s)\int_0^s A_3(\tau)d\tau\right)^{\frac{1+4\gamma}{\gamma}}, \tag{32}$$

$$\tilde{e}_2(t) = \exp\left(-\int_0^t \tilde{h}_2(s)\tilde{b}_2(s)ds\right), \tag{33}$$

$$M_2 = \frac{1}{\beta}B\left[\frac{\xi(1+4\gamma)-\gamma}{\beta(1+3\gamma)}, \frac{4\gamma^2}{1+3\gamma}\right], \tag{34}$$

$$\theta_2 = \frac{1+4\gamma}{1+3\gamma}[\beta(\gamma-1)+\xi-1]+1. \tag{35}$$

Proof. If $\beta \in (0,1]$, $\gamma \in (1/2,1)$ and $\xi \geqslant 3/2-\gamma$, let $p_1 = 1/\gamma$, $q_1 = 1/(1-\gamma)$; if $\beta \in (0,1]$, $\gamma \in (0,1/2]$ and $\xi > (1-2\gamma^2)/(1-\gamma^2)$, let $p_2 = (1+4\gamma)/(1+3\gamma)$, $q_2 = (1+4\gamma)/\gamma$, then $1/p_i + 1/q_i = 1$ for $i = 1,2$. Using Hölder's inequality in Lemma 1 to (5), we have

$$\begin{aligned} u(t) \leqslant{}& a(t) + b(t)\left[\int_0^{\alpha(t)} (\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\right]^{1/p_i}\left[\int_0^{\alpha(t)} A_1^{q_i}(s)u^{q_i}(s)ds\right]^{1/q_i} \\ &+b(t)\left[\int_0^{\alpha(t)} (\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\right]^{1/p_i} \\ &\times\left[\int_0^{\alpha(t)} \left(A_2(s)\int_0^s A_3(\tau)u(\tau)d\tau\right)^{q_i}ds\right]^{1/q_i}. \end{aligned} \tag{36}$$

Define a function $z(t)$ by the right hand side of the inequality (36), i. e.

$$z(t) = a(t)+b(t)\Big[\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\Big]^{1/p_i}\Big[\int_0^{\alpha(t)}A_1^{q_i}(s)u^{q_i}(s)ds\Big]^{1/q_i}$$
$$+b(t)\Big[\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\Big]^{1/p_i}$$
$$\times\Big[\int_0^{\alpha(t)}\Big(A_2(s)\int_0^s A_3(\tau)u(\tau)d\tau\Big)^{q_i}ds\Big]^{1/q_i}. \tag{37}$$

Then, $z(t)$ is a nondecreasing function, and $u(t)\leqslant z(t)$, we have

$$z(t) \leqslant a(t)+b(t)\Big[\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\Big]^{1/p_i}\Big[\int_0^{\alpha(t)}A_1^{q_i}(s)z^{q_i}(s)ds\Big]^{1/q_i}$$
$$+b(t)\Big[\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\Big]^{1/p_i}$$
$$\times\Big[\int_0^{\alpha(t)}\Big(A_2(s)\int_0^s A_3(\tau)d\tau\Big)^{q_i}z^{q_i}(s)ds\Big]^{1/q_i}. \tag{38}$$

Using discrete Jensen inequality (10) in Lemma 2 with $n=3$, $l=q_i$, we obtain

$$z^{q_i}(t) \leqslant 3^{q_i-1}a^{q_i}(t)+3^{q_i-1}b^{q_i}(t)\Big[\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\Big]^{q_i/p_i}$$
$$\times\int_0^{\alpha(t)}A_1^{q_i}(s)z^{q_i}(s)ds$$
$$+3^{q_i-1}b^{q_i}(t)\Big[\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{p_i(\gamma-1)}s^{p_i(\xi-1)}ds\Big]^{q_i/p_i}$$
$$\times\int_0^{\alpha(t)}\Big(A_2(s)\int_0^s A_3(\tau)d\tau\Big)^{q_i}z^{q_i}(s)ds. \tag{39}$$

Using Lemmas 4 and 5, the inequality (39) can be rewritten as

$$z^{q_i}(t) \leqslant 3^{q_i-1}a^{q_i}(t)+3^{q_i-1}b^{q_i}(t)\big(M_i\alpha^{\theta_i}(t)\big)^{q_i/p_i}$$
$$\times\int_0^{\alpha(t)}\Big[A_1^{q_i}(s)+\Big(A_2(s)\int_0^s A_3(\tau)d\tau\Big)^{q_i}\Big]z^{q_i}(s)ds, \tag{40}$$

for $t\in\mathbf{R}_+$, where

$$M_i = \frac{1}{\beta}B\Big[\frac{p_i(\xi-1)+1}{\beta},p_i(\gamma-1)+1\Big], \tag{41}$$
$$\theta_i = p_i[\beta(\gamma-1)+\xi-1]+1\geqslant 0, \tag{42}$$

for $i=1,2$. Applying Lemma 6 to (40), we obtain

$$u^{q_i}(t)\leqslant z^{q_i}(t)\leqslant \tilde{a}_i(t)+\frac{\tilde{b}_i(t)}{\tilde{e}_i(\alpha(t))}\int_0^{\alpha(t)}\tilde{h}_i(s)\tilde{a}_i(s)\tilde{e}_i(s)ds,\ \ t\in\mathbf{R}_+, \tag{43}$$

where

$$\tilde{a}_i(t) = 3^{q_i-1}a^{q_i}(t), \tag{44}$$

$$\tilde{b}_i(t) = 3^{q_i-1}b^{q_i}(t)\big(M_i\alpha^{\theta_i}(t)\big)^{q_i/p_i}, \tag{45}$$

$$\tilde{h}_i(t) = A_1^{q_i}(s) + \Big(A_2(s)\int_0^s A_3(\tau)d\tau\Big)^{q_i}, \tag{46}$$

$$\tilde{e}_i(t) = \exp\Big(-\int_0^t \tilde{h}_i(s)\tilde{b}_i(s)ds\Big). \tag{47}$$

Substituting $p_1 = 1/\gamma$, $q_1 = 1/(1-\gamma)$ and $p_2 = (1+4\gamma)/(1+3\gamma)$, $q_2 = (1+4\gamma)/\gamma$ to (43) respectively, we can get the desired estimations (22) and (29). The proof is completed. □

THEOREM 2. *Let $u(t)$, $d(t)$, $b(t)$ and $f(t)$ be nonnegative continuous functions on $\mathbf{R}_+$, and $b(t)$ is a nondecreasing function, and let $\alpha(t)$ be continuous, differentiable and increasing functions on $[t_0,+\infty)$ with $\alpha(t) \leqslant t$, $\alpha(0) = 0$. Let p, q, m, n, β, γ, ξ be positive constants with $p \geqslant m$, $p \geqslant n$, m, n, $q \in [0,1)$. If $u(t)$ satisfies the inequality (6).*

(1) If $\beta \in (0,1]$, $\gamma \in (1/2,1)$ and $\xi \geqslant 3/2-\gamma$; then for any $K > 0$, we have

$$u(t) \leqslant \Big[d(t) + \Big(\tilde{a}_1(t) + \frac{\tilde{b}_1(t)}{\tilde{e}_1(\alpha(t))}\int_0^{\alpha(t)} \tilde{h}_1(s)\tilde{a}_1(s)\tilde{e}_1(s)ds\Big)^{1-\gamma}\Big]^{1/p}, \quad t \in \mathbf{R}_+, \tag{48}$$

where $\tilde{b}_1(t)$, $\tilde{h}_1(t)$, $\tilde{e}_1(t)$ are the same as in Theorem 1.

$$\tilde{a}_1(t) = \big(3M_1\alpha^{\theta_1}(t)\big)^{\frac{\gamma}{1-\gamma}}a^{\frac{1}{1-\gamma}}(t), \tag{49}$$

$$a(t) = b(t)\int_0^{\alpha(t)} (\alpha^\beta(t) - s^\beta)^{\gamma-1}s^{\xi-1}A(s)ds, \tag{50}$$

$$\begin{aligned} A(t) &= f(t)\Big[(1-q)K^q + qK^{q-1}\Big(\frac{m}{p}K^{\frac{m-p}{p}}d(t) + \frac{p-m}{p}K^{\frac{m}{p}}\Big)\Big] \\ &\quad + qK^{q-1}f(t)\int_0^t g(\tau)\Big[\frac{n}{p}K^{\frac{n-p}{p}}d(\tau) + \frac{p-n}{p}K^{\frac{n}{p}}\Big]d\tau, \\ A_1(t) &= \frac{mq}{p}K^{\frac{m}{p}+q-2}f(t), A_2(t) = qK^{q-1}f(t), A_3(t) = \frac{n}{p}K^{\frac{n-p}{p}}g(t). \end{aligned} \tag{51}$$

(2) If $\beta \in (0,1]$, $\gamma \in (0,1/2]$ and $\xi > (1-2\gamma^2)/(1-\gamma^2)$, then for any $K > 0$, we have

$$u(t) \leqslant \Big[d(t) + \Big(\tilde{a}_2(t) + \frac{\tilde{b}_2(t)}{\tilde{e}_2(\alpha(t))}\int_0^{\alpha(t)} \tilde{h}_2(s)\tilde{a}_2(s)\tilde{e}_2(s)ds\Big)^{\frac{\gamma}{1+4\gamma}}\Big]^{1/p}, \quad t \in \mathbf{R}_+, \tag{52}$$

where $\tilde{a}_2(t)$, $\tilde{b}_2(t)$, $\tilde{h}_2(t)$, $\tilde{e}_2(t)$ are the same as in Theorem 1,

Proof. By Lemma 3, for any $K > 0$ we have

$$\Big[u^m(s) + \int_0^s g(\tau)u^n(\tau)d\tau\Big]^q \leqslant qK^{q-1}\Big[u^m(s) + \int_0^s g(\tau)u^n(\tau)d\tau\Big] + (1-q)K^q, \tag{53}$$

Substituting (53) to (6), we have

$$u^p(t) \leqslant d(t)+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}f(s) \times\Big[qK^{q-1}\Big(u^m(s)+\int_0^s g(\tau)u^n(\tau)d\tau\Big)+(1-q)K^q\Big]ds, \tag{54}$$

Define a function $w(t)$ by

$$\begin{aligned} w(t)=b(t)&\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}(1-q)K^q f(s)ds \\ &+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}qK^{q-1}f(s)u^m(s)ds \\ &+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}qK^{q-1}f(s)\int_0^s g(\tau)u^n(\tau)d\tau ds, \quad t\in\mathbf{R}_+. \end{aligned} \tag{55}$$

We have

$$u^p(t) \leqslant d(t)+w(t), \quad \text{or,} \quad u(t)\leqslant \big(d(t)+w(t)\big)^{1/p}, \quad t\in[t_0,+\infty). \tag{56}$$

By Lemma 3 and (56), for the above $K>0$, we obtain

$$u^m(t) \leqslant \big(d(t)+w(t)\big)^{m/p} \leqslant \frac{m}{p}K^{\frac{m-p}{p}}(d(t)+w(t))+\frac{p-m}{p}K^{\frac{m}{p}}, \quad t\in\mathbf{R}_+, \tag{57}$$

$$u^n(t) \leqslant \big(d(t)+w(t)\big)^{n/p} \leqslant \frac{n}{p}K^{\frac{n-p}{p}}(d(t)+w(t))+\frac{p-n}{p}K^{\frac{n}{p}}, \quad t\in\mathbf{R}_+. \tag{58}$$

Substituting the inequality (57) and (58) into (55) we have

$$
\begin{aligned}
w(t) \leqslant\ & b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}(1-q)K^q f(s)ds \\
&+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}qK^{q-1}f(s) \\
&\times\left[\frac{m}{p}K^{\frac{m-p}{p}}(d(s)+w(s))+\frac{p-m}{p}K^{\frac{m}{p}}\right]ds \\
&+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}qK^{q-1}f(s)\int_0^s g(\tau) \\
&\times\left[\frac{n}{p}K^{\frac{n-p}{p}}(d(\tau)+w(\tau))+\frac{p-n}{p}K^{\frac{n}{p}}\right]d\tau ds \\
\leqslant\ & b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}f(s) \\
&\times\left[(1-q)K^q+qK^{q-1}\left(\frac{m}{p}K^{\frac{m-p}{p}}d(s)+\frac{p-m}{p}K^{\frac{m}{p}}\right)\right]ds \\
&+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}qK^{q-1}f(s)\int_0^s g(\tau)\left[\frac{n}{p}K^{\frac{n-p}{p}}d(\tau)+\frac{p-n}{p}K^{\frac{n}{p}}\right]d\tau ds \\
&+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}\frac{mq}{p}K^{\frac{m}{p}+q-2}f(s)w(s)ds \\
&+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}qK^{q-1}f(s)\int_0^s \frac{n}{p}K^{\frac{n-p}{p}}g(\tau)w(\tau)d\tau ds \\
\leqslant\ & b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}A(s)ds+b(t)\int_0^{\alpha(t)}(t^\beta-s^\beta)^{\gamma-1}s^{\xi-1}A_1(s)w(s)ds \\
&+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}A_2(s)\int_0^s A_3(\tau)w(\tau)d\tau ds, \quad t\in\mathbf{R}_+ \\
=\ & a(t)+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}A_1(s)w(s)ds \\
&+b(t)\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\xi-1}A_2(s)\int_0^s A_3(\tau)w(\tau)d\tau ds, \quad t\in\mathbf{R}_+. \qquad (59)
\end{aligned}
$$

where

$$a(t) = b(t)\int_0^{\alpha(t)} (\alpha^\beta(t) - s^\beta)^{\gamma-1} s^{\xi-1} A(s)ds, \tag{60}$$

$$A(t) = f(t)\Big[(1-q)K^q + qK^{q-1}\Big(\frac{m}{p}K^{\frac{m-p}{p}}d(t) + \frac{p-m}{p}K^{\frac{m}{p}}\Big)\Big]$$
$$+qK^{q-1}f(t)\int_0^t g(\tau)\Big[\frac{n}{p}K^{\frac{n-p}{p}}d(\tau) + \frac{p-n}{p}K^{\frac{n}{p}}\Big]d\tau,$$

$$A_1(t) = \frac{mq}{p}K^{\frac{m}{p}+q-2}f(t),$$

$$A_2(t) = qK^{q-1}f(t),$$

$$A_3(t) = \frac{n}{p}K^{\frac{n-p}{p}}g(t).$$

Since (59) have the same form of (5) and the functions of (59) satisfy the conditions of Theorem 1, applying Theorem 1 to (59), considering the relation (54), we can get the desired estimations (48) and (52). The proof is completed. □

3. Application

Consider the following Volterra type retarded weakly singular integral equations

$$y^p(t) - \lambda t^{-\beta\delta}\int_0^{\alpha(t)} \frac{(\alpha^\beta(t) - s^\beta)^{\gamma-1}}{\Gamma(\gamma)} s^{\beta(1+\delta)-1}\Big[y(s) + \int_0^s g(\tau)y(\tau)d\tau\Big]^q ds = h(t) \tag{61}$$

which arises very often in various problems, especial describing physical processes with aftereffects. Ma and Pečarić [18] discussed the case $\alpha(t) = t$, $g(t) \equiv 0$ in (61).

THEOREM 3. *Let $y(t)$, $g(t)$ and $h(t)$ be continuous functions on $[0,+\infty)$, and let $\alpha(t)$ be continuous, differentiable and increasing functions on $[0,+\infty)$ with $\alpha(t) \leqslant t$, $\alpha(0) = 0$. Let p, q, β, γ, δ be positive constants with $p \geqslant q$. If $y(t)$ satisfies the equation (61).*

(i) If $\beta \in (0,1]$, $\gamma \in (1/2,1)$ and $\beta(1+\delta) \geqslant 3/2-\gamma$; then for any $K > 0$, we have

$$|y(t)| \leqslant \Big[|h(t)| + \Big(\tilde{a}_1(t) + \frac{\tilde{b}_1(t)}{\tilde{e}_1(\alpha(t))}\int_0^{\alpha(t)} \tilde{h}_1(s)\tilde{a}_1(s)\tilde{e}_1(s)ds\Big)^{1-\gamma}\Big]^{1/p}, \quad t \in \mathbf{R}_+, \tag{62}$$

where

$$\tilde{a}_1(t) = \big(3M_1\alpha^{\theta_1}(t)\big)^{\frac{\gamma}{1-\gamma}}\Big(\frac{|\lambda|}{\Gamma(\gamma)}t^{-\beta\xi}\Big)^{\frac{1}{1-\gamma}}\int_0^{\alpha(t)} A_1^{\frac{1}{1-\gamma}}(s)ds,$$

$$\tilde{b}_1(t) = \big(3M_1\alpha^{\theta_1}(t)\big)^{\frac{\gamma}{1-\gamma}}\Big(\frac{|\lambda|}{\Gamma(\gamma)}t^{-\beta\xi}\Big)^{\frac{1}{1-\gamma}},$$

$$\tilde{h}_1(t) = A_2^{\frac{1}{1-\gamma}}(t) + \Big(A_3(t)\int_0^t A_4(\tau)d\tau\Big)^{\frac{1}{1-\gamma}},$$

$$\begin{aligned}
\tilde{e}_1(t) &= \exp\Big(-\int_0^t \tilde{h}_1(s)\tilde{b}_1(s)ds\Big),\\
M_1 &= \frac{1}{\beta}B\Big[\frac{\gamma+\xi-1}{\beta\gamma},\frac{2\gamma-1}{\gamma}\Big],\\
\theta_1 &= \frac{1}{\gamma}[\beta(\gamma-1)+\xi-1]+1,\\
A_1(t) &= (1-q)K^q+qK^{q-1}\Big(\frac{1}{p}K^{\frac{1-p}{p}}|h(t)|+\frac{p-1}{p}K^{\frac{1}{p}}\Big)\\
&\quad +qK^{q-1}\int_0^t |g(\tau)|\Big[\frac{1}{p}K^{\frac{1-p}{p}}|h(\tau)|+\frac{p-1}{p}K^{\frac{1}{p}}\Big]d\tau,\\
A_2(t) &= \frac{q}{p}K^{\frac{1}{p}+q-2}, A_3(t)=qK^{q-1}, A_4(t)=\frac{1}{p}K^{\frac{1-p}{p}}|g(t)|.
\end{aligned}$$

(2) If $\beta\in(0,1]$, $\gamma\in(0,1/2]$ and $\xi>(1-2\gamma^2)/(1-\gamma^2)$, then for any $K>0$, we have

$$w(t)\leqslant\Big[|h(t)|+\Big(\tilde{a}_2(t)+\frac{\tilde{b}_2(t)}{\tilde{e}_2(\alpha(t))}\int_0^{\alpha(t)}\tilde{h}_2(s)\tilde{a}_2(s)\tilde{e}_2(s)ds\Big)^{\frac{\gamma}{1+4\gamma}}\Big]^{1/p},\quad t\in\mathbf{R}_+, \tag{63}$$

where

$$\begin{aligned}
\tilde{a}_2(t) &= \big(3M_2\alpha^{\theta_2}(t)\big)^{\frac{1+3\gamma}{\gamma}}\Big(\frac{|\lambda|}{\Gamma(\gamma)}t^{-\beta\xi}\Big)^{\frac{1+4\gamma}{\gamma}}\int_0^{\alpha(t)}A_1^{\frac{1+4\gamma}{\gamma}}(s)ds,\\
\tilde{b}_2(t) &= \big(3M_2\alpha^{\theta_2}(t)\big)^{\frac{1+3\gamma}{\gamma}}\Big(\frac{|\lambda|}{\Gamma(\gamma)}t^{-\beta\xi}\Big)^{\frac{1+4\gamma}{\gamma}},\\
\tilde{h}_2(t) &= A_2^{\frac{1+4\gamma}{\gamma}}(s)+\Big(A_3(s)\int_0^s A_4(\tau)d\tau\Big)^{\frac{1+4\gamma}{\gamma}},\\
\tilde{e}_2(t) &= \exp\Big(-\int_0^t \tilde{h}_2(s)\tilde{b}_2(s)ds\Big),\\
M_2 &= \frac{1}{\beta}B\Big[\frac{\xi(1+4\gamma)-\gamma}{\beta(1+3\gamma)},\frac{4\gamma^2}{1+3\gamma}\Big],\\
\theta_2 &= \frac{1+4\gamma}{1+3\gamma}[\beta(\gamma-1)+\xi-1]+1.
\end{aligned}$$

Proof. From (61), we have

$$\begin{aligned}
|y(t)|^p &\leqslant |h(t)|+\frac{|\lambda|}{\Gamma(\gamma)}t^{-\beta\xi}\int_0^{\alpha(t)}(\alpha^\beta(t)-s^\beta)^{\gamma-1}s^{\beta(1+\xi)-1}\\
&\quad\times\Big[|y(s)|+\int_0^s|g(\tau)||y(\tau)|d\tau\Big]^q ds.
\end{aligned}\tag{64}$$

Appling the Theorem 1 (with $m=n=1$, $a(t)=|h(t)|$, $b(t)=|\lambda|t^{-\beta\delta}/\Gamma(\gamma)$, $\xi=\beta(1+\delta)$) to (64), we obtain the desired estimations (48) and (52). □

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REFERENCES

[1] T. H. GRONWALL, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann Math. **20** (1919) 292–296.

[2] R. BELLMAN, *The stability of solutions of linear differential equations*, Duke Math. J. **10** (1943) 643–647.

[3] A. ABDELDAIM, *Nonlinear retarded integral inequalities of type and applications*, J. Math. Inequal. **10** (1) (2016) 285–299.

[4] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and Integral Inequalities, Theory and Applications*, Academic Press, New York, 1969.

[5] D. D. BAINOV AND P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992.

[6] R. P. AGARWAL, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1993.

[7] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.

[8] O. LIPOVAN, *A retarded Gronwall-like inequality and its applications*, J. Math. Anal. Appl. **252** (2000) 389–401.

[9] R. P. AGARWAL, S. DENG AND W. ZHANG, *Generalization of a retarded Gronwall-like inequality and its applications*, Appl. Math. Comput. **165** (2005) 599–612.

[10] W. S. CHEUNG, *Some new nonlinear inequalities and applications to boundary value problems*, Nonlinear Anal. **64** (2006) 2112–2128.

[11] A. ABDELDAIM AND M. YAKOUT, *On some new integral inequalities of Gronwall-Bellman-Pachpatte type*, Appl. Math. Comput. **217** (2011) 7887–7899.

[12] H. EL-OWAIDY, A. A. RAGAB, W. ABUELELA, A. A. EL-DEEB, *On some new nonlinear integral inequalities of Gronwall-Bellman type*, KYUNGPOOK Math. J. **54** (2014) 555–575.

[13] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math., vol. **840**, Springer-Verlag, New York/Berlin, 1981.

[14] H. SANO AND N. KUNIMATSU, *Modified Gronwall's inequality and its application to stabilization problem for semilinear parabolic systems*, Systems Control Lett. **22** (1994) 145–156.

[15] H. P. YE, J. M. GAO, Y. S. DING, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. **328** (2007) 1075–1081.

[16] M. MEDVEĎ, *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*, J. Math. Anal. Appl. **214** (1997) 349–366.

[17] M. MEDVEĎ, *Integral inequalities and global solutions of semilinear evolution equations*, J. Math. Anal. Appl. **267** (2002) 643–650.

[18] Q. H. MA AND J. PEČARIĆ, *Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations*, J. Math. Anal. Appl. **341** (2) (2008) 894–905.

[19] M. MEDVEĎ, *Nonlinear singular integral inequalities for functions in two and n independent variables*, J. Inequal. Appl. **5** (2000) 287–308.

[20] Q. H. MA AND E. H. YANG, *Estimations on solutions of some weakly singular Volterra integral inequalities*, Acta Math. Appl. Sin. **25** (2002) 505–515.

[21] S. DENG, C. PRATHER, *Generalization of an impulsive nonlinear singular Gronwall-Bihari inequality with delay*, J. Inequal. Pure Appl. Math. **9** (2008) Article 34.

[22] S. MAZOUZI AND N. TATAR, *New bounds for solutions of a singular integro-differential inequality*, Math. Inequal. Appl. **13** (2) (2010) 427–435.

[23] H. WANG AND K. ZHENG, *Some nonlinear weakly singular integral inequalities with two variables and applications*, J. Inequal. Appl. **2010** (2010) Article ID 345701.

[24] K. CHENG, C. GUO AND M. TANG, *Some nonlinear Gronwall-Bellman-Gamidov integral inequalities and their weakly singular analogues with applications*, Abstr. Appl. Anal. **2014** (2014) Article ID 562691.

[25] B. ZHENG, *Explicit bounds derived by some new inequalities and applications in fractional integral equations*, J. Inequal. Appl. **2014** (2014) Article 4.

[26] Y. OUYANG AND W. S. WANG, *A class of weakly singular nonlinear Volterra-Fredholm type iterated integral inequality with pth power and its application*, Journal of Sichuan Normal University (Natural Science) **39** (2016) 209–213.

[27] C. M. HUANG AND W. S. WANG, *Weakly singular nonlinear iterated integral inequality with its applications*, Journal of Sichuan Normal University (Natural Science) **39** (2016), 214–220.

[28] R. XU AND F. MENG, *Some new weakly singular integral inequalities and their applications to fractional differential equations*, J. Inequal. Appl. **2016** (2016) Article 78.

[29] M. KUCZMA, *An introduction to the theory of functional equations and inequalities: Cauchy's equation and Jensen's inequality*, University of Katowice, Katowice, 1985.

[30] F. C. JIANG AND F. W. MENG, *Explicit bounds on some new nonlinear integral inequalities with delay*, J. Comput. Appl. Math. **205** (2007) 479–486.

[31] D. WILLETT, *Nonlinear vector integral equations as contraction mappings*, Arch. Ration. Mech. Anal. **15** (1964) 79–86.

[32] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

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Ricai Luo
School of Mathematics and Statistics
Hechi University
Guangxi, Yizhou 546300, P. R. China

Wu-Sheng Wang
School of Mathematics and Statistics
Hechi University
Guangxi, Yizhou 546300, P. R. China
e-mail: wang4896@126.com

Zongyi Hou
School of Mathematics and Statistics
Hechi University
Guangxi, Yizhou 546300, P. R. China