Journal of Mathematical Inequalities
Volume 12, Number 1 (2018), 285–302 doi:10.7153/jmi-2018-12-21

ESTIMATES FOR THE FIRST EIGENVALUE FOR p–LAPLACIAN WITH MIXED BOUNDARY CONDITIONS

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(*Communicated by M. Plum*)

Abstract. In this article, we consider eigenvalue problems on domains with an interior hole. Precisely, we show a Cheng-type inequality on manifolds, and certain Faber-Krahn inequalities on space forms. Besides, we obtain upper and lower bounds of the eigenvalue through the classic Dirchlet eigenvalue, implying convergence of the eigenvalue as the hole tends to $\emptyset$.

1. Introduction and main results

Let D_1 and D_2 be bounded smooth domains in a complete Riemannian manifold (M^n,g) satisfying $D_1 \subset D_2$, and $\Omega = D_2 \setminus D_1$. Denote by ∂D_1 and ∂D_2 the inner boundary and the outer boundary of Ω respectively. For each $1 < p < +\infty$, one define (see for example [9])

DEFINITION 1. (Inner Neumann and outer Dirichlet boundary)

$$\lambda_{(p)}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} : u \in W^{1,p}(\Omega) \setminus 0 \text{ with } u|_{\partial D_2} = 0 \right\}; \tag{1.1}$$

DEFINITION 2. (Inner Dirichlet and outer Neumann boundary)

$$\mu_{(p)}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} : u \in W^{1,p}(\Omega) \setminus 0 \text{ with } u|_{\partial D_1} = 0 \right\}. \tag{1.2}$$

It has been shown in [8, 10, 13] that there exists a unique minimizer u (up to a multiple) solving (1.1), and $\lambda_{(p)} > 0$. Moreover u does not change sign in Ω, satisfying elliptic equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda_{(p)}|u|^{p-2}u \tag{1.3}$$

in the weak sense, which means for any $\phi \in C^\infty(\Omega)$ with $\phi = 0$ on ∂D_2,

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla\phi, \nabla u\rangle \, dx = \lambda_{(p)} \int_\Omega |u|^{p-2}u\,\phi \, dx,$$

see for example Section 6.5 in [8]. Besides, we see from [1, 18] that $u(x) \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ and $u(x)$ satisfies $u(x) = 0$ on ∂D_2 and $\frac{\partial}{\partial \nu}u = 0$ on ∂D_1, where

Mathematics subject classification (2010): Primary 53C21, secondary 35P30.
Keywords and phrases: Eigenvalue, p-Laplacian, comparison theorems, mixed boundary conditions.

ν denotes the unit normal vector field. Throughout the paper, we call u the normalized eigenfunction if it is positive in Ω with $\int_\Omega |u|^p \, dx = 1$.

Similarly, one can show $\mu_{(p)}$ is simple and positive, and the corresponding normalized minimizer $v(x) \in C^{1,\alpha}(\bar{\Omega})$, satisfying

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) = \mu_{(p)}|v|^{p-2}v \tag{1.4}$$

in the weak sense with Dirichlet condition on ∂D_1 and Neumann condition on ∂D_2.

When D_1 is empty, there are several classic estimates on eigenvalues of Dirichlet and Neumann Laplace operators. In 1923 and 1924, G. Faber and E. Krahn proved that the round balls minimize the smallest eigenvalue of Dirichlet Laplace operator among all the domains with the same volume in $\mathbb{R}^n$. More precisely,

THEOREM 1.1. (Faber-Krahn inequality, see for example [5]) *Let Ω be a smooth, bounded domain in $\mathbb{R}^n$. Then*

$$\lambda(\Omega) \geqslant \lambda(\Omega^*),$$

where Ω^* is a round ball in $\mathbb{R}^n$ with the same volume as that of Ω, and $\lambda(\Omega)$ is the smallest eigenvalue of Dirichlet Laplace operator. The equality holds if only if Ω is a round ball.

Faber-Krahn inequality was extended to the smallest eigenvalues of Dirichlet p-Laplacian by H. Takeuchi in [17] and A.-M. Matei in [14]. For the eigenvalues $\lambda_{(p)}$ and $\mu_{(p)}$ characterized in Definition 1 and 2, we proved the following estimates.

THEOREM 1.2. *Let $D_1 \subset D_2$ in $\mathbb{R}^n$ and $p > 1$. Then*

$$\lambda_{(p)}(D_2 \setminus D_1) \geqslant \lambda_{(p)}(D_2^* \setminus D_1^*), \tag{1.5}$$

and

$$\mu_{(p)}(D_2 \setminus D_1) \geqslant \mu_{(p)}(D_2^* \setminus D_1^*), \tag{1.6}$$

where D_1^* and D_2^* are the round balls in $\mathbb{R}^n$ centered at the origin with volume equal to that of D_1 and D_2 respectively. The equality holds if only if $D_2 \setminus D_1$ is isometric to $D_2^* \setminus D_1^*$.

In fact, Theorem 1.2 is also true in space forms M^n_κ by similar proofs. For manifolds with positive Ricci curvature, Berard and Meyer [3] obtained a Faber-Krahn inequality for the first nonzero Dirichlet eigenvalue for the Laplacian on bounded domains, and later generalized to the p-Laplacian by Matei [14]. Here we get the following Faber-Krahn inequality on positive manifolds similarly as in [14], extending the result in [14].

THEOREM 1.3. *Let (M^n, g) be a Riemannian manifold. Assume that the Ricci curvature of (M^n, g) is bounded from below by $(n-1)$. Denote by $\mathbb{S}^n$ the n-dimensional unit round sphere in $\mathbb{R}^{n+1}$. Let $\beta = \frac{|\mathbb{S}^n|}{|M|}$. Then for each $p > 1$,*

$$\lambda_{(p)}(D_2 \setminus D_1) \geqslant \lambda_{(p)}(\widetilde{D}_2 \setminus \widetilde{D}_1), \tag{1.7}$$

and

$$\mu_{(p)}(D_2 \setminus D_1) \geqslant \mu_{(p)}(\widetilde{D}_2 \setminus \widetilde{D}_1), \tag{1.8}$$

where $\widetilde{D}_1$ and $\widetilde{D}_2$ are geodesic balls in $\mathbb{S}^n$ with volume equal to $\beta|D_1|$ and $\beta|D_2|$ respectively.

On Rimannian manifolds, S.-Y. Cheng in 1975 proved the following comparison on the first nonzero eigenvaule of the Dirichlet Laplace operator.

THEOREM 1.4. (Cheng inequality [6]) *Let M be an n-dimensional complete Riemannian manifold.*

(1) If the Ricci curvature is bounded from below by $(n-1)\kappa$, then for each $x_0 \in M$,

$$\lambda(B_{r_0}(x_0)) \leqslant \lambda(V_{r_0,\kappa}).$$

(2) If the sectional curvature is bounded from above by κ, then for each $x_0 \in M$,

$$\lambda(B_{r_0}(x_0)) \geqslant \lambda(V_{r_0,\kappa}).$$

Moreover if the equality holds, then $B_{r_0}(x_0)$ is isometric to $V_{r_0,\kappa}$. Here and thereafter, $B_{r_0}(x_0)$ denotes the geodesic ball of radius r_0 with center x_0, $V_{r_0,\kappa}$ denotes the geodesic ball of radius r_0 on a space form M^n_κ with constant sectional curvature κ. $\lambda(\Omega)$ stands for the first nonzero eigenvalue of Dirichlet Laplace operator on Ω.

This result was extended to the smallest eigenvalues of Dirichlet p-Laplacian by H. Takeuchi in [17]. Motivated by the above results, we consider the case as the hole D_1 is non-empty, and prove the following comparisons.

THEOREM 1.5. *Assume $B_{r_2}(0)$ is an injective ball in M^n, and $0 < r_1 < r_2$.*
(1) If the Ricci curvature of (M,g) is bounded from below by $(n-1)\kappa$, then

$$\lambda_{(p)}(B_{r_2}(0) \setminus B_{r_1}(0)) \leqslant \lambda_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa}), \tag{1.9}$$

and

$$\mu_{(p)}(B_{r_2}(0) \setminus B_{r_1}(0)) \geqslant \mu_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa}). \tag{1.10}$$

(2) If the sectional curvature of (M,g) is bounded from above by κ, then

$$\lambda_{(p)}(B_{r_2}(0) \setminus B_{r_1}(0)) \geqslant \lambda_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa}), \tag{1.11}$$

and

$$\mu_{(p)}(B_{r_2}(0) \setminus B_{r_1}(0)) \leqslant \mu_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa}). \tag{1.12}$$

Moreover if the equality holds, then $B_{r_2}(0) \setminus B_{r_1}(0)$ is isometric to $V_{r_2,\kappa} \setminus V_{r_1,\kappa}$.

In 1984, S. Y. Cheng, P. Li and S. T. Yau proved a eigenvalue estimates for the Laplace operator on minimal submanifolds, see [7]. Here we prove a similar result for the eigenvalue $\mu_{(2)}$ we mentioned above. We obtained the following theorem.

THEOREM 1.6. *Let M^n be a minimally immersed submanifold of $\mathbb{R}^{n+l}$, $l > 1$. Then for each $x_0 \in M$ and $0 < r_1 < r_2$,*

$$\mu_{(2)}(D_{r_2}(x_0) \setminus D_{r_1}(x_0)) \geqslant \mu_{(2)}(V_{r_2,0} \setminus V_{r_1,0}), \tag{1.13}$$

where $D_r(x_0) := B_r^{n+l}(x_0) \bigcap M$, and $B_r^{n+l}(x_0)$ denotes the $(n+l)$-dimensional round ball with center x_0 and radius r in $\mathbb{R}^{n+l}$, and $V_{r,0}$ stands for the ball of radius r, centered at the origin in $\mathbb{R}^n$. Moreover if the equality holds, then $D_{r_2}(x_0) \setminus D_{r_1}(x_0)$ is isometric to $V_{r_2,0} \setminus V_{r_1,0}$.

Recall that the Dirichlet eigenvalue of p-Laplacian is defined as

$$\lambda_{(p),0}(D_2) = \inf\left\{ \frac{\int_{D_2} |\nabla u|^p \, dx}{\int_{D_2} |u|^p \, dx} : u \in W^{1,p}(D_2) \setminus 0 \text{ with } u|_{\partial D_2} = 0 \right\}. \tag{1.14}$$

Finally, we also study the relationships between the Dirichlet eigenvalue $\lambda_{(p),0}(D_2)$ and $\lambda_{(p)}(D_2 \setminus D_1)$. We obtain certain estimates on $\lambda_{(p)}(D_2 \setminus D_1)$ via $\lambda_{(p),0}(D_2)$, implying the convergence result as the hole goes to vanishing.

THEOREM 1.7. *(1) As the volume of D_1 goes to vanishing, we have the following sharp estimate*

$$\lambda_{(p)}(D_2 \setminus D_1) \leqslant \lambda_{(p),0}(D_2) + C_1 \operatorname{Vol}(D_1). \tag{1.15}$$

(2) Assume that D_1 is star-shaped. Then as the diameter of D_1 goes to zero,

$$\lambda_{(p),0}(D_2) \leqslant \lambda_{(p)}(D_2 \setminus D_1) + C_2 \operatorname{Area}(\partial D_1). \tag{1.16}$$

Where C_1 and C_2 are constants independent of D_1.

REMARK 1.8. We can see from (1.15) and (1.16) that as D_1 goes to vanishing, $\lambda_{(p)}(D_2 \setminus D_1)$ converges to the Dirichlet eigenvalue of $\lambda_{(p),0}(D_2)$. Thus by taking a limit, we see easily that Theorem 1.2, 1.3, 1.5 and 1.6 all generalize the classic eigenvalue comparisons: Faber-Krahn, Cheng-type inequalities. For $\mu_{(p)}$, it can be checked similarly that $\mu_{(p)}(D_2 \setminus D_1)$ tends to zero as D_1 goes to vanishing, which is the zero eigenvalue of Neumann Laplace operator.

2. Eigenfunctions on space forms

2.1. Notations

One purpose of this paper is to give certain comparisons on eigenvalues for p-Laplace operator on manifolds. We start by setting up the notations.

Let (M,g) be an n-dimensional Riemannian manifold, and ∇ be the Levi-Civita connection corresponding to the metric g. If we denote $S(TM)$ to be the set of smooth vector fields on M, the curvature tensor of the Riemannian metric is then given by

$$\operatorname{Rm}(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

for $X, Y, Z \in S(TM)$. The sectional curvature of the 2-plane section spanned by a pair of orthonormal vectors X and Y are defined by

$$\mathrm{Sect}_{\mathrm{g}}(X,Y) = \langle \mathrm{Rm}(X,Y)X, Y\rangle.$$

If we take $\{e_1, e_2, \cdots, e_n\}$ to be an orthonormal basis of the tangent space of M, then the Ricci curvature is defined to be the symmetric 2-tensor given by

$$R_{ij} = \sum_{k=1}^{n} \langle \mathrm{Rm}(e_i, e_k)e_j, e_k\rangle.$$

2.2. Volume elements

On Riemannian manifold (M^n, g) it is convenient to use the polar coordinate (t, θ) at the center o, where θ is the standard parametrization of $\mathbb{S}^{n-1}$. For $\theta \in \mathbb{S}^{n-1}$, $l(\theta)$ is so defined that geodesics $\gamma(s) = \exp_o(s\theta)$ are minimizers up to $s = l(\theta)$. In terms of polar coordinates (t, θ), we write the volume element as

$$dx = J(r, \theta)\, dr \wedge d\theta.$$

Denote by M^n_κ the space form, which is a complete and simply connected Riemannian manifold with constant sectional curvature κ (for example, Hyperbolic space $\mathbb{H}^n$ with $\kappa = -1$, Euclidean space $\mathbb{R}^n$ with $\kappa = 0$ and Euclidean sphere $\mathbb{S}^n$ with $\kappa = 1$). Denote by $d\bar{x}$ the volume element in M^n_κ at x, then it is well-known that

$$d\bar{x} = J_\kappa(r(x))\, dr \wedge d\theta = s_\kappa^{n-1}(r)\, dr \wedge d\theta,$$

where $r(x)$ denotes distance away from the fixed center $\bar{o}$, and $s_\kappa(r)$ is given by

$$s_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t, & \kappa > 0, \\ t, & \kappa = 0, \\ \frac{1}{\sqrt{|\kappa|}} \sinh \sqrt{|\kappa|} t, & \kappa < 0. \end{cases}$$

Now we collect two well-known comparisons on the volume elements those will be used in this paper.

THEOREM 2.1. *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric}_g \geqslant (n-1)\kappa g$. *Then*

$$\frac{\partial_r J(r,\theta)}{J(r,\theta)} \leqslant \frac{J'_\kappa(r)}{J_\kappa(r)}$$

within the cut-locus of the origin o in M.

THEOREM 2.2. *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Sect}_g \leqslant \kappa$. *Then*

$$\frac{\partial_r J(r,\theta)}{J(r,\theta)} \geqslant \frac{J'_\kappa(r)}{J_\kappa(r)}$$

within the cut-locus of the origin o in M.

The above two theorems are direct consequences of Bishop-Gromov comparison and Hessian comparison respectively, see for example [12].

2.3. Eigenfunctions on space forms

It is known that the first Dirichlet eigenfunction for p-Laplace operator on the geodesic ball $V_{r,\kappa}$ is a radial function, since the first eigenspace is simple [4, 13, 14] and $V_{r,\kappa}$ is radially symmetric. Then we denote by $R(r(x))$ and $Q(r(x))$ the normalized eigenfunctions corresponding to $\lambda_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa})$ and $\mu_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa})$ respectively. For $R(r)$ and $Q(r)$, we have the following properties.

PROPOSITION 2.1. *For functions $R(r)$ and $Q(r)$ mentioned above, we have following properties.*

(1) $R'(r)<0$ for $r\in(r_1,r_2]$, and

$$-(p-1)(-R')^{p-2}R''(r)-\frac{c_\kappa(r)}{s_\kappa(r)}(-R')^{p-2}R'(r)=\lambda_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa})\,R^{p-1}(r) \quad (2.1)$$

in (r_1,r_2) with $R'(r_1)=0$ and $R(r_2)=0$.

(2) $Q'(r)>0$ for $r\in[r_1,r_2)$, and

$$-(p-1)(Q')^{p-2}Q''(r)-\frac{c_\kappa(r)}{s_\kappa(r)}(Q')^{p-2}Q'(r)=\mu_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa})\,Q^{p-1}(r) \quad (2.2)$$

for $r\in(r_1,r_2)$ with $Q(r_1)=0$ and $Q'(r_2)=0$. Where

$$c_\kappa(t)=\begin{cases}\cos\sqrt{\kappa}t, & \kappa>0,\\ 1, & \kappa=0,\\ \cosh\sqrt{|\kappa|}t, & \kappa<0.\end{cases}$$

Proof. In polar coordinates, it follows from PDE (1.3) that

$$-(p-1)(-R')^{p-2}R''(r)-\frac{c_\kappa(r)}{s_\kappa(r)}(-R')^{p-2}R'(r)=\lambda_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa})\,R^{p-1}(r).$$

Now we are in position to show $R'(r)<0$ for $r\in(r_1,r_2]$. Assume by contradiction that there exists $r_0\in(r_1,r_2]$ such that $R'(r_0)=0$, then we define a testing function $\overline{R}$ on $V_{r_2,\kappa}\setminus V_{r_1,\kappa}$ by

$$\overline{R}(r)=R(r),\ \text{for } r\in[r_0,r_2),$$

and

$$\overline{R}(r)=R(r_0),\ \text{for } r\in[r_1,r_0).$$

It follows that

$$\begin{aligned}\frac{\int_{V_{r_2,\kappa}\setminus V_{r_1,\kappa}}|\nabla\overline{R}|^p\,dx}{\int_{V_{r_2,\kappa}\setminus V_{r_1,\kappa}}|\overline{R}|^p\,dx}&=\frac{\int_{V_{r_2,\kappa}\setminus V_{r_0,\kappa}}|\nabla R|^p\,dx}{\int_{V_{r_2,\kappa}\setminus V_{r_1,\kappa}}|\overline{R}|^p\,dx}\\&<\frac{\int_{V_{r_2,\kappa}\setminus V_{r_0,\kappa}}|\nabla R|^p\,dx}{\int_{V_{r_2,\kappa}\setminus V_{r_0,\kappa}}|R|^p\,dx}\\&=\lambda_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa}),\end{aligned} \quad (2.3)$$

where last equality follows from equation (2.1), and boundary condition $R'(r_0) = R(r_2) = 0$. Then we get a contradiction with the definition of $\lambda_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa})$, hence $R'(r) < 0$ holds for $r \in (r_1, r_2]$.

To prove that $Q'(r) > 0$ for $r \in [r_1, r_2)$, we assume by contradiction that there exists $r_0 \in (r_1, r_2)$ such that $Q'(r_0) = 0$. On one hand from equation (2.2) and boundary conditions $Q'(r_0) = 0$ and $Q(r_1) = 0$, it follows

$$\frac{\int_{V_{r_0,\kappa} \setminus V_{r_1,\kappa}} |\nabla Q|^p \, dx}{\int_{V_{r_0,\kappa} \setminus V_{r_1,\kappa}} |Q|^p \, dx} = \mu_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa}),$$

which immediately implies

$$\mu_{(p)}(V_{r_0,\kappa} \setminus V_{r_1,\kappa}) \leqslant \mu_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa}).$$

On the other hand, it follows from the simlar argument as (2.3) that

$$\mu_{(p)}(V_{r_0,\kappa} \setminus V_{r_1,\kappa}) > \mu_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa}),$$

giving the contradiction. Thus $Q'(r) > 0$ for $r \in [r_1, r_2)$.

REMARK 2.3. In fact, $\lambda_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa})$ and $\mu_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa})$ are given by roots of some Bessel equations, see [9, Section 3.2].

3. Proof of Theorem 1.2

In this section, we make use of Schwartz spherical rearrangement argument and co-area formula (see also [10, 15]) to prove Theorem 1.2. The proof is similar to that of the classic Faber-Krahn inequality. But for the completeness, we give the details here.

Proof of Theorem 1.2. Suppose u is the normalized eigenfunction corresponding to the eigenvalue $\lambda_{(p)}(\Omega)$, where $\Omega = D_2 \setminus D_1$. Let

$$a_0 = \max_{C(\overline{\Omega})} u > 0,$$

$$\Omega(t) = \{x \in \Omega : u(x) > t\},$$

and

$$\Gamma_t = \{x \in \Omega : u(x) = t\}.$$

For $t \in [0, a_0]$, let $D^*(t)$ be the ball centered at the origin such that

$$\mathrm{Vol}(D^*(t)) = \mathrm{Vol}(\Omega(t)) + \mathrm{Vol}(D_1), \tag{3.1}$$

and define a radial function $h(x)$ on $\Omega^* = D^*(0) \setminus D^*(a_0)$ by

$$h(x) = \sup\{t : x \in D^*(t)\}.$$

Denote $\Omega^*(t)=\{x\in\Omega^*: h(x)>t\}$ and $\Gamma_t^*=\partial D^*(t)$. From the definition of $\Omega^*(t)$, we see that

$$\mathrm{Vol}(\Omega^*(t))=\mathrm{Vol}(\Omega(t)).$$

Using the co-area formula, we rewrite the above identity as

$$\int_t^{a_0}\int_{\Gamma_r}\frac{1}{|\nabla u|}\,dA_r dr=\int_t^{a_0}\int_{\Gamma_r^*}\frac{1}{|\nabla h|}\,dA_r^* dr,$$

where dA_r and dA_r^* denote the area element of Γ_r and Γ_r^* respectively. Taking the derivative of t yields

$$\int_{\Gamma_t}\frac{1}{|\nabla u|}\,dA_t=\int_{\Gamma_t^*}\frac{1}{|\nabla h|}\,dA_t^*. \tag{3.2}$$

Using the co-area formula again, we obtain

$$\begin{aligned}\int_\Omega u^p\,dx&=\int_0^{a_0}t^p\int_{\Gamma_t}\frac{1}{|\nabla u|}\,dA_t dt\\&=\int_0^{a_0}t^p\int_{\Gamma_t^*}\frac{1}{|\nabla h|}\,dA_t^* dt\\&=\int_{\Omega^*}h^p\,dx.\end{aligned} \tag{3.3}$$

It follows from (3.1) and the isoperimetric inequality that

$$\mathrm{Area}(\Gamma_t^*)\leqslant\mathrm{Area}(\Gamma_t). \tag{3.4}$$

Using Hölder inequality, we have

$$\mathrm{Area}(\Gamma_t)=\int_{\Gamma_t}dA_t\leqslant\Big(\int_{\Gamma_t}\frac{1}{|\nabla u|}\,dA_t\Big)^{\frac{p-1}{p}}\Big(\int_{\Gamma_t}|\nabla u|^{p-1}\,dA_t\Big)^{\frac{1}{p}}. \tag{3.5}$$

Since h is a radial function, we conculde

$$\mathrm{Area}(\Gamma_t^*)=\int_{\Gamma_t^*}dA_t^*=\Big(\int_{\Gamma_t^*}\frac{1}{|\nabla h|}\,dA_t^*\Big)^{\frac{p-1}{p}}\Big(\int_{\Gamma_t^*}|\nabla h|^{p-1}\,dA_t^*\Big)^{\frac{1}{p}}. \tag{3.6}$$

Putting (3.2), (3.4–3.6) together, we arrive at

$$\int_{\Gamma_t^*}|\nabla h|^{p-1}\,dA_t^*\leqslant\int_{\Gamma_t}|\nabla u|^{p-1}\,dA_t,$$

giving

$$\int_{\Omega^*}|\nabla h|^p\,dx\leqslant\int_\Omega|\nabla u|^p\,dx. \tag{3.7}$$

Hence by (3.3) and (3.7), we get

$$\lambda_{(p)}(\Omega^*)\leqslant\frac{\int_{\Omega^*}|\nabla h|^p\,dx}{\int_{\Omega^*}h^p\,dx}\leqslant\frac{\int_\Omega|\nabla u|^p\,dx}{\int_\Omega u^p\,dx}=\lambda_{(p)}(\Omega).$$

When equality holds, we see from (3.4) that Γ_t is a round sphere for each t. Hence Ω is isometric to Ω^*.

One can follow the same argument as above to show

$$\mu_{(p)}(\Omega^*) \leqslant \mu_{(p)}(\Omega),$$

and we omit the details here. □

REMARK 3.1. The proof of Theorem 1.3 is using the Gromov's isoperimetric inequality [2] and following the similar process as that of Theorem 1.2, and we omit it here.

4. Proof of Theorem 1.5

In this section, we combine Cheng's argument of transplanting and applying the comparison theorems, Theorem 2.1 and 2.2, to show Cheng-type eigenvalue comparison theorems.

4.1. Proof of (1.9)

For $0 < r_1 < r_2$, denote by u_0 the normalized eigenfunction with respect to eigenvalue $\lambda_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa})$. For the sake of convenience, we denote $\lambda_{(p)}(V_{r_2,\kappa} \setminus V_{r_1,\kappa})$ by $\lambda_{(p),\kappa}$. Recall from (1) of Proposition 2.1 that $u_0 = R(r(x))$ is a radial function satisfying

$$-\mathrm{div}(|\nabla u_0|^{p-2}\nabla u_0) = \lambda_{(p),\kappa}\, |u_0|^{p-2}u_0$$

for $r \in (r_1, r_2)$ and boundary conditions

$$R'(r_1) = 0, \quad R(r_2) = 0, \quad \text{and} \quad R'(r) < 0 \quad \text{with} \quad r \in (r_1, r_2].$$

We define a trial function $u(x)$ on $B_{r_2}(0) \setminus B_{r_1}(0)$ as

$$u(x) = R(d(x)), \tag{4.1}$$

where $d(x)$ is the distance function originated from o in M.

To prove (1.9), it suffices to check that for the testing function u

$$\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla u|^p \, dx \leqslant \lambda_{(p),\kappa} \int_{B_{r_2}(0)\setminus B_{r_1}(0)} |u|^p \, dx.$$

Observing that

$$J(r,\theta)\nabla u = \nabla\Big(\frac{Ju}{J_\kappa}\Big)J_\kappa - \frac{\nabla J}{J_\kappa}J_\kappa u + \frac{\nabla J_\kappa}{J_\kappa}Ju, \tag{4.2}$$

we have that

$$\begin{aligned}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla u|^p\,dx &= \int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} |\nabla u|^p J(r,\theta)\,drd\theta\\ &= \int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} |\nabla u|^{p-2}\Big\langle \nabla u, \nabla\Big(\frac{Ju}{J_\kappa}\Big)\Big\rangle J_\kappa\,drd\theta\\ &\quad+\int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} |\nabla u|^{p-2}\Big(-u_r\Big(\frac{\partial_r J}{J}-\frac{J'_\kappa}{J_\kappa}\Big)\Big)uJ\,drd\theta\end{aligned} \tag{4.3}$$

where $l(\theta)$ is defined in Section 2.2.

Since $\mathrm{Ric}_g \geqslant (n-1)\kappa g$, Theorem 2.1 gives

$$\frac{\partial_r J(r,\theta)}{J(r,\theta)} \leqslant \frac{J'_\kappa(r)}{J_\kappa(r)}, \tag{4.4}$$

combining with $u_r = R'(r) < 0$, we see

$$-u_r\Big(\frac{\partial_r J}{J}-\frac{J'_\kappa}{J_\kappa}\Big) \leqslant 0. \tag{4.5}$$

It follows from (4.3–4.5) that

$$\begin{aligned}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla u|^p\,dx &\leqslant \int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} |\nabla R|^{p-2}\Big\langle \nabla R, \nabla\Big(\frac{JR}{J_\kappa}\Big)\Big\rangle J_\kappa\,drd\theta\\ &= \int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} |\nabla u_0|^{p-2}\Big\langle \nabla u_0, \nabla\Big(\frac{Ju_0}{J_\kappa}\Big)\Big\rangle\,d\bar{x}\\ &= \int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} -\mathrm{div}(|\nabla u_0|^{p-2}\nabla u_0)\frac{Ju_0}{J_\kappa}\,d\bar{x}\\ &\quad+\int_{\{\theta: r_1<l(\theta)<r_2\}} |\nabla u_0|^{p-2}\frac{\partial u_0}{\partial r}(l(\theta),\theta)u_0(l(\theta),\theta)J(l(\theta),\theta)d\theta.\end{aligned}$$

From Proposition 2.1, it follows that for $l(\theta)\in(r_1,r_2)$

$$\frac{\partial u_0}{\partial r}(l(\theta),\theta) = R'(l(\theta)) \leqslant 0,$$

then it follows that

$$\begin{aligned}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla u|^p\,dx &\leqslant \int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} -\mathrm{div}(|\nabla u_0|^{p-2}\nabla u_0)\frac{Ju_0}{J_\kappa}\,d\bar{x}\\ &= \int_{\mathbb{S}^{n-1}}\int_{\min\{r_1,l(\theta)\}}^{\min\{r_2,l(\theta)\}} \lambda_{(p),\kappa}\,|R|^p J\,drd\theta\\ &= \lambda_{(p),\kappa}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |u|^p\,dx,\end{aligned}$$

proving the claim, hence the inequality (1.9).

When equality holds, from (4.4) we know that $\frac{\partial_r J(r,\theta)}{J(r,\theta)} = \frac{J'_\kappa(r)}{J_\kappa(r)}$. Hence we get that $B_{r_2}(0)$ is isometric to $V_{r_2,\kappa}$ (see for example [15]).

4.2. Proof of (1.11)

Now we assume that sectional curvature of M is bounded from above by κ, then Theorem 2.2 gives that

$$\frac{\partial_r J(r,\theta)}{J(r,\theta)} \geqslant \frac{J_\kappa'(r)}{J_\kappa(r)}. \tag{4.6}$$

To prove the theorem, we claim firstly for u given by (4.1) that

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \geqslant \lambda_{(p),\kappa} u^{p-1}$$

in $B_{r_2}(0)\setminus B_{r_1}(0)$ in the weak sense, that is for each nonnegative function $\eta \in W^{1,p}(B_{r_2}(0)\setminus B_{r_1}(0))$ with $\eta = 0$ on $\partial B_{r_2}(0)$

$$\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla u|^{p-2}\langle \nabla u, \nabla \eta\rangle \, dx \geqslant \lambda_{(p),\kappa}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |u|^{p-1}\eta \, dx. \tag{4.7}$$

Using (4.2) again, we calculate that

$$\begin{aligned}
\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla u|^{p-2}\langle \nabla u, \nabla \eta\rangle \, dx &= \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla u|^{p-2}\langle \nabla u, \nabla \eta\rangle J(r,\theta)\, d\theta dr \\
&= \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla u|^{p-2}\Big\langle \nabla u, \nabla\Big(\frac{J\eta}{J_\kappa}\Big)\Big\rangle J_\kappa \, d\theta dr \\
&\quad + \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla u|^{p-2}\Big(-u_r\Big(\frac{\partial_r J}{J} - \frac{J_\kappa'}{J_\kappa}\Big)\Big)\eta J \, d\theta dr \\
&\geqslant \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla u|^{p-2}\Big\langle \nabla u, \nabla\Big(\frac{J\eta}{J_\kappa}\Big)\Big\rangle J_\kappa \, d\theta dr \\
&= \int_{V_{r_2,\kappa}\setminus V_{r_1,\kappa}} |\nabla u_0|^{p-2}\Big\langle \nabla u_0, \nabla\Big(\frac{J\eta}{J_\kappa}\Big)\Big\rangle \, d\bar{x} \\
&= \lambda_{(p),\kappa}\int_{V_{r_2,\kappa}\setminus V_{r_1,\kappa}} |u_0|^{p-1}\eta\frac{J}{J_\kappa}\, d\bar{x} \\
&= \lambda_{(p),\kappa}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |u|^{p-1}\eta \, dx,
\end{aligned}$$

where we used that (4.6) and $\frac{\partial}{\partial r}u = R'(r) < 0$, which proves (4.7), hence the claim.

Now we turn to prove (1.11). Let $g(x)$ be the normalized eigenfunction with respect to eigenvalue $\lambda_{(p)}(B_{r_2}(0)\setminus B_{r_1}(0))$ and choose a testing function as

$$\eta(x) = \frac{g^p}{u^{p-1}},$$

which is clearly nonnegative in $W^{1,p}(B_{r_2}(0)\setminus B_{r_1}(0))$ with $\eta = 0$ on $\partial B_{r_2}(0)$, see [16, Appendix A].

Substituting $\eta(x)$ to inequality (4.7), we deduce

$$\begin{aligned}\lambda_{(p),\kappa}\int_{B_{r_2}(0)\backslash B_{r_1}(0)} g^p\,dx &\leqslant \int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla u|^{p-2}\Big\langle \nabla u, \nabla \frac{g^p}{u^{p-1}}\Big\rangle\,dx\\ &= -(p-1)\int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla u|^p \frac{g^p}{u^p}\,dx\\ &\quad + p\int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla u|^{p-2}\langle \nabla u, \nabla g\rangle \frac{g^{p-1}}{u^{p-1}}\,dx. \end{aligned} \tag{4.8}$$

Using Young inequality, we have that

$$\begin{aligned}&\int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla g|^p\,dx + (p-1)\int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla u|^p \frac{g^p}{u^p}\,dx\\ \geqslant\ & p\int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla u|^{p-2}\langle \nabla u, \nabla g\rangle \frac{g^{p-1}}{u^{p-1}}\,dx. \end{aligned} \tag{4.9}$$

Putting (4.8) and (4.9) together, we conclude that

$$\lambda_{(p),\kappa}\int_{B_{r_2}(0)\backslash B_{r_1}(0)} g^p\,dx \leqslant \int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla g|^p\,dx,$$

which gives (1.11).

4.3. Proof of (1.10)

The proof of (1.10) is similar to that of (1.11). Let v_0 be the normalized eigenfunction with respect to eigenvalue $\mu_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa})$. For the sake of convenience, we denote $\mu_{(p)}(V_{r_2,\kappa}\setminus V_{r_1,\kappa})$ by $\mu_{(p),\kappa}$. Recall from (2) of Proposition 2.1 that $v_0 = Q(r(x))$ is a radial function satisfying

$$-\mathrm{div}(|\nabla v_0|^{p-2}\nabla v_0) = \mu_{(p),\kappa}\,|v_0|^{p-2}v_0$$

for $r\in(r_1,r_2)$ and boundary conditions

$$Q(r_1)=0,\quad Q'(r_2)=0,\quad \text{and}\quad Q'(r)>0\quad \text{with}\quad r\in(r_1,r_2].$$

Define a trial function $v(x)$ on $B_{r_2}(0)\setminus B_{r_1}(0)$ as

$$v(x) = Q(d(x)), \tag{4.10}$$

where $d(x)$ is the distance function originated from o in M.

Firstly, we show for v given by (4.10) that

$$-\mathrm{div}(|\nabla v|^{p-2}\nabla v) \geqslant \mu_{(p),\kappa} v^{p-1}$$

in $B_{r_2}(0)\setminus B_{r_1}(0)$ in the weak sense, that is for each nonnegative function $\eta\in W^{1,p}(B_{r_2}(0)\setminus B_{r_1}(0))$ with $\eta=0$ on $\partial B_{r_1}(0)$

$$\int_{B_{r_2}(0)\backslash B_{r_1}(0)} |\nabla v|^{p-2}\langle \nabla v, \nabla \eta\rangle\,dx \geqslant \mu_{(p),\kappa}\int_{B_{r_2}(0)\backslash B_{r_1}(0)} |v|^{p-1}\eta\,dx. \tag{4.11}$$

In fact, using (4.2) again, we calculate that

$$\begin{aligned}
\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla v|^{p-2}\langle \nabla v, \nabla \eta\rangle\, dx &= \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla v|^{p-2}\langle \nabla v, \nabla \eta\rangle J(r,\theta)\, d\theta dr\\
&= \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla v|^{p-2}\Big\langle \nabla v, \nabla\Big(\frac{J\eta}{J_\kappa}\Big)\Big\rangle J_\kappa\, d\theta dr\\
&\quad + \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla v|^{p-2}\Big(-v_r\Big(\frac{\partial_r J}{J}-\frac{J'_\kappa}{J_\kappa}\Big)\Big)\eta J\, d\theta dr\\
&\geqslant \int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} |\nabla v|^{p-2}\Big\langle \nabla v, \nabla\Big(\frac{J\eta}{J_\kappa}\Big)\Big\rangle J_\kappa\, d\theta dr\\
&= \int_{V_{r_2,\kappa}\setminus V_{r_1,\kappa}} |\nabla v_0|^{p-2}\Big\langle \nabla v_0, \nabla\Big(\frac{J\eta}{J_\kappa}\Big)\Big\rangle\, d\bar{x}\\
&= \mu_{(p),\kappa}\int_{V_{r_2,\kappa}\setminus V_{r_1,\kappa}} |v_0|^{p-1}\eta\frac{J}{J_\kappa}\, d\bar{x}\\
&= \mu_{(p),\kappa}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |v|^{p-1}\eta\, dx,
\end{aligned}$$

where we used that (4.4) and $\frac{\partial}{\partial r}v = Q'(r) > 0$, which proves (4.11).

Now we turn to prove (1.10). Let $h(x)$ be the normalized eigenfunction with respect to eigenvalue $\mu_{(p)}(B_{r_2}(0)\setminus B_{r_1}(0))$ and choose a testing function as

$$\eta(x) = \frac{h^p}{v^{p-1}},$$

which is nonnegative in $W^{1,p}(B_{r_2}(0)\setminus B_{r_1}(0))$ with $\eta = 0$ on $\partial B_{r_1}(0)$. Substituting $\eta(x)$ to inequality (4.11), we deduce

$$\begin{aligned}
\mu_{(p),\kappa}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} h^p\, dx &\leqslant \int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla v|^{p-2}\Big\langle \nabla v, \nabla\frac{h^p}{v^{p-1}}\Big\rangle\, dx\\
&= -(p-1)\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla v|^p\frac{h^p}{v^p}\, dx\\
&\quad + p\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla v|^{p-2}\langle \nabla v, \nabla h\rangle\frac{h^{p-1}}{v^{p-1}}\, dx. \qquad (4.12)
\end{aligned}$$

Using Young inequality, we have that

$$\begin{aligned}
&\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla h|^p\, dx + (p-1)\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla v|^p\frac{h^p}{v^p}\, dx\\
&\geqslant p\int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla v|^{p-2}\langle \nabla v, \nabla h\rangle\frac{h^{p-1}}{v^{p-1}}\, dx. \qquad (4.13)
\end{aligned}$$

From (4.12) and (4.13) we deduce that

$$\mu_{(p),\kappa}\int_{B_{r_2}(0)\setminus B_{r_1}(0)} h^p\, dx \leqslant \int_{B_{r_2}(0)\setminus B_{r_1}(0)} |\nabla h|^p\, dx,$$

which gives (1.10).

REMARK 4.1. Since the proof of (1.12) is almost the same as that of (1.9), we omit the details here.

5. Proof of Theorem 1.6

In this section, we assume that M^n is a minimally immersed manifold of $\mathbb{R}^{n+l}$, $l>1$. For fixed $o \in M^n$, we denote by $d_o(x)=|o-x|$ the distance function in $\mathbb{R}^{n+l}$. Denote by ∇ and $\triangle$ the covariant derivative and Laplace operator on M^n respectively. A comparison inequality of Dirichlet eigenvalue was proved by Cheng, Li and Yau, see [7, Corollary 3]. Now we use Cheng's argument of transplanting to prove the analogous inequality for μ_2. By minimality of M in $\mathbb{R}^{n+l}$, we observe that (c.f. [7])

$$\triangle d_o^2(x)=2n, \tag{5.1}$$

for any $x\in M$. Now we prove Theorem 1.6.

Proof of Theorem 1.6. Recall that $D(o,r)=\{x\in M, d_o(x)<r\}$. Define a trial function $u(x)$ on $D(o,r_2)\setminus D(o,r_1)$ by

$$u(x)=Q(d_o(x)),$$

where $Q(x)$ is the normalized eigenfunction for $V_{r_2,0}\setminus V_{r_1,0}$ in $\mathbb{R}^n$, characterized in Proposition 2.1. We see from (5.1) that

$$\begin{aligned}\triangle u(x) &= Q''(d_o(x))|\nabla d_o(x)|^2+Q'(d_o(x))\triangle d_o(x)\\ &=\Big(Q''-\frac{Q'}{d_o(x)}\Big)|\nabla d_o(x)|^2+n\frac{Q'}{d_o(x)}.\end{aligned}$$

Now we claim that $f(r):=Q''-\frac{Q'}{r}\leqslant 0$ for $r\in(r_1,r_2)$. Then from the fact $|\nabla d|\leqslant 1$, we deduce that

$$\triangle u(x)\geqslant\Big(Q''-\frac{Q'}{d_o(x)}\Big)+n\frac{Q'}{d_o(x)}=-\mu_{(2)}(V_{r_2,0}\setminus V_{r_1,0})\,u(x),$$

where we used equation (2.2) for the case $p=2$. Then we conclude that

$$\mu_{(2)}(D(o,r_2)\setminus D(o,r_1))\leqslant\mu_{(2)}(V_{r_2,0}\setminus V_{r_1,0}),$$

proving the theorem. When equality holds, we see that $|\nabla d_o|=1$. Combining with (5.1), we get that $D(o,r_2)\setminus D(o,r_1)$ is totally in $\mathbb{R}^n$.

Now we prove the claim. If $f(t)\geqslant 0$, then

$$-\mu_{(2)}(V_{r_2,0}\setminus V_{r_1,0})\,Q\geqslant\frac{n}{t}Q'.$$

Then we deduce

$$\begin{aligned}f'(t)&=-\mu_{(2)}(V_{r_2,0}\setminus V_{r_1,0})\,Q'+\frac{n}{t^2}Q'-\frac{n}{t}Q''\\ &=-\mu_{(2)}(V_{r_2,0}\setminus V_{r_1,0})\,Q'+\frac{n}{t^2}Q'+\frac{n}{t}\Big(\frac{n-1}{t}Q'+\mu_{(2)}(V_{r_2,0}\setminus V_{r_1,0})\,Q\Big)\\ &\leqslant-\mu_{(2)}(V_{r_2,0}\setminus V_{r_1,0})\,Q'\\ &\leqslant 0.\end{aligned}$$

Observing $f(r_1) = -\frac{n}{r_1}Q'(r_1) < 0$, then we conclude $f(r) \leqslant 0$ for $r \in (r_1, r_2)$. This proves the claim. $\square$

6. Proof of Theorem 1.7

Let $D_\varepsilon \subset D_2$ in $\mathbb{R}^n$. In this section, we denote $\lambda_{(p)}(D_2 \setminus D_\varepsilon)$ and $\mu_{(p)}(D_2 \setminus D_\varepsilon)$ by $\lambda_{(p)}(\varepsilon)$ and $\mu_{(p)}(\varepsilon)$ respectively.

6.1. Proof of estimate (1.15)

Let u_ε be the corresponding normalized eigenfunction of $D_2 \setminus D_\varepsilon$ with eigenvalue $\lambda_{(p)}(\varepsilon)$ and D_ε goes to empty as $\varepsilon \to 0$. Denote by u_0 the normalized eigenfunction on D_2 with respect to the first Dircichlet eigenvalue $\lambda_{(p),0}$ defined in (1.14). It is obviously that

$$|u_0|_{C^1(\overline{D_2})} \leqslant C.$$

Using u_0 as a trial function for $\lambda_{(p)}(\varepsilon)$, we get

$$\begin{aligned}
\lambda_{(p)}(\varepsilon) &\leqslant \frac{\int_{D_2 \setminus D_\varepsilon} |\nabla u_0|^p \, dx}{\int_{D_2 \setminus D_\varepsilon} |u_0|^p \, dx} \\
&\leqslant \frac{\int_{D_2} |\nabla u_0|^p \, dx}{\int_{D_2} |u_0|^p \, dx - \int_{D_\varepsilon} |u_0|^p \, dx} \\
&\leqslant \frac{\lambda_{(p),0}}{1 - C \operatorname{Vol}(D_\varepsilon)} \\
&\leqslant \lambda_{(p),0} + C \operatorname{Vol}(D_\varepsilon),
\end{aligned} \tag{6.1}$$

where C is a constant independent of ε.

6.2. The uniform bound for u_ε

To prove the lower bound of $\lambda_{(p)}(\varepsilon)$, we need that u_ε is uniformly bounded, which means the bound is independent of ε as $\varepsilon \to 0$. More precisely, we prove the following lemma.

LEMMA 6.1.

$$\max_{x \in \partial D_\varepsilon} u_\varepsilon \leqslant C, \tag{6.2}$$

where C is a constant which is independent of ε.

Proof of Lemma 6.1. Choose a fixed small ball $B_{2\eta}(x_0) \subset D_2$ centered at x_0 with radius 2η such that $D_\varepsilon \subset B_\eta(x_0)$ and $\lambda_{(p),0}(B_{2\eta}(x_0)) \geqslant \lambda_{(p)}(\varepsilon) + 1$. This can be done due to (1.15).

Let $w(x)$ be the corresponding normalized Dirichlet eigenfunction. Define a function $h(x)$ in $B_{2\eta}(x_0) \setminus D_\varepsilon$ by

$$h(x) = \frac{u_\varepsilon}{w}.$$

First we claim that $h(x)$ can not attain the maximum in the interior of $B_\eta(x_0)\setminus D_\varepsilon$. Assume by contradiction that $h(x)$ attains a maximum at $x_1\in B_\eta(x_0)\setminus D_\varepsilon$. Then at x_1 we have that

$$\frac{\nabla u_\varepsilon}{u_\varepsilon}=\frac{\nabla w}{w}\neq 0.$$

Thus in a small neighborhood of x_1, equation (1.3) is strictly elliptic. That means in a neighborhood of x_1, u_ε is twice differentiable and satisfies

$$-\langle\nabla|\nabla u_\varepsilon|^{p-2},\nabla u_\varepsilon\rangle-|\nabla u_\varepsilon|^{p-2}\triangle u_\varepsilon=\lambda_{(p)}(\varepsilon)\,u_\varepsilon^{p-1}.$$

Using $\nabla h(x_1)=0$, we have that at x_1

$$-\mathrm{div}(|\nabla w|^{p-2}\nabla w)h-|\nabla w|^{p-2}w\left(\triangle h+(p-2)\nabla^2 h\Big(\frac{\nabla w}{|\nabla w|},\frac{\nabla w}{|\nabla w|}\Big)\right)=\lambda_{(p)}(\varepsilon)hw^{p-1}.$$

Since $\lambda_{(p),0}(B_o(2\eta))>\lambda_{(p)}(\varepsilon)$, then we have

$$\triangle h+(p-2)\nabla^2 h\left(\frac{\nabla w}{|\nabla w|},\frac{\nabla w}{|\nabla w|}\right)>0,$$

contradicting with the assumption that x_1 is a maximum point of h. Hence the claim comes true.

Now we use the maximum principle to prove the lemma. Let ν be the outward normal direction of D_ε. Then on ∂D_ε taking the normal derivative of h yields

$$\frac{\partial}{\partial\nu}h=-\frac{h}{w}\frac{\partial}{\partial\nu}w=-\frac{h}{w}\Big\langle\nu,\frac{\partial}{\partial r}\Big\rangle\frac{\partial}{\partial r}w>0,$$

where the last inequality we used that the star-shapeness and w is a radial function satisfying $\frac{\partial}{\partial r}w<0$ for $r>0$ (see for example [14]). Then we conclude that h can not achieve the maximum on ∂D_ε, and therefore

$$\max_{B_\eta(x_0)\setminus D_\varepsilon}\frac{u_\varepsilon}{w}\leqslant\max_{\partial B_\eta(x_0)}\frac{u_\varepsilon}{w}.$$

The standard Moser's iteration argument (see for example [12]) gives that

$$\max_{\partial B_\eta(x_0)}u_\varepsilon\leqslant C.$$

Therefore, we complete the proof of the lemma. □

6.3. Proof of (1.16)

Similar computations as (4.7) and (4.8) give that

$$\begin{aligned}\lambda_{(p),0}\int_{D_2\setminus D_\varepsilon}u_\varepsilon^p\,dx&=-\int_{\partial D_\varepsilon}\frac{u_\varepsilon^p}{u_0^{p-1}}|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial\nu}\,dA_\varepsilon+\int_{D_2\setminus D_\varepsilon}\Big\langle\nabla\frac{u_\varepsilon^{p-1}}{u_0^{p-1}},|\nabla u_0|^{p-2}\nabla u_0\Big\rangle\,dx\\&\leqslant\int_{\partial D_\varepsilon}\frac{u_\varepsilon^p}{u_0^{p-1}}|\nabla u_0|^{p-1}\,dA_\varepsilon+\int_{D_2\setminus D_\varepsilon}|\nabla u_\varepsilon|^p\,dx\\&=O(|\partial D_\varepsilon|)+\int_{D_2\setminus D_\varepsilon}|\nabla u_\varepsilon|^p\,dx,\end{aligned}$$

where we used fact u_ε is bounded from Lemma 6.1. Then we conclude

$$\lambda_{(p)}(\varepsilon) \geqslant \lambda_{(p),0} - O(|\partial D_\varepsilon|).$$

6.4. Proof of the sharpness of (1.15)

We now explain that inequality (1.15) is sharp in the decay rate of the volume of D_ε. By direct calculations, one can show the following variational formula for $\lambda_{(p)}(\varepsilon)$ (see for example [11, Section 6])

$$\dot{\lambda}_{(p)}(\varepsilon) = \int_{\partial D_\varepsilon} (|\nabla u_\varepsilon|^p - \lambda_{(p)}(\varepsilon) u_\varepsilon^p)\langle X, \nu\rangle dA, \tag{6.3}$$

where u_ε is the normalized eigenfunction, $\lambda_{(p)}(\varepsilon)$ is the eigenvalue of $D_2 \setminus D_\varepsilon$ and X is the variation field of ∂D_ε. We consider D_2 as the round unit ball centered at the origin and D_ε as the round ball centered at the origin with radius ε. We see from Proposition 2.1 that u_ε is radial with Neumann condition on ∂D_ε, and then $|\nabla u_\varepsilon| = 0$ on ∂D_ε. Consequently it follows from (6.3) that

$$\dot{\lambda}_{(p)}(\varepsilon) = \int_{\partial D_\varepsilon} (-\lambda_{(p)}(\varepsilon) u_\varepsilon^p)\langle X, \nu\rangle dA = O(|\partial D_\varepsilon|) = O(\varepsilon^{n-1}).$$

Here in the second equality we used the lower bound of $\lambda_{(p)}(\varepsilon)$ given in (1.16). Thus, by elementary calculus we have

$$\lambda_{(p)}(\varepsilon) = \lambda_{(p)}(0) + O(\varepsilon^n) = \lambda_{(p),0} + O(|D_\varepsilon|),$$

which gives the sharpness of (1.15). Where we have used equality $\lambda_{(p)}(0) = \lambda_{(p),0}$, deduced from estimates (1.15) and (1.16).

Acknowledgements. The authors are very grateful to the anonymous referees for the careful reading and the valuable suggestions. The research of this paper is partially supported by NSFC grant No. 11601359, NSF of Jiangsu Province No. BK20160301, NSF of Education Committee of Jiangsu Province grant No. 16KJB110018, and China Postdoctoral Foundation grant No. 2016M591900.

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(Received October 23, 2016)

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