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MAXIMAL NUMERICAL RANGE OF A COMPACT SET AND APPLICATIONS TO SOME DRAGOMIR'S INEQUALITIES

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Abstract. Let K, A be respectively a compact and an element of $B(H)$ the algebra of all bounded linear operators acting on a complex Hilbert space H. In this paper we define the maximal numerical range of the set $A^*K = \{A^*B : B \in K\}$ relatively to K by

$$W_K(A^*K) = co(\bigsqcup_{B\in K} W_B(A^*B)).$$

Where $W_B(A^*B)$ is the maximal numerical range of A^*B relatively to B defined by Magajna [6] and which coincides with the maximal numerical range $W_0(B)$ of B defined by Stampfli [7] if A is the unit element I. Our new definition will generalize the results of Stampfli [7] and Barraa-Boumazguour [1] over the distance of an element B to $Vect(A)$. It also will generalize and improve several inequalities established by Dragomir [4, 5] linking the norm and the numerical radius of B.

1. Introduction

Let H be a complexe Hilbert space. Let $H_1 = \{x \in H : \|x\| = 1\}$ and $B(H)$ be the algebra of all bounded linear operators acting on H. For $B \in B(H)$, Stampfli [7] defined the maximal numerical range $W_0(B)$ of B by

$$W_0(B) = \{\lambda = \lim_n \langle Bx_n, x_n\rangle : (x_n)_n \subseteq H_1,\ \lim_n \|Bx_n\| = \|B\|\},$$

and the center of mass of B as the unique scalar c_B such that

$$\|B - c_B I\| = \inf_{\lambda\in\mathbb{C}} \|B - \lambda I\|.$$

He established the equivalence of the following assertions

(1) $0 \in W_0(B - c_B I)$,

(2) $\|B - c_B I\| \leqslant \|B - c_B I - \lambda I\|$, for all $\lambda \in \mathbb{C}$,

(3) $\|B - c_B I\|^2 + |\lambda|^2 \leqslant \|B - c_B I - \lambda I\|^2$, for all $\lambda \in \mathbb{C}$.

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He showed the continuity of the map $B \longmapsto c_B$, $B \in B(H)$. Magajna [6] defined the maximal numerical range $W_B(A^*B)$ of A^*B relatively to B as the following convex compact

$$W_B(A^*B) = \{\lambda = \lim_n \langle A^*Bx_n, x_n\rangle : (x_n)_n \subseteq H_1,\ \lim_n \|Bx_n\| = \|B\|\}.$$

Note that if $A = I$, then $W_B(A^*B) = W_0(B)$. Barraa and Boumazgour [1] showed the equivalence of the following assertions

(1) $0 \in W_B(A^*B)$,

(2) $\|B\| \leqslant \|B + \lambda A\|$, for all $\lambda \in \mathbb{C}$,

(3) $\|B\|^2 + |\lambda|^2 m^2(A) \leqslant \|B + \lambda A\|^2$, for all $\lambda \in \mathbb{C}$, where $m(A) = \inf\limits_{x \in H_1} \|Ax\|$.

When $m(A) > 0$, they showed the existence and the uniqueness of a scalar denoted $c_B(A^*B)$ such that

$$\|B - c_B(A^*B)A\| = \inf_{\lambda \in \mathbb{C}} \|B - \lambda A\|.$$

They called it the center of mass of A^*B relatively to B. They also showed the continuity of the map $B \longmapsto c_B(A^*B)$, $B \in B(H)$. Bouchen and Chraibi Kaadoud [2] defined the numerical range of a compact K of $B(H)$ by

$$W(K) = co(\bigsqcup_{B \in K} W(B)).$$

Here, $co(S)$ denotes the convex hull of a set S in a vector space and $W(B)$ is the numerical range of B defined by

$$W(B) = \{\langle Bx, x\rangle : x \in H_1\}.$$

In this direction, for a compact K and an element A of $B(H)$, we define the maximal numerical range $W_K(A^*K)$ of the set $A^*K = \{A^*B : B \in K\}$ relatively to K by

$$W_K(A^*K) = co(\bigsqcup_{B \in K} W_B(A^*B)).$$

In the second section we show (Theorem 1) that $W_K(A^*K)$ is a compact set. For a scalar z_0, we show (Proposition 1) the equivalence of the following assertions

(1) $0 \in W_{(K - z_0A)'}(A^*(K - z_0A)')$,

(2) $|K - z_0A| \leqslant |(K - z_0A)' - \lambda A|$, for all $\lambda \in \mathbb{C}$,

(3) $|K - z_0A|^2 + |\lambda|^2 m^2(A) \leqslant |(K - z_0A)' - \lambda A|^2$, for all $\lambda \in \mathbb{C}$.

Where, for any compact S of $B(H)$, $S' = \{B \in S : \|B\| = |S|\}$ with $|S|$ is the modulus of S defined by $|S| = \sup\limits_{B \in S} \|B\|$. Note that S' is also a compact set of $B(H)$ and $|S| = |S'|$.

In the case where $m(A) > 0$, we show the existence of a class, denoted $\mathscr{K}_A$, of compacts K of $B(H)$ such that there exists a unique scalar c_K checking one of the three previous equivalent assertions. A such scalar will be called the center of mass of A^*K relatively to K, or simply the center of A^*K. We also show the continuity in the sens of Hausdorff of the mapp $K \longmapsto c_K$, $K \in \mathscr{K}_A$.

In the last section we show some inequalities linking $|K|$ and the positive real $w'_{K'}(A^*K')$ defined by

$$w'_{K'}(A^*K') = \inf_{B \in K'} \{|z| : z \in W_B(A^*B)\}.$$

The inequalities we get in the multivoque case generalize and improve several inequalities established by Dragomir [4, 5] linking the norm and the numerical radius of B in the univoque case (i.e., K is reduced to a singleton).

Throughout, K and A will denote respectively a compact and a non zero element of $B(H)$.

2. Maximal numerical range of a compact set

DEFINITION 1. The maximal numerical range of A^*K relatively to K, denoted $W_K(A^*K)$, is defined by

$$W_K(A^*K) = co(\bigsqcup_{B \in K} W_B(A^*B)),$$

where

$$W_B(A^*B) = \{\lambda = \lim_n \langle A^*Bx_n, x_n\rangle : (x_n) \subseteq H_1,\ \lim_n \|Bx_n\| = \|B\|\}.$$

THEOREM 1. *The set $W_K(A^*K)$ is compact.*

Proof. It suffices to show that $\bigsqcup_{B \in K} W_B(A^*B)$ is compact. For any $z \in \bigsqcup_{B \in K} W_B(A^*B)$, we have $|z| \leqslant \|A\|\,|K|$. It follows that $\bigsqcup_{B \in K} W_B(A^*B)$ is bounded. Now we show that it is closed. Let $(\lambda_m)_m$ be a sequence of elements of $\bigsqcup_{B \in K} W_B(A^*B)$ converging to λ. Then

$$\forall p \in \mathbb{N}^*,\ \exists m_p \in \mathbb{N},\ \forall m \geqslant m_p :\ |\lambda_m - \lambda| < \frac{1}{2p}. \tag{2.1}$$

Since $\lambda_{m_p} \in \bigsqcup_{B \in K} W_B(A^*B)$, then $\lambda_{m_p} \in W_{B_{m_p}}(A^*B_{m_p})$ for some $B_{m_p} \in K$, thus

$$\exists (x_{m_p}^{(n)})_n \subset H_1 : \lim_n \langle A^*B_{m_p}x_{m_p}^{(n)}, x_{m_p}^{(n)}\rangle = \lambda_{m_p} \text{ and } \lim_n \|Bx_{m_p}^{(n)}\| = \|B_{m_p}\|.$$

Then

$$\forall p \in \mathbb{N}^*,\ \exists n'_{m_p} \in \mathbb{N},\ \forall n \geqslant n'_{m_p} :\ |\langle A^*B_{m_p}x_{m_p}^{(n)}, x_{m_p}^{(n)}\rangle - \lambda_{m_p}| < \frac{1}{2p}, \tag{2.2}$$

$$\forall p \in \mathbb{N}^*,\ \exists n''_{m_p} \in \mathbb{N},\ \forall n \geqslant n''_{m_p}:\ |\|B_{m_p}x_{m_p}^{(n)}\| - \|B_{m_p}\|| < \frac{1}{p}. \tag{2.3}$$

For all $p \in \mathbb{N}^*$, put $n_p = max(n'_{m_p}, n''_{m_p})$, $x_p = x_{m_p}^{(n_p)}$ and $B_p = B_{m_p}$. We have

$$(x_p)_{p\geqslant 1} \subseteq H_1 \text{ and } (B_p)_{p\geqslant 1} \subseteq K.$$

By (2.1), (2.2) and using the triangular inequality, we have

$$\forall p \in \mathbb{N}^* : |\langle A^*B_p x_p, x_p\rangle - \lambda| < \frac{1}{p}. \tag{2.4}$$

By (2.3), we obtain

$$\forall p \in \mathbb{N}^* : |\| B_p x_p \| - \| B_p \|| < \frac{1}{p}. \tag{2.5}$$

The sequenses $(B_p)_{p\geqslant 1}$, $(\| B_p x_p \|)_{p\geqslant 1}$ and $(\| B_p \|)_{p\geqslant 1}$ are bounded. We can assume they converge. Let B be the limit of $(B_p)_{p\geqslant 1}$, then we have

$$\lim_p \|B_p\| = \|B\|. \tag{2.6}$$

Thus

$$|\langle A^*Bx_p, x_p\rangle - \langle A^*B_p x_p, x_p\rangle| \leqslant \|A\|\|B - B_p\| \xrightarrow[p\to+\infty]{} 0.$$

We can also assume that the sequense $(\langle A^*Bx_p, x_p\rangle)_{p\geqslant 1}$ converge (it is bounded).

By (2.4), $\lim_p \langle A^*B_p x_p, x_p\rangle = \lambda$. Then

$$\lim_p \langle A^*Bx_p, x_p\rangle = \lambda. \tag{2.7}$$

By (2.5) and (2.6), we have

$$\lim_p \|B_p x_p\| = \lim_p \|B_p\| = \|B\|.$$

Since $\|B_p x_p - Bx_p\| \leqslant \|B_p - B\| \xrightarrow[p\to+\infty]{} 0$, then the sequenses $(B_p x_p)_{p\geqslant 1}$ and $(Bx_p)_{p\geqslant 1}$ have the same limit. We obtain

$$\lim_p \|Bx_p\| = \|B\|. \tag{2.8}$$

From (2.7) and (2.8), it follows that $\lambda \in W_B(A^*B) \subseteq \bigsqcup_{B\in K} W_B(A^*B)$. □

REMARK 1. We find again the compactness of $W_B(A^*B)$ by taking $K = \{B\}$.

REMARK 2. If K is a non compact set, $W_K(A^*K)$ fail to be compact. For example, set $K = \{\alpha I : \alpha \in (-1,1)\}$. It is clear that K is a non compact subset of $B(H)$. Take $A = I$; $W_K(A^*K) = (-1,1)$ is non compact.

EXAMPLE 1. Let $\lambda_0 \in \mathbb{C}$. Define the compact S of $\mathbb{C}$ by $S = \{\lambda_0 + 1,\ \lambda_0 + i,\ \lambda_0 - 1,\ \lambda_0 - i\}$ and the compact K of $B(H)$ by $K = \{zA : z \in S\}$. Note that the smallest disc containing $S - \lambda_0$ is centered at 0. By Chraibi [3, Proposition 4],

$$|S - \lambda_0| \leqslant |S - \lambda_0 - \lambda|, \text{ for all } \lambda \in \mathbb{C}.$$

It follows, since $(K - \lambda_0 A)' = K - \lambda_0 A$, that

$$|K - \lambda_0 A| \leqslant |(K - \lambda_0 A)' - \lambda A|, \text{ for all } \lambda \in \mathbb{C}.$$

On the other hand, $W_{(K-\lambda_0 A)'}(A^*(K - \lambda_0 A)') = co(\{1,\ i,\ -1,\ -i\}W_A(A^*A))$; we see that $0 \in W_{(K-\lambda_0 A)'}(A^*(K - \lambda_0 A)')$.

The previous example suggests the following theorem.

THEOREM 2. *Let $K \subseteq B(H)$ a compact. The following are equivalent*

(1) $0 \in W_{K'}(A^*K')$,

(2) $|K| \leqslant |K' - \lambda A|$, *for all* $\lambda \in \mathbb{C}$,

(3) $|K|^2 + |\lambda|^2 m^2(A) \leqslant |K' - \lambda A|^2$, *for all* $\lambda \in \mathbb{C}$.

Proof. $(1) \Rightarrow (3)$. Assume that $0 \in W_{K'}(A^*K')$. By the Caratheodory theorem, 0 is a combinaison convexe of at most three elements of $\bigsqcup_{B \in K'} W_B(A^*B)$. Then there exists $\lambda_i \in W_{B_i}(A^*B_i),\ \alpha_i \in \mathbb{R}^+$ for $i \in \{1,2,3\}$ such that

$$B_i \in K', \ \sum_{i=1}^{i=3} \alpha_i = 1 \text{ and } 0 = \sum_{i=1}^{i=3} \alpha_i \lambda_i.$$

For $i \in \{1,2,3\}$, there exists a sequence $(x_{i,n})_n$ of elements of H_1 such that

$$\lambda_i = \lim_n \langle A^* B_i x_{i,n}, x_{i,n} \rangle \text{ and } \lim_n \|B_i x_{i,n}\| = \|B_i\|.$$

Let $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$ and $i \in \{1,2,3\}$

$$\begin{aligned} \|B_i + \lambda A\|^2 &\geqslant \|(B_i + \lambda A)x_{i,n}\|^2 \\ &= \|B_i x_{i,n}\|^2 + |\lambda|^2 \|A x_{i,n}\|^2 + 2Re(\overline{\lambda} \langle A^* B_i x_{i,n}, x_{i,n} \rangle) \\ &\geqslant \|B_i x_{i,n}\|^2 + |\lambda|^2 m^2(A) + 2Re(\overline{\lambda} \langle A^* B_i x_{i,n}, x_{i,n} \rangle). \end{aligned}$$

Since $|K' + \lambda A| \geqslant \|B_i + \lambda A\|$ for $i \in \{1,2,3\}$, then

$$|K' + \lambda A|^2 \geqslant |\lambda|^2 m^2(A) + \sum_{i=1}^{i=3} \alpha_i \|B_i x_{i,n}\|^2 + 2\sum_{i=1}^{i=3} \alpha_i Re(\overline{\lambda} \langle A^* B_i x_{i,n}, x_{i,n} \rangle). \tag{2.9}$$

But

$$\lim_n \sum_{i=1}^{i=3} \alpha_i Re(\overline{\lambda}\langle A^* B_i x_{i,n}, x_{i,n}\rangle) = \sum_{i=1}^{i=3} \alpha_i Re(\overline{\lambda}\lambda_i) = Re(\overline{\lambda}\sum_{i=1}^{i=3} \alpha_i \lambda_i) = 0,$$

by passing to the limit in (2.9), we obtain $|K|^2 + |\lambda|^2 m^2(A) \leqslant |K' - \lambda A|^2$.

$(3) \Rightarrow (2)$. is trivial.

$(2) \Rightarrow (1)$. Assume that $0 \notin W_{K'}(A^*K')$. By rotating, we can assume that $Re(W_{K'}(A^*K')) \geqslant t$ for some $t > 0$. Then $Re(W_B(A^*B)) \geqslant t$, for all $B \in K'$.

For $B \in K'$, consider the set $G_B = \{x \in H_1 : Re(\langle A^*Bx, x\rangle) \leqslant \frac{t}{2}\}$ and put

$$\eta = sup\{\|Bx\| : B \in K',\ x \in G_B\}.$$

The following of the proof is in two steps.

First step: we show that $\eta < |K'|$.

It is clair that $\eta \leqslant |K'|$. Assume that $\eta = |K'|$. then there exists a sequence $(B_n)_n$ of elements of K' and a sequence $(x_n)_n$ with $x_n \in G_{B_n}$ for all $n \in \mathbb{N}$ such that

$$\lim_n \|B_n x_n\| = |K'|.$$

Since K' is compact, there exists a subsequence $(B_{\varphi(n)})_n$ which converges to some $B \in K'$. In addition we have

$$x_{\varphi(n)} \in G_{B_{\varphi(n)}} \text{ for all } n \in \mathbb{N} \text{ and } \lim_n \|B_{\varphi(n)} x_{\varphi(n)}\| = |K'|.$$

But $|\|Bx_{\varphi(n)}\| - \|B_{\varphi(n)}x_{\varphi(n)}\|| \leqslant \|(B - B_{\varphi(n)})x_{\varphi(n)}\| \leqslant \|B - B_{\varphi(n)}\| \xrightarrow[n\to+\infty]{} 0$. We deduce that

$$\lim_n \|Bx_{\varphi(n)}\| = |K'|.$$

Since $x_{\varphi(n)} \in G_{B_{\varphi(n)}}$, then $Re(\langle A^*B_{\varphi(n)}x_{\varphi(n)}, x_{\varphi(n)}\rangle) \leqslant \frac{t}{2}$. The sequence $(\langle A^*Bx_{\varphi(n)}, x_{\varphi(n)}\rangle)_n$ is bounded, we can assume it converges. Let λ be its limit, by continuity of the map $z \in \mathbb{C} \longmapsto Re(z)$ we have $Re(\lambda) \leqslant \frac{t}{2}$. We obtain

$$\lim_n \langle A^*Bx_{\varphi(n)}, x_{\varphi(n)}\rangle = \lambda, \quad \lim_n \|Bx_{\varphi(n)}\| = |K'| \text{ and } Re(\lambda) \leqslant \frac{t}{2}.$$

We deduce that $\lambda \in W_B(A^*B)$ and $Re(\lambda) \leqslant \frac{t}{2}$. This contradicts the fact that $Re(W_B(A^*B)) \geqslant t$.

Second step: take $\mu = \inf\left(\frac{t}{\|A\|^2}, \frac{|K'| - \eta}{2\|A\|}\right)$, we show that $|K' - \mu A| < |K|$. This completes the proof.

By compactness of the set $K' - \mu A$, there exists $B_\mu \in K'$ such that $|K' - \mu A| = \|B_\mu - \mu A\|$. Let $x \in H_1$, we study two cases.

First case: $x \in G_{B_\mu}$.

$$\begin{aligned}\|(B_\mu - \mu A)x\| &\leqslant \|B_\mu x\| + \mu\|Ax\| \leqslant \eta + \mu\|A\| \\ &\leqslant \eta + \frac{|K'| - \eta}{2} = \frac{|K'| + \eta}{2} \\ &< |K'|.\end{aligned}$$

Second case: $x \notin G_{B_\mu}$. We have $Re(\langle A^* B_\mu x, x\rangle) > \frac{t}{2}$. Put

$$A^* B_\mu x = (a + ib)x + y, \text{ with } a,\ b \in \mathbb{R} \text{ and } \langle x, y\rangle = 0.$$

Then we obtain $Re(\langle A^* B_\mu x, x\rangle) = a > \frac{t}{2}$.

$$\begin{aligned}\|(B_\mu - \mu A)x\|^2 &= \|B_\mu x\|^2 + \mu^2\|Ax\|^2 - 2\mu Re(\langle A^* B_\mu x, x\rangle) \\ &\leqslant \|B_\mu\|^2 + \mu(\mu\|A\|^2 - 2a) \\ &< \|B_\mu\|^2 + \mu(\mu\|A\|^2 - 2t) \\ &< |K'|^2,\end{aligned}$$

since $\mu\|A\|^2 - 2t < 0$. In total, $|K' - \mu A| < |K'|$. $\square$

PROPOSITION 1. *Let K be a compact of $B(H)$ and z_0 be a scalar. The following are equivalent:*

(1) $0 \in W_{(K - z_0 A)'}(A^*(K - z_0 A)')$,

(2) $|K - z_0 A| \leqslant |(K - z_0 A)' - \lambda A|$, *for all* $\lambda \in \mathbb{C}$,

(3) $|K - z_0 A|^2 + |\lambda|^2 m^2(A) \leqslant |(K - z_0 A)' - \lambda A|^2$, *for all* $\lambda \in \mathbb{C}$.

Proof. It suffices to replace K by $K - z_0 A$ in Theorem 2. $\square$

DEFINITION 2. Let $A \in B(H)$ and $m(A) > 0$. Let $\mathscr{K}_A$ be the class of compacts K of $B(H)$ such that

$$|K - \lambda_K A| \leqslant |(K - \lambda_K A)' - \lambda A|, \text{ for all } \lambda \in \mathbb{C}$$

holds for some scalar λ_K.

This class is not empty since it contains the compacts $K = \{B\}$ with $B \in B(H)$ ($\lambda_K = c_B(A^* B)$, center of mass of $A^* B$ relatively to B). We give other examples later.

PROPOSITION 2. *Assume that $m(A) > 0$. Let $K \in \mathscr{K}_A$, then there exists a unique scalar c_K such that*

$$|K - c_K A| \leqslant |(K - c_K A)' - \lambda A|, \text{ for all } \lambda \in \mathbb{C}. \tag{2.10}$$

Proof. By definition of $\mathscr{K}_A$, the scalar c_K exists. Assume that c'_K is an other scalar checking the inequality (2.10). Let λ be a scalar, from Proposition 1, we have ,

$$\begin{aligned}|K - c'_K|^2 + |\lambda|^2 m^2(A) &\leqslant |(K - c'_K A)' - \lambda A|^2,\\ &\leqslant |K - c'_K A - \lambda A|^2.\end{aligned}$$

Take $\lambda = c_K - c'_K$, we have

$$\begin{aligned}|K - c'_K A|^2 + |c_K - c'_K|^2 m^2(A) &\leqslant |K - c_K A|^2\\ &\leqslant |(K - c_K A)' - \alpha A|^2, \text{ for all } \alpha \in \mathbb{C}\\ &\leqslant |K - c_K A - \alpha A|^2, \text{ for all } \alpha \in \mathbb{C}.\end{aligned}$$

Take $\alpha = c_K + c'_K$, we obtain

$$|K - c'_K A|^2 + |c_K - c'_K|^2 m^2(A) \leqslant |K - c'_K A|^2.$$

We deduce that $|c_K - c'_K|^2 m^2(A) = 0$. Since $m(A) \neq 0$, then $c'_K = c_K$. □

DEFINITION 3. Let K, A and c_K be as in Proposition 2. The scalar c_K is called center of A^*K.

EXAMPLE 2. In Example 1, $K \in \mathscr{K}_A$ and λ_0 is the center of A^*K.

The following proposition and example give a sufficient condition not necessary for a compact K of $B(H)$ to be in $\mathscr{K}_A$.

PROPOSITION 3. *Let K be a compact of $B(H)$. If there exists $B \in K$ and a scalar λ such that*

$$|K - \lambda A| = \|B - c_B(A^*B)A\|,$$

then $K \in \mathscr{K}_A$ and $c_K = c_B(A^*B)$.

Proof. By hypothesis

$$\begin{aligned}\|B - c_B(A^*B)A\| &= |K - \lambda A|\\ &\geqslant \|B - \lambda A\|.\end{aligned}$$

From the definition of center of A^*B, we deduce that $\lambda = c_B(A^*B)$ and $B - c_B(A^*B)A \in (K - c_B(A^*B)A)'$. Therefore

$$\begin{aligned}|K - c_B(A^*B)A| &= \|B - c_B(A^*B)A\|\\ &\leqslant \|B - c_B(A^*B)A - \lambda A\|, \text{ for all } \lambda \in \mathbb{C}\\ &\leqslant |(K - c_B(A^*B)A)' - \lambda A|, \text{ for all } \lambda \in \mathbb{C},\end{aligned}$$

which implies that $K \in \mathscr{K}_A$ and that $c_B(A^*B)$ is the center of A^*K. □

EXAMPLE 3. Consider the compact $K = \{e^{i\theta}A \ : \ \theta \in [0,2\pi]\}$ of $B(H)$. By Chraibi [3, proposition 4],

$$1 \leqslant \sup_{\theta\in[0,2\pi]} |e^{i\theta} - \lambda|, \text{ for all } \lambda \in \mathbb{C}.$$

We deduce

$$\|A\| \leqslant \sup_{\theta\in[0,2\pi]} \|e^{i\theta}A - \lambda A\|, \text{ for all } \lambda \in \mathbb{C}.$$

Since $K' = K$ and $|K| = \|A\|$, then

$$|K| \leqslant |K' - \lambda A|, \text{ for all } \lambda \in \mathbb{C}.$$

This implies that $K \in \mathscr{K}_A$ and $c_K = 0$. For all $B \in K$, $\|B - c_B(A^*B)A\| = 0$ and $|K - c_K| = \|A\| \neq 0$.

DEFINITION 4. Given two compacts K and S of $B(H)$. The Hausdorff distance is defined by

$$h(K,S) = max(e(K,S), e(S,K))$$

where

$$e(K,S) = \sup_{B\in K}\inf_{C\in S} \|B - C\|.$$

REMARK 3. If $T \in B(H)$, then

$$e(K,\{T\}) = \sup_{B\in K} \|B - T\| = |K - T| \text{ and } e(\{T\},K) = \inf_{B\in K} \|B - T\|.$$

Therefore, we have $h(K,\{T\}) = |K - T|$. Note that $h(K,\{T\}) \leqslant |K| + \|T\|$.

PROPOSITION 4. *let K, S be two elements of $\mathscr{K}_A$, and c_K, c_S be the centers of A^*K and A^*S respectively. Then*

$$|c_K - c_S| \leqslant \frac{1}{2m^2(A)}\left(h(K,S)\|A\| + \sqrt{h^2(K,S)\|A\|^2 + 8h(K,S)h(S,\{c_SA\})m^2(A)}\,\right).$$

Consequently, the map $K \mapsto c_K$, $K \in \mathscr{K}_A$ is continuous in the sens of Hausdorff.

Proof. Assume that $c_S = 0$. Since c_K is the center of A^*K, then

$$|K - c_KA| \leqslant |(K - c_KA)' - \lambda A|, \text{ for all } \lambda \in \mathbb{C}.$$

By Proposition 1, we have

$$|K - c_KA|^2 + |\lambda|^2 m^2(A) \leqslant |(K - c_KA)' - \lambda A|^2, \text{ for all } \lambda \in \mathbb{C}.$$

But $(K-c_KA)' \subseteq K-c_KA$, then $|K-c_KA-\lambda A| \geqslant |(K-c_KA)'-\lambda A|$ for all $\lambda \in \mathbb{C}$. Take $\lambda = -c_K$, we have

$$\begin{aligned}
|K|^2 &\geqslant |(K-c_KA)'-c_KA|^2 \\
&\geqslant m^2(A)|c_K|^2 + |K-c_KA|^2 \\
&= m^2(A)|c_K|^2 + h^2(K,\{c_KA\}) \\
&\geqslant m^2(A)|c_K|^2 + \Big(h(K,S)-h(S,\{c_KA\})\Big)^2 \\
&= m^2(A)|c_K|^2 + h^2(S,\{c_KA\}) - 2h(K,S)h(S,\{c_KA\}) + h^2(K,S).
\end{aligned}$$

Since $c_S = 0$, by Proposition 1 $|S-c_KA|^2 \geqslant |S'-c_KA|^2 \geqslant m^2(A)|c_K|^2 + |S|^2$. Thus

$$\begin{aligned}
|K|^2 &\geqslant 2m^2(A)|c_K|^2 + |S|^2 + h^2(K,S) - 2h(K,S)h(S,\{c_KA\}) \\
&= 2m^2(A)|c_K|^2 + \Big(|S|+h(K,S)\Big)^2 - 2h(K,S)\Big(|S|+h(S,\{c_KA\})\Big) \\
&\geqslant 2m^2(A)|c_K|^2 + |K|^2 - 2h(K,S)\Big(2|S|+\|A\||c_K|\Big),
\end{aligned}$$

whence

$$m^2(A)|c_K|^2 - h(K,S)\|A\||c_K| - 2h(K,S)|S| \leqslant 0.$$

Consequently

$$|c_K| \leqslant \frac{1}{2m^2(A)}\Big(h(K,S)\|A\| + \sqrt{h^2(K,S)\|A\|^2 + 8h(K,S)|S|m^2(A)}\Big).$$

For the case $c_S \neq 0$, we replace K and S by $K-c_SA$ and $S-c_SA$ respectively. We easily check by the assertion (2) of Proposition 1 that the center of $A^*(K-c_SA)$ is $c_K - c_S$ and the center of $A^*(S-c_SA)$ is 0. We obtain the desired result. $\square$

3. Applications

Let A and B two elements of $B(H)$. The numerical radius of B is defined by

$$w(B) = \sup\{|z| : z \in W(B)\}.$$

Put

$$w_0'(B) = \inf\{|z| : z \in W_0(B)\} \text{ and } w_B'(A^*B) = \inf\{|z| : z \in W_B(A^*B)\}.$$

Since $W_0(B) \subseteq \overline{W(B)}$ and $W_B(A^*B) \subseteq \overline{W(A^*B)}$, we have

$$w_0'(B) \leqslant w(B) \text{ and } w_B'(A^*B) \leqslant w(A^*B).$$

These inequalities can be strict. Indeed, let $B \in B(H)$ such that $B \neq \lambda I$ for all $\lambda \in \mathbb{C}$. Let z_0 be the center of mass of B and put $C = B - z_0I$. Then the center of mass of C is 0. We deduce that $0 \in W_0(C)$, thus $w_0'(C) = 0$. Since $C \neq 0$, then $w(C) > 0$. Let A

be a selfadjoint operator with numerical range the segment $[1,2]$ and take $B = I$. We have $w'_B(A^*B) = 1$ and $w(A^*B) = 2$.

In the following we assume that $m(A) > 0$. We give a generalization and improvement of each of the following five theorems established by Dragomir [4, 5].

THEOREM 3. [4] *If* $0 \neq \lambda \in \mathbb{C}$ *and* $r > 0$ *are such that* $\|B - \lambda I\| \leqslant r$, *then*

$$\|B\| \leqslant w(B) + \frac{1}{2}\frac{r^2}{|\lambda|^2}.$$

THEOREM 4. [4] *If* $0 \neq \lambda \in \mathbb{C}$ *and* $r > 0$ *are such that* $\|B - \lambda I\| \leqslant r$ *and* $|\lambda| \geqslant r$, *then*

$$\sqrt{1 - \frac{r^2}{|\lambda|^2}} \leqslant \frac{w(B)}{\|B\|}.$$

THEOREM 5. [5] *If* A *is invertible and* $r > 0$ *are such that* $\|B - A| \leqslant r$, *then*

$$\|B\| \leqslant \|A^{-1}\|\left(w(A^*B) + \frac{1}{2}r^2\right).$$

THEOREM 6. [5] *If* A *is invertible and* $r > 0$ *are such that*

$$\|B - A| \leqslant r \ \ and \ \ \frac{1}{\sqrt{r^2+1}} \leqslant \|A^{-1}\| \leqslant \frac{1}{r},$$

then

$$\|B\|^2 \leqslant w^2(A^*B) + 2w(A^*B)\frac{\|A^{-1}\| - \sqrt{1 - r^2\|A^{-1}\|^2}}{\|A^{-1}\|}.$$

THEOREM 7. [5] *If* A *is invertible and* $r > 0$ *are such that*

$$\|B - A| \leqslant r \ \ and \ \ \|A^{-1}\| \leqslant \frac{1}{r},$$

then

$$\|A\|^2\|B\|^2 - w^2(A^*B) \leqslant 2w(A^*B)\frac{\|A\|}{\|A^{-1}\|}\left(\|A\|\|A^{-1}\| - \sqrt{1 - r^2\|A^{-1}\|^2}\right).$$

Let K be a compact of $B(H)$. Put

$$w'_{K'}(A^*K') = \inf_{B \in K'}\{|z| : \ z \in W_B(A^*B)\}.$$

THEOREM 8. *If* $0 \neq \lambda \in \mathbb{C}$ *and* $r > 0$ *are such that*

$$|K - \lambda A| \leqslant r,$$

then

$$|K| \leqslant \frac{1}{m(A)}\left(w'_{K'}(A^*K') + \frac{1}{2}\frac{r^2}{|\lambda|}\right).$$

Proof. By Theorem 1, $W_{K'}(A^*K')$ is compact. Let $\alpha \in W_{K'}(A^*K')$ such that $|\alpha| = w'_{K'}(A^*K')$, then $\alpha \in W_B(A^*B)$ for some $B \in K'$. Thus we can find a sequence $(x_n)_n \subseteq H_1$ such that

$$\alpha = \lim_n \langle A^*Bx_n, x_n\rangle \text{ and } \lim_n \|Bx_n\| = \|B\|.$$

Since $|K - \lambda A| \leqslant r$, then $\|(B-\lambda A)x_n\|^2 \leqslant r^2$, for all $n \in \mathbb{N}$. Then

$$\|Bx_n\|^2 + |\lambda|^2\|Ax_n\|^2 \leqslant 2Re(\overline{\lambda}\langle A^*Bx_n, x_n\rangle) + r^2,$$

and we have

$$\|Bx_n\|^2 + |\lambda|^2 m^2(A) \leqslant 2|\lambda||\langle A^*Bx_n, x_n\rangle)| + r^2.$$

By passing to the limit, we obtain

$$|K|^2 + |\lambda|^2 m^2(A) \leqslant 2|\lambda| w'_{K'}(A^*K') + r^2. \tag{3.1}$$

Thus

$$|K| \leqslant \frac{1}{m(A)}\left(w'_{K'}(A^*K') + \frac{1}{2}\frac{r^2}{|\lambda|}\right). \quad \square$$

REMARK 4. (i) Theorem 8 generalizes and improves Theorem 3. Indeed, if $0 \neq \lambda \in \mathbb{C}$ and $r > 0$ are such that

$$\|B - \lambda A\| \leqslant r.$$

Take $K = \{B\}$,

$$\|B\| \leqslant \frac{1}{m(A)}\left(w'_B(A^*B) + \frac{1}{2}\frac{r^2}{|\lambda|}\right).$$

If in addition $A = I$, then $m(A) = 1$ and

$$\|B\| \leqslant w'_0(B) + \frac{1}{2}\frac{r^2}{|\lambda|} \leqslant w(B) + \frac{1}{2}\frac{r^2}{|\lambda|}.$$

(ii) If A is invertible and $x \in H_1$, then

$$1 = \|x\| = \|A^{-1}Ax\| \leqslant \|A^{-1}\|\|Ax\|.$$

We have

$$\frac{1}{\|Ax\|} \leqslant \|A^{-1}\|, \text{ for all } x \in H_1.$$

We deduce

$$\frac{1}{m(A)} \leqslant \|A^{-1}\|. \tag{3.2}$$

If $r > 0$ and $\|B - A\| \leqslant r$, then by Theorem 8,

$$\|B\| \leqslant \frac{1}{m(A)}\left(w'_B(A^*B) + \frac{1}{2}r^2\right) \leqslant \|A^{-1}\|\left(w(A^*B) + \frac{1}{2}r^2\right).$$

This improves Theorem 5.

THEOREM 9. *If* $0 \neq \lambda \in \mathbb{C}$ *et* $r > 0$ *are such that*

$$|K - \lambda A| \leqslant r \ \text{ and } \ |\lambda| m(A) > r,$$

then

$$m(A)\sqrt{1 - \frac{r^2}{|\lambda|^2 m^2(A)}} \leqslant \frac{w'_{K'}(A^*K')}{|K|}.$$

Proof. Put $\omega = \sqrt{|\lambda|^2 m^2(A) - r^2}$ and $\delta = |\lambda| w'_{K'}(A^*K')$. By the inequality (3.1), we have

$$|K|^2 + \omega^2 \leqslant 2\delta. \tag{3.3}$$

Divide both sides by the positive real ω, we obtain

$$\frac{|K|^2}{\omega} + \omega \leqslant \frac{2\delta}{\omega}.$$

Thus

$$|K| \leqslant \frac{\delta}{\omega}.$$

Whence

$$m(A)\sqrt{1 - \frac{r^2}{|\lambda|^2 m^2(A)}} \leqslant \frac{w'_{K'}(A^*K')}{|K|}. \qquad \square$$

REMARK 5. If $\|B - \lambda A\| \leqslant r$ and $m(A)|\lambda| > r > 0$. Take $K = \{B\}$, then

$$m(A)\sqrt{1 - \frac{r^2}{|\lambda|^2 m^2(A)}} \leqslant \frac{w'_B(A^*B)}{\|B\|}.$$

If in addition $A = I$, then $m(A) = 1$ and

$$\sqrt{1 - \frac{r^2}{|\lambda|^2}} \leqslant \frac{w'_0(B)}{\|B\|} \leqslant \frac{w(B)}{\|B\|}.$$

This generalizes and improves Theorem 4.

THEOREM 10. *If* $0 \neq \lambda \in \mathbb{C}$ *et* $r > 0$ *are such that*

$$|K - \lambda A| \leqslant r \ \text{ and } \ |\lambda| m(A) \geqslant r,$$

then

$$|K|^2 \leqslant |\lambda|^2 (w'_{K'}(A^*K'))^2 + 2|\lambda| w'_{K'}(A^*K')\left(1 - \sqrt{|\lambda|^2 m^2(A) - r^2}\right)$$

Proof. Using the same notations in the proof of Theorem 9 and the inequality (3.3), we have

$$|K|^2 \leqslant 2\delta - \omega^2.$$

It follows that

$$\begin{aligned} |K|^2 - \delta^2 &\leqslant 2\delta - (\delta^2 + \omega^2) \\ &= 2\delta - 2\delta\omega - (\delta - \omega)^2 \\ &\leqslant 2\delta(1-\omega). \end{aligned}$$

Whence

$$|K|^2 \leqslant |\lambda|^2 (w'_{K'}(A^*K'))^2 + 2|\lambda| w'_{K'}(A^*K')\Big(1 - \sqrt{|\lambda|^2 m^2(A) - r^2}\Big). \qquad \square$$

REMARK 6. Theorem 10 generalizes and improves Theorem 6 with fewer conditions. Indeed, if $\|B - A\| \leqslant r$, $r > 0$ and $\|A^{-1}\| \leqslant \dfrac{1}{r}$, then $m(A) \geqslant r$. Apply Theorem 10 for $K = \{B\}$ and $\lambda = 1$, we obtain

$$\|B\|^2 \leqslant (w'_B(A^*B))^2 + 2w'_B(A^*B)\Big(1 - \sqrt{m^2(A) - r^2}\Big).$$

A simple calculation gives

$$1 - \sqrt{m^2(A) - r^2} \leqslant \frac{\|A^{-1}\| - \sqrt{1 - r^2\|A^{-1}\|^2}}{\|A^{-1}\|}.$$

THEOREM 11. *If* $0 \neq \lambda \in \mathbb{C}$ *et* $r > 0$ *are such that*

$$|K - \lambda A| \leqslant r \ \text{ and } \ m(A)|\lambda| \geqslant r,$$

then

$$\|A\|^2 |K|^2 \leqslant |\lambda|^2 (w'_{K'}(A^*K'))^2 + 2|\lambda| w'_{K'}(A^*K')\|A\|\Big(\|A\| - \sqrt{|\lambda|^2 m^2(A) - r^2}\Big).$$

Proof. Using the same notations in the proof of Theorem 9 and the inequality (3.3), we have

$$|K|^2 \leqslant 2\delta - \omega^2.$$

It follows that

$$\begin{aligned} |K|^2 - \frac{\delta^2}{\|A\|^2} &\leqslant 2\delta - \Big(\frac{\delta^2}{\|A\|^2} + \omega^2\Big) \\ &= 2\delta - 2\frac{\delta\omega}{\|A\|} - \Big(\frac{\delta}{\|A\|} - \omega\Big)^2 \\ &\leqslant 2\delta\Big(1 - \frac{\omega}{\|A\|}\Big). \end{aligned}$$

Thus

$$\|A\|^2|K|^2 \leqslant \delta^2 + 2\delta\|A\|(\|A\| - \omega).$$

Whence

$$\|A\|^2|K|^2 \leqslant |\lambda|^2(w'_{K'}(A^*K'))^2 + 2|\lambda|w'_{K'}(A^*K')\|A\|\left(\|A\| - \sqrt{|\lambda|^2m^2(A) - r^2}\right). \quad \square$$

10 generalizes and improves Theorem 6

REMARK 7. Theorem 11 generalizes and improves Theorem 7. Indeedt, if $\|B - A\| \leqslant r$, $r > 0$, A is invertible and $\|A^{-1}\| \leqslant \frac{1}{r}$, take $K = \{B\}$ and $\lambda = 1$, then $m(A) \geqslant r$. Apply Theorem 11, we obtain

$$\|A\|^2\|B\|^2 \leqslant (w'_B(A^*B))^2 + 2w'_B(A^*B)\|A\|\left(\|A\| - \sqrt{m^2(A) - r^2}\right).$$

Using the inequality (3.2), it is easily to check sthat

$$\|A\| - \sqrt{m^2(A) - r^2} \leqslant \|A\| - \frac{1}{\|A^{-1}\|}\sqrt{1 - r^2\|A^{-1}\|^2}.$$

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