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REFINEMENTS AND GENERALIZATIONS OF MAJORIZATION, FAVARD AND BERWALD–TYPE INEQUALITIES VIA FINK IDENTITY

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Abstract. In this paper we present refinements of the majorization-type inequalities via an inequality obtained from a Fink's identity as well as the refinements of the Favard-Berwald type inequalities by using monotonic sequence and positive weights.

1. Introduction and preliminaries

In [1], Berwald proved the following important generalization of Favard's inequality (see also [4, pp. 413–414]):

THEOREM 1. *Let ϕ be a non-negative, continuous concave function, not identically zero on $[a,b]$, and ζ be continuous and strictly monotonic function defined on $[0,x_0]$, where x_0 is sufficiently large. If $\bar{z}$ is the unique positive root of the equation*

$$\frac{1}{\bar{z}}\int_0^{\bar{z}} \zeta(x)\,dx = \frac{1}{b-a}\int_a^b \zeta(\phi(z))\,dz,$$

then for every function $f:[0,x_0]\to\mathbb{R}$ which is convex with respect to ζ i.e., $f\circ\zeta^{-1}$ is convex, we have

$$\int_0^1 f(s\,\bar{z})\,ds = \frac{1}{\bar{z}}\int_0^{\bar{z}} f(x)\,dx \geqslant \frac{1}{b-a}\int_a^b f(\phi(z))\,dz.$$

Favard's inequality (see [2]) follows from Theorem 1 using $\zeta = id$. The most important consequence (of the Berwald's inequality) is the following corollary which is a reversed Hölder-type inequality.

COROLLARY 1. *Let ϕ be a non-negative concave function defined on $[a,b]\subset\mathbb{R}$. If $s>q>0$, then*

$$\left(\frac{q+1}{b-a}\int_a^b \phi^q(x)dx\right)^{\frac{1}{q}} \geqslant \left(\frac{s+1}{b-a}\int_a^b \phi^s(x)dx\right)^{\frac{1}{s}}. \tag{1}$$

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Notice that (1) is a sharp inequality. Equality is obtained for $\phi(x) = x - a$.

In [9], Maligranda, Pečarić and Persson presented the weighted Favard's and Berwald's inequality, of the form given in Theorem 1.

The following theorem, proven in [6], is a discrete weighted version of the integral Berwald's inequality given in Theorem 1.

THEOREM 2. *Let $\mathbf{p}$, $\mathbf{x}$ and $\mathbf{y}$ be positive m-tuples. Suppose $\psi, \varphi : [0,\infty) \to \mathbb{R}$ are such that ψ is a continuous and strictly increasing function and φ is a convex function with respect to ψ. Let z_1 be such that*

$$\sum_{i=1}^{m} p_i \psi(x_i) = \sum_{i=1}^{m} p_i \psi(z_1 y_i).$$

Let $\mathbf{x}/\mathbf{y}$ be a decreasing m-tuple. If $\mathbf{x}$ is an increasing m-tuple, then

$$\sum_{i=1}^{m} p_i \varphi(x_i) \leqslant \sum_{i=1}^{m} p_i \varphi(z_1 y_i). \tag{2}$$

If $\mathbf{y}$ is a decreasing m-tuple, then the reversed inequality holds in (2).

Let $\mathbf{x}/\mathbf{y}$ be an increasing m-tuple. If $\mathbf{y}$ is an increasing m-tuple, then the reversed inequality holds in (2). If $\mathbf{x}$ is a decreasing m-tuple, then inequality (2) holds.

For a non-negative m-tuple $\mathbf{p}$ and a non-negative, increasing, concave m-tuple $\mathbf{x}$ it was proven in [10] (using different method) that if $0 < s \leqslant r$, then

$$M_m^{[r]}(\mathbf{x};\mathbf{p}) \leqslant \frac{M_m^{[r]}(\mathbf{I}_m;\mathbf{p})}{M_m^{[s]}(\mathbf{I}_m;\mathbf{p})} M_m^{[s]}(\mathbf{x};\mathbf{p}), \tag{3}$$

where $M_m^{[r]}(\mathbf{x};\mathbf{p}) = \left(\frac{1}{\sum_{i=1}^m p_i}\sum_{i=1}^m p_i x_i^r\right)^{1/r}$, $\mathbf{I}_m = (0,1,\dots,m-1)$. Equality in (3) is obtained for $\mathbf{x} = \mathbf{I}_m$. Analogous inequality holds for a decreasing, non-negative, concave $\mathbf{x}$. Inequality (3) is a discrete analogue of inequality (1) and it can be obtained from Theorem 2 in the same manner as Corollary 1 follows from Theorem 1 (see Lemma 2).

For an increasing concave $(m+1)$-tuple $(0,\mathbf{x}) = (0,x_1,\dots,x_m)$, the following inequality

$$M_m^{[1]}(\mathbf{x},\mathbf{1}) \leqslant \frac{m+1}{2(m!)^{1/m}} M_m^{[0]}(\mathbf{x},\mathbf{1}) \tag{4}$$

was proven in [8], where $\mathbf{1} = (1,\dots,1)$. This is a non-weighted version of (3), but for a different class of concave m-tuples.

The following lemma is given in [6]. It is an important technical tool used while dealing with the monotonicity in mean of sequences.

LEMMA 1. *Let $\mathbf{u} = (u_1,\dots,u_m)$ be a positive m-tuple.*

(i) If $\mathbf{v} = (v_1,\dots,v_m)$ is a decreasing real m-tuple, then

$$\sum_{i=1}^{k} v_i u_i \sum_{i=1}^{m} u_i \geqslant \sum_{i=1}^{m} v_i u_i \sum_{i=1}^{k} u_i, \qquad k = 1,\dots,m.$$

(ii) If $\mathbf{v}$ *is an increasing real* m*-tuple, then the reversed inequality holds in* (i).

The weighted majorization will be frequently used in the text. For a given non-negative m-tuple $\mathbf{p}$, an m-tuple $\mathbf{x}$ p-majorize an m-tuple $\mathbf{y}$ if

$$\sum_{i=1}^{k} p_i x_i \geqslant \sum_{i=1}^{k} p_i y_i, \quad k = 1, \ldots, m-1 \quad \text{and} \quad \sum_{i=1}^{m} p_i x_i = \sum_{i=1}^{m} p_i y_i. \tag{5}$$

In this case we write $\mathbf{x} \succ_p \mathbf{y}$.

The following theorem is given in [6] (see also [11]).

THEOREM 3. *Let* $f : I \to \mathbb{R}$ *be a convex function, where* $I \subseteq \mathbb{R}$ *is an interval. Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *be a positive* m*-tuple and let* $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m) \in I^m$ *such that* $\mathbf{x} \succ_p \mathbf{y}$.

(i) If $\mathbf{y}$ *is a decreasing* m*-tuple, then*

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant \sum_{i=1}^{m} p_i f(y_i). \tag{6}$$

(ii) If $\mathbf{x}$ *is an increasing* m*-tuple, then the reversed inequality holds in* (6).

If f *is concave, then the reversed inequalities hold in* (i) *and* (ii).

Both the claims (i) and (ii) in Theorem 3 also hold for f which is convex with respect to a strictly increasing function $\zeta : I \to \mathbb{R}$ (that is $f \circ \zeta^{-1}$ is convex) under the assumption $\zeta(\mathbf{x}) \succ_p \zeta(\mathbf{y})$, where $\zeta(\mathbf{x}) = (\zeta(x_1), \ldots, \zeta(x_m))$.

The following theorem is proved by S. Khalid, J. Pečarić and A. Vukelić in [5], which is a consequence of a Fink's identity given in [3].

THEOREM 4. *Let* $f : [a,b] \to \mathbb{R}$, $n \geqslant 1$, *be such that* $f^{(n-1)}$ *is absolutely continuous. Let* $x_i, y_i \in [a,b]$, $p_i \in \mathbb{R}$, $i = 1, \ldots, m$ *and let* $k(t,x)$ *be defined as*

$$k(t,x) = \begin{cases} t-a, & a \leqslant t \leqslant x \leqslant b, \\ t-b, & a \leqslant x < t \leqslant b. \end{cases} \tag{7}$$

If

$$\sum_{i=1}^{m} p_i (x_i - t)^{n-1} k(t, x_i) \geqslant \sum_{i=1}^{m} p_i (y_i - t)^{n-1} k(t, y_i) \tag{8}$$

holds and if f *is* n*-convex, then*

$$\begin{aligned} &\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \\ &\geqslant \frac{1}{b-a} \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(f^{(k-1)}(a) \left(\sum_{i=1}^{m} p_i (y_i - a)^k - \sum_{i=1}^{m} p_i (x_i - a)^k \right) \right. \\ &\left. - f^{(k-1)}(b) \left(\sum_{i=1}^{m} p_i (y_i - b)^k - \sum_{i=1}^{m} p_i (x_i - b)^k \right) \right). \end{aligned} \tag{9}$$

If the reversed inequality holds in (8), *then the reversed inequality holds in* (9).

n-Convexity of a function is defined in the usual way by using the non-negativity of its divided differences of order n.

The organization of the paper is the following: in Section 2, we present the main results of this paper which consist of the refinements and generalizations of the weighted majorization-type inequalities for the two real m-tuples $\mathbf{x}$ and $\mathbf{y}$. In Sections 3 and 4, refinements and generalizations of the discrete Favard and Berwald-type inequalities are given respectively. An outline of the analogous results in the continuous case is also presented in Section 5.

2. Refinements of the majorization-type inequalities

The following theorem presents a refinement of the weighted majorization-type inequality as well as a refinement of the reversed weighted majorization-type inequality for monotonous m-tuples.

THEOREM 5. *Let $f:[a,b]\to\mathbb{R}$, $n\geqslant 1$, be such that $f^{(n-1)}$ is absolutely continuous. Suppose that $\mathbf{p}$ is a positive m-tuple, $\mathbf{x}$, $\mathbf{y}\in[a,b]^m$ such that $\mathbf{x}\succ_p\mathbf{y}$.*

- (i) *If $\mathbf{y}$ is a decreasing m-tuple, $n\geqslant 2$ is even and f is n-convex, then the inequality (9) holds.*
- (ii) *Let $\mathbf{y}$ be a decreasing m-tuple and let $F:[a,b]\to\mathbb{R}$ be defined by*

$$F(x)=\frac{1}{b-a}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left((x-b)^k f^{(k-1)}(b)-(x-a)^k f^{(k-1)}(a)\right). \tag{10}$$

  If F is a convex function and the inequality (9) holds, then the inequality (6) holds.
- (iii) *If $\mathbf{x}$ is an increasing m-tuple, $n\geqslant 2$ is even and f is n-convex, then the reversed inequality holds in (9).*
- (iv) *Let $\mathbf{x}$ be an increasing m-tuple. If F given by (10) is a convex function and the reversed inequality holds in (9), then the reversed inequality holds in (6).*

Proof. Set $\eta(x):=(x-t)^{n-1}k(t,x)$. For $n=2$, this is obviously a convex function. For $n\geqslant 3$, η is differentiable with $\eta'(x)=(n-1)(x-t)^{n-2}k(x,t)$ which is an increasing function for even n. This shows that η is a convex function for even $n\geqslant 2$.

- (i) Apply Theorem 3(i) for the convex function η, it follows (8) for even n, $n\geqslant 2$. As by assumption f is n-convex, apply Theorem 4, the inequality (9) is immediate.
- (ii) It is easy to see that the inequality (9) is equivalent to

$$\sum_{i=1}^{m}p_if(x_i)-\sum_{i=1}^{m}p_if(y_i)\geqslant\sum_{i=1}^{m}p_iF(x_i)-\sum_{i=1}^{m}p_iF(y_i). \tag{11}$$

  The claim follows by using Theorem 3(i) for the function F.

(iii) Apply Theorem 3(ii) for the convex function η. The reversed inequality in (8) follows for even n, $n \geqslant 2$, and as f is n-convex by assumption, apply Theorem 4, the reversed inequality in (9) is immediate.

(iv) As the reverse of the inequality (9) is equivalent to the reverse of the inequality (11), by using F in Theorem 3(ii), the claim follows. □

REMARK 1. Theorem 5 is a generalization and a refinement of the Theorem 3. In the convex case (that is for $n = 2$), the function F defined in (10) is a linear function, which clearly implies (under the assumption $\mathbf{x} \succ_p \mathbf{y}$) that the right-hand side of the inequality (9) vanishes, which gives inequality (6). For functions f which, besides being convex, poses some higher convexity of even order, Theorem 5 gives refinements of inequality (6).

REMARK 2. It is obvious that the function $f(x) = x^2$ is n-convex for any $n \geqslant 2$. The function F defined in (10) in this case is given by:

$$F(x) = \frac{1}{b-a}\Bigg[(n-1)\left(b^2(x-b) - a^2(x-a)\right) + \frac{n-2}{2}\left(2b(x-b)^2 - 2a(x-a)^2\right) + \frac{n-3}{3!}\left(2(x-b)^3 - 2(x-a)^3\right)\Bigg]$$

and obviously $F''(x) = 2$. This shows that Theorem 5 generates genuine refinements of the weighted majorization inequality (6).

3. Refinements and generalizations of the Favard-type inequalities

The aim of this section is to present some refinements of the well known results including generalized Favard's inequality.

If $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ are two m-tuples, then we define $\mathbf{xy} = (x_1y_1, \ldots, x_my_m)$ and $\frac{\mathbf{x}}{\mathbf{y}} = \left(\frac{x_1}{y_1}, \ldots, \frac{x_m}{y_m}\right)$ with each $y_i \neq 0$ for $i = 1, \ldots, m$.

The following theorem presents refinement of the generalized discrete weighted Favard's inequality.

THEOREM 6. *Let* $f : [a,b] \to \mathbb{R}$, $[a,b] \subset (0,\infty)$, $n \geqslant 1$, *be such that* $f^{(n-1)}$ *is absolutely continuous. Let* $\mathbf{p}$, $\mathbf{x}$ *and* $\mathbf{y}$ *be positive* m*-tuples such that* $\tilde{x}_i = \frac{x_i}{\sum_{j=1}^m p_j x_j}$, $\tilde{y}_i = \frac{y_i}{\sum_{j=1}^m p_j y_j} \in [a,b]$, $i = 1, \ldots, m$.

Let $\frac{\mathbf{x}}{\mathbf{y}}$ *be a decreasing* m*-tuple.*

(i) *If* $\mathbf{y}$ *is a decreasing* m*-tuple and if* f *is* n*-convex for even* n, $n \geqslant 2$, *then*

$$\sum_{i=1}^m p_i f(\tilde{x}_i) - \sum_{i=1}^m p_i f(\tilde{y}_i) \geqslant \sum_{i=1}^m p_i F(\tilde{x}_i) - \sum_{i=1}^m p_i F(\tilde{y}_i), \tag{12}$$

where F *is defined in* (10).

(ii) *Let* $\mathbf{y}$ *be a decreasing* m*-tuple and let the inequality* (12) *be satisfied. If* F *is convex, then*

$$\sum_{i=1}^{m} p_i f(\tilde{x}_i) \geqslant \sum_{i=1}^{m} p_i f(\tilde{y}_i). \tag{13}$$

(iii) *If* $\mathbf{x}$ *is an increasing* m*-tuple and if* f *is* n*-convex for even* n, $n \geqslant 2$, *then the reversed inequality holds in* (12).

(iv) *Let* $\mathbf{x}$ *be an increasing* m*-tuple and let the reverse of the inequality* (12) *be satisfied. If* F *is convex, then the reversed inequality holds in* (13).

Analogous statements hold if $\frac{\mathbf{x}}{\mathbf{y}}$ *is an increasing* m*-tuple.*

Proof. The idea of the proof is the same as in the proof of Theorem 2.3 in [6].

Obviously $\sum_{i=1}^{m} p_i\tilde{x}_i = \sum_{i=1}^{m} p_i\tilde{y}_i = 1$. For the positive m-tuple $\mathbf{u} = \mathbf{py}$ and for the decreasing m-tuple $\mathbf{v} = \frac{\mathbf{x}}{\mathbf{y}}$, apply Lemma 1 (i), it follows $\sum_{i=1}^{k} p_i\tilde{x}_i \geqslant \sum_{i=1}^{k} p_i\tilde{y}_i$, $k = 1,\ldots,m-1$ and we have

$$\tilde{\mathbf{x}} \succ_p \tilde{\mathbf{y}}. \tag{14}$$

(i) The assumption that $\mathbf{y}$ is decreasing obviously implies that $\tilde{\mathbf{y}}$ is decreasing. Apply Theorem 5 (i) with (14), the inequality (12) is immediate.

(ii) As (14) holds, apply Theorem 3 (i) for the convex function F, the right hand side of the inequality (12) is non-negative.

(iii) By following the proof of (i) and by applying Theorem 5 (iii), the reverse of the inequality (12) follows.

(iv) Apply Theorem 3 (ii) for the convex function F, the claim follows. □

The following definition is given in [12, p. 6].

DEFINITION 1. An m-tuple $\mathbf{x} = (x_1,\ldots,x_m) \in \mathbb{R}^m$ is said to be convex if $\frac{x_{k-1}+x_{k+1}}{2} \geqslant x_k$ holds for $k = 2,\ldots,m-1$. If the reversed inequality holds, then $\mathbf{x}$ is said to be concave.

By simple induction, it is easy to prove the following lemma.

LEMMA 2.

1. *If* $\mathbf{x} = (x_1,\ldots,x_m)$ *is a non-negative and concave* m*-tuple, then* $\mathbf{y} = \left(\frac{x_2}{1}, \frac{x_3}{2}, \ldots, \frac{x_m}{m-1}\right)$ *is a decreasing* $(m-1)$*-tuple. If* $\mathbf{x}$ *is non-negative and convex with* $x_1 = 0$, *then* $\mathbf{y}$ *is increasing.*

2. *If* $\mathbf{x} = (x_0,x_1,\ldots,x_m)$ *is non-negative and concave (alt. convex) with* $x_0 = 0$, *then* $\left(\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_m}{m}\right)$ *is decreasing (alt. increasing).*

3. *If* $\mathbf{x} = (x_1,\ldots,x_m)$ *is a non-negative and concave* m*-tuple, then* $\mathbf{z} = \left(\frac{x_1}{m-1}, \frac{x_2}{m-2}, \ldots, \frac{x_{m-1}}{1}\right)$ *is increasing. If* $\mathbf{x}$ *is non-negative and convex with* $x_m = 0$, *then* $\mathbf{z}$ *is decreasing.*

4. *If* $(x_1,\dots,x_m,x_{m+1})$ *is non-negative and concave* (*alt. convex*) *with* $x_{m+1}=0$, *then* $\left(\frac{x_1}{m},\frac{x_2}{m-1},\dots,\frac{x_m}{1}\right)$ *is increasing* (*alt. decreasing*).

COROLLARY 2. *Let* $f:[a,b]\to\mathbb{R}$, $[a,b]\subset(0,\infty)$, $n\geqslant 1$, *be such that* $f^{(n-1)}$ *is absolutely continuous and let* f *be* n*-convex for even* n, $n\geqslant 2$. *Suppose that* $\mathbf{p}$ *and* $(x_1,\dots,x_m)$ *are positive* m*-tuples such that* $\tilde{x}_i=\frac{x_i}{\sum_{j=1}^m p_jx_j}$, $\tilde{u}_i=\frac{i}{\sum_{j=1}^m p_j j}$, $\tilde{v}_i=\frac{m+1-i}{\sum_{j=1}^m p_j(m+1-j)}\in[a,b]$, $i=1,\dots,m$.

(*i*) *If* $\mathbf{x}=(x_0,x_1,\dots,x_m)$ *is an increasing concave* $(m+1)$*-tuple with* $x_0=0$, *then*

$$\sum_{i=1}^m p_i f(\tilde{x}_i)-\sum_{i=1}^m p_i f(\tilde{u}_i)\leqslant\sum_{i=1}^m p_i F(\tilde{x}_i)-\sum_{i=1}^m p_i F(\tilde{u}_i), \tag{15}$$

where F *is defined in* (10).

(*ii*) *If* $\mathbf{x}=(x_0,x_1,\dots,x_m)$ *is a convex* $(m+1)$*-tuple with* $x_0=0$, *then the reversed inequality holds in* (15).

(*iii*) *If* $\mathbf{x}=(x_1,\dots,x_m,x_{m+1})$ *is a decreasing concave* $(m+1)$*-tuple with* $x_{m+1}=0$, *then*

$$\sum_{i=1}^m p_i f(\tilde{x}_i)-\sum_{i=1}^m p_i f(\tilde{v}_i)\leqslant\sum_{i=1}^m p_i F(\tilde{x}_i)-\sum_{i=1}^m p_i F(\tilde{v}_i), \tag{16}$$

where F *is defined in* (10).

(*iv*) *If* $\mathbf{x}=(x_1,\dots,x_m,x_{m+1})$ *is a convex* $(m+1)$*-tuple with* $x_{m+1}=0$, *then the reversed inequality holds in* (16).

Proof. The idea of the proof is the same as that of the proof of Corollary 2.4 in [6].

(*i*) By using the concavity of $\mathbf{x}$ with $x_0=0$, it follows that $\left(\frac{x_1}{1},\frac{x_2}{2},\dots,\frac{x_m}{m}\right)$ is decreasing by Lemma 2 (2). As $\mathbf{x}$ is increasing by assumption, apply Theorem 6 (*iii*), inequality (15) is immediate.

(*ii*) Use Lemma 2 (2) and apply Theorem 6 (12), the claim follows.

(*iii*) As the concavity of $(m+1)$-tuple $\mathbf{x}$ with $x_{m+1}=0$ implies (by Lemma 2 (4)) that $\left(\frac{x_1}{m},\frac{x_2}{m-1},\dots,\frac{x_m}{1}\right)$ is increasing and as by assumption $\mathbf{x}$ is decreasing, use the reverse of the inequality (12), we obtain inequality (16).

(*iv*) By using Lemma 2 (4) and by applying Theorem 6 (*i*), the claim follows. □

REMARK 3. For $f(x)=x^p$, where $x\in(0,\infty)$, it is easy to see that the function f is

(*i*) 1-convex for $p\in(0,\infty)$

(ii) 2-convex or convex for $p \in (-\infty,0) \cup (1,\infty)$

(iii) n-convex for odd $n \geqslant 3$ for $p \in (0,1) \cup (2,3) \cup \ldots \cup (n-3,n-2) \cup (n-1,\infty)$

(iv) n-convex for even $n \geqslant 4$ for $p \in (-\infty,0) \cup (1,2) \cup (3,4) \cup \ldots \cup (n-3,n-2) \cup (n-1,\infty)$.

The following corollary is an application of Theorem 6.

COROLLARY 3. *Let $f(x) = x^p$, $x \in (0,\infty)$, be n-convex for even n, $n \geqslant 2$. Let $\mathbf{p}$, $\mathbf{x}$ and $\mathbf{y}$ be positive m-tuples and let $[a,b] \subseteq (0,\infty)$ such that $\tilde{x}_i = \frac{x_i}{\sum_{j=1}^m p_j x_j}$, $\tilde{y}_i = \frac{y_i}{\sum_{j=1}^m p_j y_j} \in [a,b]$, $i = 1,\ldots,m$. Consider the inequality*

$$\frac{\sum_{i=1}^m p_i x_i^p}{\sum_{i=1}^m p_i y_i^p} - \left(\frac{\sum_{i=1}^m p_i x_i}{\sum_{i=1}^m p_i y_i}\right)^p \geqslant \frac{(\sum_{i=1}^m p_i x_i)^p}{\sum_{i=1}^m p_i y_i^p} \cdot \frac{1}{b-a} \sum_{k=1}^{n-1} \frac{n-k}{k!} (p)_{k-1} \times \left(a^{p-k+1} \left(\sum_{i=1}^m p_i (\tilde{y}_i - a)^k - \sum_{i=1}^m p_i (\tilde{x}_i - a)^k \right) - b^{p-k+1} \left(\sum_{i=1}^m p_i (\tilde{y}_i - b)^k - \sum_{i=1}^m p_i (\tilde{x}_i - b)^k \right) \right), \tag{17}$$

where $(p)_{k-1} = p(p-1)\ldots(p-(k-2))$ is the Pochhammer symbol.

Let $\frac{\mathbf{x}}{\mathbf{y}}$ be a decreasing m-tuple.

(i) *If $\mathbf{y}$ is a decreasing m-tuple, then the inequality (17) holds.*

(ii) *If $\mathbf{x}$ is an increasing m-tuple, then the reversed inequality holds in (17).*

Analogous statements hold if $\frac{\mathbf{x}}{\mathbf{y}}$ is an increasing m-tuple.

An application of Corollary 2 states that:

COROLLARY 4. *Let $f(x) = x^p$, $x \in (0,\infty)$, be n-convex for even n, $n \geqslant 2$. Let $\mathbf{p}$ be a positive m-tuple and let $[a,b] \subset (0,\infty)$ such that $\tilde{x}_i = \frac{x_i}{\sum_{j=1}^m p_j x_j}$, $\tilde{u}_i = \frac{i}{\sum_{j=1}^m p_j j}$, $\tilde{v}_i = \frac{m+1-i}{\sum_{j=1}^m p_j (m+1-j)} \in [a,b]$, $i = 1,\ldots,m$.*

(i) *If $\mathbf{x} = (x_0, x_1, \ldots, x_m)$ is an increasing concave $(m+1)$-tuple with $x_0 = 0$, then*

$$\frac{\sum_{i=1}^m p_i x_i^p}{\sum_{i=1}^m p_i i^p} - \left(\frac{\sum_{i=1}^m p_i x_i}{\sum_{i=1}^m p_i i}\right)^p \leqslant \frac{(\sum_{i=1}^m p_i x_i)^p}{\sum_{i=1}^m p_i i^p} \cdot \frac{1}{b-a} \sum_{k=1}^{n-1} \frac{n-k}{k!} (p)_{k-1} \times \left(a^{p-k+1} \left(\sum_{i=1}^m p_i (\tilde{u}_i - a)^k - \sum_{i=1}^m p_i (\tilde{x}_i - a)^k \right) - b^{p-k+1} \left(\sum_{i=1}^m p_i (\tilde{u}_i - b)^k - \sum_{i=1}^m p_i (\tilde{x}_i - b)^k \right) \right). \tag{18}$$

(ii) *If* $\mathbf{x}=(x_0,x_1,\ldots,x_m)$ *is a convex* $(m+1)$*-tuple with* $x_0=0$*, then the reversed inequality holds in* (18).

(iii) *If* $\mathbf{x}=(x_1,\ldots,x_m,x_{m+1})$ *is a decreasing concave* $(m+1)$*-tuple with* $x_{m+1}=0$*, then*

$$\begin{aligned}
&\frac{\sum_{i=1}^m p_i x_i^p}{\sum_{i=1}^m p_i (m+1-i)^p} - \left(\frac{\sum_{i=1}^m p_i x_i}{\sum_{i=1}^m p_i (m+1-i)}\right)^p \\
&\leqslant \frac{\left(\sum_{i=1}^m p_i x_i\right)^p}{\sum_{i=1}^m p_i (m+1-i)^p}\cdot\frac{1}{b-a}\sum_{k=1}^{n-1}\frac{n-k}{k!}(p)_{k-1} \\
&\times\left(a^{p-k+1}\left(\sum_{i=1}^m p_i(\tilde{v}_i-a)^k-\sum_{i=1}^m p_i(\tilde{x}_i-a)^k\right)\right. \\
&\left.-b^{p-k+1}\left(\sum_{i=1}^m p_i(\tilde{v}_i-b)^k-\sum_{i=1}^m p_i(\tilde{x}_i-b)^k\right)\right). \qquad (19)
\end{aligned}$$

(iv) *If* $\mathbf{x}=(x_1,\ldots,x_m,x_{m+1})$ *is a convex* $(m+1)$*-tuple with* $x_{m+1}=0$*, then the reversed inequality holds in* (19).

REMARK 4. By using the method given in [6], it is easy to prove an analogue of inequality (18) for the m-tuples $\mathbf{x}$ which are non-negative, increasing and concave. In this case set $\tilde{u}_i=\frac{i-1}{\sum_{i=1}^m p_i(i-1)}$, $i=1,\ldots,m$, $a=0$ and $p\geqslant n-1$. Analogous statement holds for the inequality (19) by setting $\tilde{v}_i=\frac{m-i}{\sum_{i=1}^m p_i(m-i)}$, $i=1,\ldots,m$, $a=0$ and $p\geqslant n-1$.

4. Refinements and generalizations of the Berwald-type inequalities

By using the analogous methods as in the previous sections, refinements and generalizations of Berwald-type inequalities are obtained.

THEOREM 7. *Let* $f,\zeta:[a,b]\to\mathbb{R}$, $n\geqslant 1$, *be such that* ζ *is strictly increasing and* $\left(f\circ\zeta^{-1}\right)^{(n-1)}$ *is absolutely continuous. Let* $\mathbf{p}$ *be a positive* m*-tuple and* $\mathbf{x}$, $\mathbf{y}\in[a,b]^m$ *such that* $\zeta(\mathbf{x})\succ_p\zeta(\mathbf{y})$ *holds.*

(i) *If* $\mathbf{y}$ *is a decreasing* m*-tuple and if* $f\circ\zeta^{-1}$ *is* n*-convex for even* n, $n\geqslant 2$*, then*

$$\begin{aligned}
&\sum_{i=1}^m p_i f(x_i)-\sum_{i=1}^m p_i f(y_i)\geqslant\frac{1}{\zeta(b)-\zeta(a)}\sum_{k=1}^{n-1}\frac{n-k}{k!} \\
&\times\left(\left(f\circ\zeta^{-1}\right)^{(k-1)}(\zeta(a))\left(\sum_{i=1}^m p_i(\zeta(y_i)-\zeta(a))^k-\sum_{i=1}^m p_i(\zeta(x_i)-\zeta(a))^k\right)\right. \\
&\left.-\left(f\circ\zeta^{-1}\right)^{(k-1)}(\zeta(b))\left(\sum_{i=1}^m p_i(\zeta(y_i)-\zeta(b))^k-\sum_{i=1}^m p_i(\zeta(x_i)-\zeta(b))^k\right)\right). \qquad (20)
\end{aligned}$$

(*ii*) *Let* $\mathbf{y}$ *be a decreasing* m*-tuple and let the inequality* (20) *be satisfied. Suppose that* G *is a function defined by*

$$G(x) = \frac{1}{\zeta(b)-\zeta(a)} \sum_{k=1}^{n-1} \frac{n-k}{k!} \Big((x-\zeta(b))^k \left(f\circ\zeta^{-1}\right)^{(k-1)} (\zeta(b)) - (x-\zeta(a))^k \left(f\circ\zeta^{-1}\right)^{(k-1)} (\zeta(a)) \Big). \tag{21}$$

If G *is convex, then the inequality* (6) *holds.*

(*iii*) *If* $\mathbf{x}$ *is an increasing* m*-tuple and if* $f\circ\zeta^{-1}$ *is* n*-convex for even* n, $n \geqslant 2$, *then the reversed inequality holds in* (20).

(*iv*) *Let* $\mathbf{x}$ *be an increasing* m*-tuple and let the reverse of the inequality* (20) *be satisfied. If* G *is defined in* (21) *and if* G *is convex, then the reversed inequality holds in* (6).

Proof.

(*i*) Apply Theorem 5 (*i*) for the n-convex function $f\circ\zeta^{-1}$ and for the m-tuples $\tilde{\mathbf{x}} = (\zeta(x_1),\ldots,\zeta(x_m))$ and $\tilde{\mathbf{y}} = (\zeta(y_1),\ldots,\zeta(y_m))$.

(*ii*) It is easy to see that the inequality (20) is equivalent to

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \geqslant \sum_{i=1}^{m} p_i G(\zeta(x_i)) - \sum_{i=1}^{m} p_i G(\zeta(y_i)). \tag{22}$$

The claim follows by using Theorem 3 (*i*).

(*iii*) For the n-convex function $f\circ\zeta^{-1}$ and for the m-tuples $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$, apply Theorem 5 (*iii*), the reverse of the inequality (20) follows.

(*iv*) As the reverse of the inequality (20) is equivalent to the reverse of the inequality (22), apply Theorem 3 (*ii*) for the convex function G, the claim follows. □

The following theorem presents refinement of the generalized discrete weighted Berwald's inequality.

THEOREM 8. *Let* $f,\zeta : (0,\infty) \to \mathbb{R}$, $n \geqslant 1$, *be such that* ζ *is continuous and strictly increasing and* $\left(f\circ\zeta^{-1}\right)^{(n-1)}$ *is absolutely continuous. Let* $\mathbf{p}$, $\mathbf{x}$ *and* $\mathbf{y}$ *be positive* m*-tuples and let* z_1 *be such that*

$$\sum_{i=1}^{m} p_i \zeta(x_i) = \sum_{i=1}^{m} p_i \zeta(z_1 y_i). \tag{23}$$

Suppose that $[a,b] \subset (0,\infty)$ *is such that* $x_i, z_1 y_i \in [a,b]$, $i = 1,\ldots,m$.

Let $\frac{\mathbf{x}}{\mathbf{y}}$ *be a decreasing* m*-tuple.*

(i) *If* $\mathbf{y}$ *is a decreasing* m*-tuple and if* $f \circ \zeta^{-1}$ *is* n*-convex for even* n, $n \geqslant 2$, *then*

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(z_1 y_i) \geqslant \sum_{i=1}^{m} p_i G(\zeta(x_i)) - \sum_{i=1}^{m} p_i G(\zeta(z_1 y_i)), \tag{24}$$

where G *is given by* (21).

(ii) *Let* $\mathbf{y}$ *be a decreasing* m*-tuple and let the inequality* (24) *be satisfied. If* G *is defined in* (21) *and if* G *is convex, then*

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant \sum_{i=1}^{m} p_i f(z_1 y_i) \tag{25}$$

holds.

(iii) *If* $\mathbf{x}$ *is an increasing* m*-tuple and if* $f \circ \zeta^{-1}$ *is* n*-convex for even* n, $n \geqslant 2$, *then the reversed inequality holds in* (24).

(iv) *Let* $\mathbf{x}$ *be an increasing* m*-tuple and let the reverse of the inequality* (24) *be satisfied. If* G *is defined in* (21) *and if* G *is convex, then the reversed inequality holds in* (25).

Analogous statements hold if $\frac{\mathbf{x}}{\mathbf{y}}$ *is an increasing* m*-tuple.*

Proof. The existence of z_1 is shown in [6, Theorem 2.8]. Obviously $z_1 \in \left[\min_j \frac{x_j}{y_j}, \max_j \frac{x_j}{y_j}\right]$.

As $\frac{\mathbf{x}}{\mathbf{y}}$ is a decreasing m-tuple, ζ is strictly increasing and as (23) holds, there is an $l \in \{1, \ldots, m\}$ such that

$$\frac{x_i}{y_i} \geqslant z_1, \quad i = 1, \ldots, l \quad \text{and} \quad z_1 \geqslant \frac{x_i}{y_i}, \quad i = l+1, \ldots, m. \tag{26}$$

Since ζ is strictly increasing, from (26) it follows that

$$\sum_{i=1}^{k} p_i \zeta(x_i) \geqslant \sum_{i=1}^{k} p_i \zeta(z_1 y_i), \quad k = 1, \ldots, l \tag{27}$$

and

$$\sum_{i=l+1}^{k} p_i \zeta(x_i) \leqslant \sum_{i=l+1}^{k} p_i \zeta(z_1 y_i), \quad k = l+1, \ldots, m. \tag{28}$$

By using (23) and inequality (28), it follows $\sum_{i=1}^{k} p_i \zeta(x_i) \geqslant \sum_{i=1}^{k} p_i \zeta(z_1 y_i)$, $k = l+1, \ldots, m$, which combined together with (27) yields that

$$\sum_{i=1}^{k} p_i \zeta(x_i) \geqslant \sum_{i=1}^{k} p_i \zeta(z_1 y_i), \quad k = 1, \ldots, m,$$

and the above inequality combined together with (23) implies that

$$\zeta(\mathbf{x}) \succ_p \zeta(z_1 \mathbf{y}). \tag{29}$$

(i) If $\mathbf{y}$ is a decreasing m-tuple, then apply Theorem 7 (i) with (29), we have inequality (24).

(ii) As (29) holds, apply Theorem 3 (i), the claim follows.

(iii) If $\mathbf{x}$ is an increasing m-tuple, then apply Theorem 7 (iii) with (29), reverse of the inequality (24) is immediate.

(iv) Apply Theorem 3 (ii) with (29), the claim follows.

For the second case the idea of the proof is the same as discussed above. □

COROLLARY 5. *Let $f,\zeta:(0,\infty)\to\mathbb{R}$, $n\geqslant 1$, be such that ζ is continuous and strictly increasing and $\left(f\circ\zeta^{-1}\right)^{(n-1)}$ is absolutely continuous. Let $f\circ\zeta^{-1}$ be n-convex for even n, $n\geqslant 2$. Let $\mathbf{p}$ be a positive m-tuple and let z_1 and z_2 be such that*

$$\sum_{i=1}^{m} p_i\zeta(x_i)=\sum_{i=1}^{m} p_i\zeta(z_1 i),\quad \sum_{i=1}^{m} p_i\zeta(x_i)=\sum_{i=1}^{m} p_i\zeta(z_2(m+1-i)) \tag{30}$$

hold. Suppose that $[a_1,b_1]$, $[a_2,b_2]\subset(0,\infty)$ are such that $x_i, z_1 i\in[a_1,b_1]$, x_i, $z_2(m+1-i)\in[a_2,b_2]$, $i=1,\ldots,m$.

(i) *If $\mathbf{x}=(x_0,x_1,\ldots,x_m)$ is an increasing concave $(m+1)$-tuple with $x_0=0$, then*

$$\sum_{i=1}^{m} p_i f(x_i)-\sum_{i=1}^{m} p_i f(z_1 i)\leqslant\sum_{i=1}^{m} p_i G(\zeta(x_i))-\sum_{i=1}^{m} p_i G(\zeta(z_1 i)), \tag{31}$$

where G is defined in (21) (for the interval $[a_1,b_1]$).

(ii) *If $\mathbf{x}=(x_0,x_1,\ldots,x_m)$ is a convex $(m+1)$-tuple with $x_0=0$, then the reversed inequality holds in (31).*

(iii) *If $\mathbf{x}=(x_1,\ldots,x_m,x_{m+1})$ is a decreasing concave $(m+1)$-tuple with $x_{m+1}=0$, then*

$$\begin{aligned}&\sum_{i=1}^{m} p_i f(x_i)-\sum_{i=1}^{m} p_i f(z_2(m+1-i))\\&\leqslant\sum_{i=1}^{m} p_i G(\zeta(x_i))-\sum_{i=1}^{m} p_i G(\zeta(z_2(m+1-i))),\end{aligned} \tag{32}$$

where G is defined in (21) (for the interval $[a_2,b_2]$).

(iv) *If $\mathbf{x}=(x_1,\ldots,x_m,x_{m+1})$ is a convex $(m+1)$-tuple with $x_{m+1}=0$, then the reversed inequality holds in (32).*

Proof.

(i) By the same arguing as given in the Corollary 2 (i), apply Theorem 8 (iii), inequality (31) is immediate.

(ii) As $\frac{x_i}{i}$, $i=1,\ldots,m$, is increasing, apply Theorem 8 (24), the claim follows.

(iii) For a decreasing m-tuple $\mathbf{x}$ and an increasing m-tuple $\frac{x_i}{m+1-i}$, $i=1,\ldots,m$, apply reverse of the inequality (24) such that (30) holds, inequality (32) is immediate.

(iv) As $\frac{x_i}{m+1-i}$, $i=1,\ldots,m$ is decreasing, apply Theorem 8 (i) such that (30) holds, the result is immediate. □

The following corollary is an application of Theorem 8.

COROLLARY 6. *Let $f(x)=x^p$, $\zeta(x)=x^q$, $q>0$, be such that $x\mapsto x^{\frac{p}{q}}$ is n-convex for even n, where $x\in(0,\infty)$ and $n\geqslant 2$. Let $\mathbf{p}$, $\mathbf{x}$ and $\mathbf{y}$ be positive m-tuples and let $[a,b]\subseteq(0,\infty)$ be such that x_i, $\left(\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i y_i^q}\right)^{\frac{1}{q}} y_i\in[a,b]$, $i=1,\ldots,m$.*

Let $\frac{\mathbf{x}}{\mathbf{y}}$ be a decreasing m-tuple.

(i) *If $\mathbf{y}$ is a decreasing m-tuple, then*

$$\begin{aligned}\frac{\sum_{i=1}^m p_i x_i^p}{\sum_{i=1}^m p_i y_i^p}-\left(\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i y_i^q}\right)^{\frac{p}{q}}&\geqslant\frac{1}{\sum_{i=1}^m p_i y_i^p}\cdot\frac{1}{b^q-a^q}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\frac{p}{q}\right)_{k-1}\\&\times\left(a^{p-(k-1)q}\left(\sum_{i=1}^m p_i\left(y_i^q\cdot\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i y_i^q}-a^q\right)^k-\sum_{i=1}^m p_i\left(x_i^q-a^q\right)^k\right)\right.\\&\left.-b^{p-(k-1)q}\left(\sum_{i=1}^m p_i\left(y_i^q\cdot\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i y_i^q}-b^q\right)^k-\sum_{i=1}^m p_i\left(x_i^q-b^q\right)^k\right)\right).\end{aligned}\tag{33}$$

(ii) *If $\mathbf{x}$ is an increasing m-tuple, then the reversed inequality holds in (33).*

Analogous statements hold if $\frac{\mathbf{x}}{\mathbf{y}}$ is an increasing m-tuple.

Proof. It's proof is an immediate consequence of Theorem 8. Notice that in this case $z_1=\left(\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i y_i^q}\right)^{\frac{1}{q}}$. □

An application of Corollary 5 states that:

COROLLARY 7. *Let $f(x)=x^p$, $\zeta(x)=x^q$, $q>0$, be such that $x\mapsto x^{\frac{p}{q}}$ is n-convex for even n, where $x\in(0,\infty)$ and $n\geqslant 2$. Let $\mathbf{p}$ be a positive m-tuple.*

(i) *If $\mathbf{x}=(x_0,x_1,\ldots,x_m)$ is an increasing concave $(m+1)$-tuple with $x_0=0$, then*

$$\begin{aligned}\frac{\sum_{i=1}^m p_i x_i^p}{\sum_{i=1}^m p_i i^p}-\left(\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i i^q}\right)^{\frac{p}{q}}&\leqslant\frac{1}{\sum_{i=1}^m p_i i^p}\cdot\frac{1}{b^q-a^q}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\frac{p}{q}\right)_{k-1}\\&\times\left(a^{p-(k-1)q}\left(\sum_{i=1}^m p_i\left(i^q\cdot\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i i^q}-a^q\right)^k-\sum_{i=1}^m p_i\left(x_i^q-a^q\right)^k\right)\right.\\&\left.-b^{p-(k-1)q}\left(\sum_{i=1}^m p_i\left(i^q\cdot\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i i^q}-b^q\right)^k-\sum_{i=1}^m p_i\left(x_i^q-b^q\right)^k\right)\right),\end{aligned}\tag{34}$$

where $[a,b]\subset(0,\infty)$ *is such that* $x_i,\ \left(\frac{\sum_{i=1}^m p_i x_i^q}{\sum_{i=1}^m p_i i^q}\right)^{\frac{1}{q}} i\in[a,b],\ i=1,\dots,m.$

(*ii*) *If* $\mathbf{x}=(x_0,x_1,\dots,x_m)$ *is a convex* $(m+1)$*-tuple with* $x_0=0$*, then the reversed inequality holds in* (34).

(*iii*) *If* $\mathbf{x}=(x_1,\dots,x_m,x_{m+1})$ *is a decreasing concave* $(m+1)$*-tuple with* $x_{m+1}=0$*, then inequality* (34) *holds, where the* m*-tuple* $(m,m-1,\dots,2,1)$ *is used instead of the* m*-tuple* $(1,2,\dots,m)$.

(*iv*) *If* $\mathbf{x}=(x_1,\dots,x_m,x_{m+1})$ *is a convex* $(m+1)$*-tuple with* $x_{m+1}=0$*, then the reversed inequality holds in* (34)*, where the* m*-tuple* $(m,m-1,\dots,2,1)$ *is used instead of the* m*-tuple* $(1,2,\dots,m)$.

Claims given in Remark 4 also hold in the setting of the Corollary 7, that is, inequality (34) also holds for non-negative, increasing and concave m-tuples $\mathbf{x}$ by using the m-tuple $(0,1,\dots,m-1)$ instead of the m-tuple $(1,2,\dots,m)$ for $a=0$ and $\frac{p}{q}\geqslant n-1$. Analogous claims hold for non-negative, decreasing and concave m-tuples.

The most interesting case is $0<q\leqslant p$ in which Corollary 7 holds for $n=2$. In this case the right-hand side of the inequality (34) vanishes. By rearranging and expressing in terms of power means, it follows that

$$M_m^{[p]}(\mathbf{x};\mathbf{p})\leqslant\frac{M_m^{[p]}(\mathbf{J}_m;\mathbf{p})}{M_m^{[q]}(\mathbf{J}_m;\mathbf{p})}M_m^{[q]}(\mathbf{x};\mathbf{p}),\tag{35}$$

where $\mathbf{J}_m=(1,2,\dots,m)$, which holds for m-tuples $\mathbf{x}$ for which $(m+1)$-tuples $(0,\mathbf{x})$ are increasing and concave. Obviously inequality (35) is sharp and equality is attained for $\mathbf{x}=\mathbf{J}_m$. By limiting argument inequality (35) also holds for $q=0$. It is easy to see that the inequality (4) follows from inequality (35) for $p=1,\ q=0,\ \mathbf{p}=\mathbf{1}=(1,1,\dots,1)$.

By using the above remarks (as given in Remark 4), an analogous inequality also holds for m-tuples $\mathbf{x}$ which are non-negative, increasing and concave, that is

$$M_m^{[p]}(\mathbf{x};\mathbf{p})\leqslant\frac{M_m^{[p]}(\mathbf{I}_m;\mathbf{p})}{M_m^{[q]}(\mathbf{I}_m;\mathbf{p})}M_m^{[q]}(\mathbf{x};\mathbf{p}),\tag{36}$$

which is exactly estimation (3). Notice that since (36) holds for $\mathbf{x}=\mathbf{J}_m$, it follows that

$$M_m^{[p]}(\mathbf{J}_m;\mathbf{p})\leqslant\frac{M_m^{[p]}(\mathbf{I}_m;\mathbf{p})}{M_m^{[q]}(\mathbf{I}_m;\mathbf{p})}M_m^{[q]}(\mathbf{J}_m;\mathbf{p}),$$

which shows that the estimation in (35) is better than the estimation in (36), but the first estimation holds for smaller class of concave functions.

OPEN PROBLEM. Find the best possible estimation of the form (35) for all the non-negative concave m-tuples $\mathbf{x}$.

5. The continuous case

Here we just give an outline of the continuous analogous of the main results of this paper. The integral analogue of Theorems 2 and 3 and Lemma 1 are given in [7].

The following theorem is the integral version of Theorem 4 (see [5]).

THEOREM 9. *Let $f:[a,b]\to\mathbb{R}$, $n\geqslant 1$, be such that $f^{(n-1)}$ is absolutely continuous and let $k(t,x)$ be defined in (7). Let $p:[c,d]\to\mathbb{R}$ and $\varphi,\psi:[c,d]\to[a,b]$ be continuous functions. If*

$$\int_c^d p(z)(\varphi(z)-t)^{n-1}k(t,\varphi(z))\,dz \geqslant \int_c^d p(z)(\psi(z)-t)^{n-1}k(t,\psi(z))\,dz \tag{37}$$

holds and if f is n-convex, then

$$\begin{aligned}&\int_c^d p(z)f(\varphi(z))\,dz-\int_c^d p(z)f(\psi(z))\,dz\\ \geqslant&\ \frac{1}{b-a}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left(f^{(k-1)}(a)\left(\int_c^d p(z)(\psi(z)-a)^k\,dz-\int_c^d p(z)(\varphi(z)-a)^k\,dz\right)\right.\\ &\left.-f^{(k-1)}(b)\left(\int_c^d p(z)(\psi(z)-b)^k\,dz-\int_c^d p(z)(\varphi(z)-b)^k\,dz\right)\right).\end{aligned} \tag{38}$$

If the reversed inequality holds in (37), then the reversed inequality holds in (38).

The integral version of Theorem 5 is as follows:

THEOREM 10. *Let $f:[a,b]\to\mathbb{R}$, $n\geqslant 1$, be such that $f^{(n-1)}$ is absolutely continuous. Let $p:[c,d]\to\mathbb{R}$ and let $\varphi,\psi:[c,d]\to[a,b]$ be continuous functions such that $\varphi\succ_p\psi$ (the obvious continuous variant of (5)).*

(*i*) *If ψ is a decreasing function and if f is n-convex for even $n\geqslant 2$, then the inequality (38) holds.*

(*ii*) *Let ψ be a decreasing function and let the inequality (38) be satisfied. If F defined in (10) is convex, then*

$$\int_c^d p(z)f(\varphi(z))\,dz \geqslant \int_c^d p(z)f(\psi(z))\,dz. \tag{39}$$

(*iii*) *If φ is an increasing function and if f is n-convex for even $n\geqslant 2$, then the reversed inequality holds in (38).*

(*iv*) *Let φ be an increasing function and let the reverse of the inequality (38) be satisfied. If F defined in (10) is convex, then the reversed inequality holds in (39).*

Proof. The proof is analogous to the proof of Theorem 5 but we apply Theorem 9 and [7, Theorem 2.1] instead of Theorems 3 and 4. □

The following theorem is the integral version of Theorem 8.

THEOREM 11. *Let $f, \zeta : [0,\infty) \to \mathbb{R}$ be such that for $n \geqslant 1$, $\left(f \circ \zeta^{-1}\right)^{(n-1)}$ is absolutely continuous and ζ is continuous and strictly increasing. Let $p, \varphi, \psi : [c,d] \to [0,\infty)$ be integrable functions. Let z_1 be defined as*

$$\int_c^d p(z)\,\zeta(\varphi(z))\,dz = \int_c^d p(z)\,\zeta(z_1\psi(z))\,dz. \tag{40}$$

Suppose that $[a,b] \subset [0,\infty)$ is such that $\varphi(z),\ z_1\psi(z) \in [a,b]$ for every $z \in [c,d]$.

Let $\frac{\varphi}{\psi}$ be a decreasing function.

(i) *If ψ is a decreasing function and if $f \circ \zeta^{-1}$ is n-convex for even $n \geqslant 2$, then*

$$\begin{aligned}&\int_c^d p(z)\,f(\varphi(z))\,dz - \int_c^d p(z)\,f(z_1\psi(z))\,dz\\ &\geqslant \int_c^d p(z)\,G(\zeta(\varphi(z)))\,dz - \int_c^d p(z)\,G(\zeta(z_1\psi(z)))\,dz\end{aligned} \tag{41}$$

holds, where G is defined by (21).

(ii) *Let ψ be a decreasing function and let the inequality (41) be satisfied. If G defined in (21) is convex, then the inequality*

$$\int_c^d p(z)\,f(\varphi(z))\,dz \geqslant \int_c^d p(z)\,f(z_1\psi(z))\,dz \tag{42}$$

holds.

(iii) *If φ is an increasing function and if $f \circ \zeta^{-1}$ is n-convex for even $n \geqslant 2$, then the reversed inequality holds in (41).*

(iv) *Let φ be an increasing function and let the reverse of the inequality (41) be satisfied. If G defined in (21) is convex, then the reversed inequality holds in (42).*

Analogous claims hold if $\frac{\varphi}{\psi}$ is an increasing function.

Proof. The idea of the proof is the same as that of the proof of Theorem 8. Apply an obvious variant of Theorem 7. □

REMARK 5. Let $z_1 > 0$, where z_1 is defined in (40) and let φ be a positive increasing concave function defined on $[c,d]$.

(i) If we substitute $\psi(z) = \frac{z-c}{d-c}$ in the reverse of the inequality (41), then Theorem 11 (iii) combined together with (iv) gives refinement of the weighted Berwald's inequality.

(ii) If we substitute $\psi(z) = \frac{d-z}{d-c}$ in the reverse of the inequality (41), then Theorem 11 (iii) combined together with (iv) gives refinement of the weighted Berwald's inequality.

The following corollary is an application of Theorem 11.

COROLLARY 8. *Let* $f(x) = x^p$ *and* $\zeta(x) = x^q$, $q > 0$, *where* $x \in (0,\infty)$ *be such that* $x \mapsto x^{\frac{p}{q}}$ *is* n*-convex for even* $n \geqslant 2$. *Let* $p, \varphi, \psi : [c,d] \to (0,\infty)$ *be integrable functions. Let* $[a,b] \subset (0,\infty)$ *be such that* $\varphi(z)$, $\left(\frac{\int_c^d p(z)\varphi^q(z)dz}{\int_c^d p(z)\psi^q(z)dz}\right)^{1/q}\psi(z) \in [a,b]$ *for every* $z \in [a,b]$. *Consider the inequality*

$$\begin{aligned}
&\frac{\int_c^d p(z)\varphi^p(z)dz}{\int_c^d p(z)\psi^p(z)dz} - \left(\frac{\int_c^d p(z)\varphi^q(z)dz}{\int_c^d p(z)\psi^q(z)dz}\right)^{\frac{p}{q}} \\
&\geqslant \frac{1}{\int_c^d p(z)\psi^p(z)dz} \cdot \frac{1}{b^q - a^q} \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(\frac{p}{q}\right)_{k-1} \\
&\times \left(a^{p-(k-1)q} \left(\int_c^d p(z) \left(\psi^q(z) \frac{\int_c^d p(z)\varphi^q(z)dz}{\int_c^d p(z)\psi^q(z)dz} - a^q \right)^k dz - \int_c^d p(z) \left(\varphi^q(z) - a^q\right)^k dz \right) \right. \\
&\left. - b^{p-(k-1)q} \left(\int_c^d p(z) \left(\psi^q(z) \frac{\int_c^d p(z)\varphi^q(z)dz}{\int_c^d p(z)\psi^q(z)dz} - b^q \right)^k dz - \int_c^d p(z) \left(\varphi^q(z) - b^q\right)^k dz \right) \right).
\end{aligned} \tag{43}$$

Let $\frac{\varphi}{\psi}$ *be a decreasing function.*

(i) *If* ψ *is a decreasing function, then the inequality* (43) *holds.*

(ii) *If* φ *is an increasing function, then the reversed inequality holds in* (43).

Analogous claims hold if $\frac{\varphi}{\psi}$ *is an increasing function.*

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