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AN INTERPOLATION OF JENSEN'S INEQUALITY AND ITS APPLICATIONS TO MEAN INEQUALITIES

JADRANKA MIĆIĆ HOT AND YUKI SEO

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Abstract. In this paper, we show operator versions of the inequality due to Cho, Matić and Pečarić in connection to Jensen's inequality for convex functions. As applications, we obtain an interpolation of the weighted arithmetic-geometric mean inequality for the Karcher mean of positive invertible operators on a Hilbert space. Moreover, we obtain an interpolation between the quasi-arithmetic means.

1. Introduction

A classical Jensen's inequality says that if a real valued function $f(t)$ is convex on an interval $[m,M]$, then for each $x_1,\dots,x_n \in [m,M]$ and for each positive real numbers $\omega_1,\dots,\omega_n$ with $\sum_{j=1}^n \omega_j = 1$

$$f\left(\sum_{j=1}^n \omega_j x_j\right) \leqslant \sum_{j=1}^n \omega_j f(x_j). \tag{1.1}$$

In [1], Cho, Matić and Pečarić considered the map which connects both sides of Jensen's inequality (1.1). It is a generalization of the result with respect to the Hadamard-Hermite inequality due to Dragomir [2]. We state the following discrete case of [1, Theorem 2.1]:

THEOREM A. *If a real valued function $f(t)$ is convex on an interval $[m,M]$, then for each $x_1,\dots,x_n \in [m,M]$ and for each positive real numbers $\omega_1,\dots,\omega_n$ with $\sum_{j=1}^n \omega_j = 1$*

$$f\left(\sum_{j=1}^n \omega_j x_j\right) \leqslant \sum_{j=1}^n \omega_j f\left(tx_j + (1-t)\sum_{k=1}^n \omega_k x_k\right) \leqslant \sum_{j=1}^n \omega_j f(x_j)$$

for all $t \in [0,1]$. Moreover, the function $F : [0,1] \mapsto \mathbb{R}$ defined by

$$F(t) = \sum_{j=1}^n \omega_j f\left(tx_j + (1-t)\sum_{k=1}^n \omega_k x_k\right),$$

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is monotonically nondecreasing and convex on $[0,1]$.

Let $\mathscr{B}(H)$ (resp. $\mathscr{B}_h(H)$) be the set of all bounded linear operators (resp. all selfadjoint operators) on a Hilbert space H. A real valued continuous function f defined on an interval $[m,M]$ is said to be operator convex if $f(tA+(1-t)B) \leqslant tf(A)+(1-t)f(B)$ for all selfadjoint operators A,B in $\mathscr{B}_h(H)$ with $m \leqslant A,B \leqslant M$. We recall the Davis-Choi-Jensen inequality for operator convex functions [3, Theorem 1.20], which is regarded as a noncommutative version of Jensen's inequality (1.1): *Let* $\Phi : \mathscr{B}(H) \mapsto \mathscr{B}(K)$ *be a unital positive linear map and* f *an operator convex function on an interval* $[m,M]$. *Then*

$$f(\Phi(A)) \leqslant \Phi(f(A))$$

for every selfadjoint operator A *with* $m \leqslant A \leqslant M$. In this content, Mićić and Pečarić in [5] defined Jensen's map as follows:

$$F_\Phi(f,A,B,t) = \Phi(f(tA+(1-t)B))$$

for all selfadjoint operators A,B with $m \leqslant A,B \leqslant M$ and $t \in [0,1]$. They showed some results related to Jensen's map.

In this paper, by virtue of Jensen's map, we show two operator versions of Theorem A. As applications, we obtain an interpolation of the weighted arithmetic-geometric mean inequality for the Karcher mean of positive invertible operators on a Hilbert space. Moreover, we obtain an interpolation between the quasi-arithmetic means.

2. Operator versions

Mond-Pečarić [8] showed the following operator version of Jensen's inequality (1.1): Let A be a selfadjoint operator on a Hilbert space H such that $m \leqslant A \leqslant M$ for some scalars $m < M$ and x a unit vector in H. If $f(t)$ is convex on $[m,M]$, then

$$f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle. \tag{2.1}$$

The multiple operator version of (2.1) is as follows: Let $A_1,\ldots,A_n$ be selfadjoint operators such that $m \leqslant A_j \leqslant M$ for all $j=1,\ldots,n$ and $x_1,\ldots,x_n \in H$ such that $\sum_{j=1}^n \|x_j\|^2 = 1$. If $f(t)$ is convex on $[m,M]$, then

$$f\left(\sum_{j=1}^n \langle A_jx_j,x_j\rangle\right) \leqslant \sum_{j=1}^n \langle f(A_j)x_j,x_j\rangle. \tag{2.2}$$

We show an operator version of Theorem A which connects both sides of (2.2):

THEOREM 2.1. *Let* $A_1,\ldots,A_n$ *be selfadjoint operators such that* $m \leqslant A_j \leqslant M$ *for all* $j=1,\ldots,n$ *and some scalars* $m<M$, *and* $x_1,\ldots,x_n \in H$ *such that* $\sum_{j=1}^n \|x_j\|^2 = 1$.

If $f(t)$ is convex on $[m,M]$, then

$$\begin{aligned} f\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right) &\leqslant \sum_{j=1}^{n}\Big\langle f\Big(tA_j+(1-t)\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle\Big)x_j,x_j\Big\rangle \\ &\leqslant \sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle \end{aligned}$$

for all $t\in[0,1]$. Moreover, the function $F:[0,1]\mapsto\mathbb{R}$ defined by

$$F(t)=\sum_{j=1}^{n}\Big\langle f\Big(tA_j+(1-t)\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle\Big)x_j,x_j\Big\rangle,$$

is monotonically nondecreasing and convex on $[0,1]$.

Proof. Firstly, we have

$$\begin{aligned} &\sum_{j=1}^{n}\Big\langle f\Big(tA_j+(1-t)\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle\Big)x_j,x_j\Big\rangle \\ &\qquad\leqslant \sum_{j=1}^{n}\Big\langle\Big(tf(A_j)+(1-t)f(\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle)\Big)x_j,x_j\Big\rangle \quad \text{by the convexity of } f \\ &\qquad\leqslant \sum_{j=1}^{n}\Big\langle\Big(tf(A_j)+(1-t)\sum_{k=1}^{n}\langle f(A_k)x_k,x_k\rangle\Big)x_j,x_j\Big\rangle \qquad \text{by (2.2)} \\ &\qquad= t\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle+(1-t)\sum_{k=1}^{n}\langle f(A_k)x_k,x_k\rangle\sum_{j=1}^{n}\langle x_j,x_j\rangle \\ &\qquad= \sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle. \end{aligned}$$

On the other hand, it follows that

$$\begin{aligned} &\sum_{j=1}^{n}\Big\langle f\Big(tA_j+(1-t)\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle\Big)x_j,x_j\Big\rangle \\ &\qquad\geqslant f\Big(\sum_{j=1}^{n}\Big\langle(tA_j+(1-t)\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle)x_j,x_j\Big\rangle\Big) \quad \text{by (2.2)} \\ &\qquad= f\Big(t\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle+(1-t)\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle\sum_{j=1}^{n}\langle x_j,x_j\rangle\Big) \\ &\qquad= f\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right). \end{aligned}$$

For the convexity of F, if we put $\alpha=\sum_{k=1}^{n}\langle A_kx_k,x_k\rangle$, then for $t_1,t_2\in[0,1]$ and $0<$

$t < 1$

$$
\begin{aligned}
tF(t_1) + (1-t)F(t_2) &= \sum_{j=1}^{n} \left\langle \left[tf(t_1A_j + (1-t_1)\alpha) + (1-t)f(t_2A_j + (1-t_2)\alpha) \right] x_j, x_j \right\rangle \\
&\geqslant \sum_{j=1}^{n} \left\langle f\Big(t(t_1A_j + (1-t_1)\alpha) + (1-t)(t_2A_j + (1-t_2)\alpha) \Big) x_j, x_j \right\rangle \\
&= \sum_{j=1}^{n} \left\langle f\Big((tt_1 + (1-t)t_2)A_j + (1 - tt_1 - (1-t)t_2)\alpha \Big) x_j, x_j \right\rangle \\
&= F(tt_1 + (1-t)t_2)
\end{aligned}
$$

and so F is convex on $[0,1]$.

For monotonically nondecreasing of F, if $0 < s < t < 1$, then $s = \frac{t-s}{t} \cdot 0 + \frac{s}{t} \cdot t$ and hence the convexity of F implies

$$
F(s) = F\left(\frac{t-s}{t} \cdot 0 + \frac{s}{t} \cdot t \right) \leqslant \frac{t-s}{t} F(0) + \frac{s}{t} F(t) \leqslant F(t)
$$

beacuse $F(0) \leqslant F(t)$. Therefore F is monotonically nondecreasing on $[0,1]$. We have just completed the proof of Theorem 2.1. $\square$

Next, we state a multiple version of Davis-Choi-Jensen's inequality for unital positive linear maps: Let $A_1, \dots, A_n$ be selfadjoint operators in $\mathscr{B}_h(H)$ with $m \leqslant A_j \leqslant M$ for all $j = 1, \dots, n$ and some scalars $m < M$, $\Phi_1, \dots, \Phi_n$ unital positive linear maps and $\omega_1, \dots, \omega_n$ any finite number of positive real numbers such that $\sum_{j=1}^{n} \omega_j = 1$. If $f(t)$ is operator convex on $[m, M]$, then

$$
f\left(\sum_{j=1}^{n} \omega_j \Phi_j(A_j) \right) \leqslant \sum_{j=1}^{n} \omega_j \Phi_j(f(A_j)), \tag{2.3}
$$

also see [3, Lemma 2.1].

By replacing the convexity of f by the operator convexity of f in Theorem 2.1, we have the following positive linear map version of Theorem A which connects both sides of (2.3), also see [5, Theorem 2.2]:

THEOREM 2.2. *Let $A_1, \dots, A_n$ be selfadjoint operators in $\mathscr{B}_h(H)$ with $m \leqslant A_j \leqslant M$ for all $j = 1, \dots, n$ and some scalars $m < M$, $\Phi_1, \dots, \Phi_n$ unital positive linear maps, and $\omega_1, \dots, \omega_n$ any finite number of positive real numbers such that $\sum_{j=1}^{n} \omega_j = 1$. If $f(t)$ is operator convex on $[m, M]$, then*

$$
\begin{aligned}
f\left(\sum_{j=1}^{n} \omega_j \Phi_j(A_j) \right) &\leqslant \sum_{j=1}^{n} \omega_j f\left(t\Phi_j(A_j) + (1-t) \sum_{k=1}^{n} \omega_k \Phi_k(A_k) \right) \\
&\leqslant \sum_{j=1}^{n} \omega_j \Phi_j(f(A_j))
\end{aligned}
$$

for all $t \in [0,1]$. *Moreover, the mapping* $F : [0,1] \mapsto \mathscr{B}(H)$ *defined by*

$$F(t) = \sum_{j=1}^{n} \omega_j f\left(t\Phi_j(A_j) + (1-t)\sum_{k=1}^{n} \omega_k \Phi_k(A_k)\right),$$

is monotonically nondecreasing and convex on $[0,1]$.

Proof. We omit the proof of this theorem because it is proved in a way similar to that of Theorem 2.1. $\square$

3. Applications

In this section, as applications, we show an interpolation of two typical mean inequalities of positive invertible operators on a Hilbert space. Firstly, we consider an interpolation of the weighted arithmetic-geometric mean inequality for the Karcher mean of positive invertible operators on a Hilbert space. The weighted arithmetic-geometric mean inequality says that for positive real numbers $x_1, x_2, \dots, x_n$ and $\omega_1, \omega_2, \dots, \omega_n$ with $\sum_{j=1}^{n} \omega_j = 1$

$$x_1^{\omega_1} x_2^{\omega_2} \cdots x_n^{\omega_n} \leqslant \sum_{j=1}^{n} \omega_j x_j.$$

By Theorem A, we have a new interpolation of the weighted arithmetic-geometrc mean inequality:

THEOREM 3.1. *For positive real numbers* $x_1, x_2, \dots, x_n$ *and* $\omega_1, \omega_2, \dots, \omega_n$ *with* $\sum_{j=1}^{n} \omega_j = 1$

$$\begin{aligned} & x_1^{\omega_1} x_2^{\omega_2} \cdots x_n^{\omega_n} \\ & \leqslant \left(tx_1 + (1-t)\sum_{j=1}^{n} \omega_j x_j\right)^{\omega_1} \left(tx_2 + (1-t)\sum_{j=1}^{n} \omega_j x_j\right)^{\omega_2} \cdots \left(tx_n + (1-t)\sum_{j=1}^{n} \omega_j x_j\right)^{\omega_n} \\ & \leqslant \sum_{j=1}^{n} \omega_j x_j \end{aligned}$$

for all $t \in [0,1]$.

Proof. If we put $f(x) = -\log x$ in Theorem A, then we get Theorem 3.1. $\square$

We try to study an operator version of Theorem 3.1. For this, we recall the Karcher mean of positive invertible operators: In 2014, Lawson and Lim [4] established the formulation of the geometric mean for n $(\geqslant 3)$ positive invertible operators on a separable Hilbert space which is a nice extension of the geometric operator mean in the Kubo-Ando theory. They showed that there exists the unique positive invertible solution of the Karcher equation

$$\sum_{i=1}^{n} \omega_i \log X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} = 0 \tag{3.1}$$

for given n positive invertible operators $A_1,\dots,A_n$, where $\omega=(\omega_1,\dots,\omega_n)$ is a weight vector, i.e., $\omega_1,\dots,\omega_n \geqslant 0$ and $\sum_{i=1}^n \omega_i = 1$. We say that the solution X of (3.1) is the Karcher mean for n positive invertible operators $A_1,\dots,A_n$ and denote it by $G_{\mathrm{K}}(\omega;A_1,\dots,A_n)$. In the case of $n=2$, the Karcher mean $G_{\mathrm{K}}((1-t,t);A,B)$ coincides with the geometric operator mean $A \sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ for $t\in[0,1]$. We list some properties of the Karcher mean which we need later:

(P1) Consistency with scalars: $G_{\mathrm{K}}(\omega;A_1,\dots,A_n)=A_1^{\omega_1}A_2^{\omega_2}\cdots A_n^{\omega_n}$ if the A_i's commute;

(P2) Joint homogeneity: $G_{\mathrm{K}}(\omega;a_1A_1,\dots,a_nA_n)=a_1^{\omega_1}\cdots a_n^{\omega_n}G_{\mathrm{K}}(\omega;A_1,\dots,A_n)$;

(P3) Joint concavity:

$$\begin{aligned}&G_{\mathrm{K}}(\omega;(1-t)A_1+tB_1,\dots,(1-t)A_n+tB_n)\\ &\geqslant (1-t)G_{\mathrm{K}}(\omega;A_1,\dots,A_n)+tG_{\mathrm{K}}(\omega;B_1,\dots,B_n) \quad \text{for } 0\leqslant t\leqslant 1;\end{aligned}$$

(P4) Self-duality: $G_{\mathrm{K}}(\omega;A_1^{-1},\dots,A_n^{-1})^{-1}=G_{\mathrm{K}}(\omega;A_1,\dots,A_n)$;

(P5) AGH weighted mean inequality:

$$\left(\sum_{j=1}^n \omega_j A_j^{-1}\right)^{-1} \leqslant G_{\mathrm{K}}(\omega;A_1,\dots,A_n) \leqslant \sum_{j=1}^n \omega_j A_j.$$

By virtue of Theorem A, we have an interpolation of the weighted arithmetic-geometric mean inequality (P5):

THEOREM 3.2. *Let $A_1,A_2,\dots,A_n$ be positive invertible operators on a Hilbert space and $\omega=(\omega_1,\dots,\omega_n)$ a weight vector. Then*

$$\begin{aligned}&G_K(\omega;A_1,\dots,A_n)\\ &\leqslant G_K\Big(\omega;tA_1+(1-t)\sum_{j=1}^n\omega_jA_j,tA_2+(1-t)\sum_{j=1}^n\omega_jA_j,\dots,tA_n+(1-t)\sum_{j=1}^n\omega_jA_j\Big)\\ &\leqslant \sum_{j=1}^n \omega_jA_j\end{aligned}$$

for all $t\in[0,1]$. Moreover, the mapping $F:[0,1]\mapsto \mathscr{B}(H)$ defined by

$$F(t)=G_K\Big(\omega;tA_1+(1-t)\sum_{j=1}^n\omega_jA_j,tA_2+(1-t)\sum_{j=1}^n\omega_jA_j,\dots,tA_n+(1-t)\sum_{j=1}^n\omega_jA_j\Big),$$

is monotonically nonincreasing and concave on $[0,1]$.

Proof. It follows from the joint concavity (P3) and AGH weighted mean inequality (P5) of the Karcher mean that

$$\begin{aligned}
&G_K\Big(\omega; tA_1+(1-t)\sum_{j=1}^n \omega_j A_j, tA_2+(1-t)\sum_{j=1}^n \omega_j A_j, \dots, tA_n+(1-t)\sum_{j=1}^n \omega_j A_j\Big)\\
&\geqslant tG_K(\omega;A_1,\dots,A_n)+(1-t)G_K\Big(\omega;\sum_{j=1}^n \omega_j A_j,\dots,\sum_{j=1}^n \omega_j A_j\Big)\\
&= tG_K(\omega;A_1,\dots,A_n)+(1-t)\sum_{j=1}^n \omega_j A_j \qquad \text{by (P1)}\\
&\geqslant G_K(\omega;A_1,\dots,A_n) \qquad \text{by (P5).}
\end{aligned}$$

The second inequality in Theorem 3.2 follows from the AGH weighted mean inequality (P5). In fact, we have

$$\begin{aligned}
&G_K\Big(\omega; tA_1+(1-t)\sum_{j=1}^n \omega_j A_j, tA_2+(1-t)\sum_{j=1}^n \omega_j A_j, \dots, tA_n+(1-t)\sum_{j=1}^n \omega_j A_j\Big)\\
&\leqslant \omega_1\Big(tA_1+(1-t)\sum_{j=1}^n \omega_j A_j\Big)+\cdots+\omega_n\Big(tA_n+(1-t)\sum_{j=1}^n \omega_j A_j\Big)\\
&=\sum_{j=1}^n \omega_j A_j.
\end{aligned}$$

For the cancavity of F, if $A_0=\sum_{j=1}^n \omega_j A_j$, then for $t_1,t_2\in[0,1]$ and $0<t<1$

$$\begin{aligned}
&tF(t_1)+(1-t)F(t_2)\\
&\leqslant G_K\Big(\omega; t(t_1A_1+(1-t_1)A_0)+(1-t)(t_2A_1+(1-t_2)A_0),\dots,\\
&\qquad\qquad t(t_1A_n+(1-t_1)A_0)+(1-t)(t_2A_n+(1-t_2)A_0)\Big)\\
&= G_K\Big(\omega; (tt_1+(1-t)t_2)A_1+(t(1-t_1)+(1-t)(1-t_2))A_0,\dots,\\
&\qquad\qquad (tt_1+(1-t)t_2)A_n+(t(1-t_1)+(1-t)(1-t_2))A_0\Big)\\
&= F(tt_1+(1-t)t_2)
\end{aligned}$$

and so F is concave on $[0,1]$.

For monotonically nonincreasing of F, if $0<s<t<1$, then $s=\frac{t-s}{t}\cdot 0+\frac{s}{t}\cdot t$ and hence the concavity of F implies

$$F(s)\geqslant \frac{t-s}{t}F(0)+\frac{s}{t}F(t)\geqslant F(t)$$

beacuse $F(0)=\sum_{j=1}^n \omega_j A_j\geqslant F(t)\geqslant F(1)=G_K(\omega;A_1,\dots,A_n)$ for all $0<t<1$. Therefore F is monotonically nonincreasing on $[0,1]$, and the proof is complete. □

We have the following interpolation which connects both sides of the geometric-harmonic mean inequality (P5) by Theorem 3.2 and the self-duality (P4) of the Karcher mean:

COROLLARY 3.3. *Let $A_1, A_2, \ldots, A_n$ be positive invertible operators and $\omega = (\omega_1, \ldots, \omega_n)$ a weight vector. Then*

$$\begin{aligned}&\left(\sum_{j=1}^{n} \omega_j A_j^{-1}\right)^{-1}\\ &\leqslant G_K\left(\omega; \left(tA_1^{-1} + (1-t)\sum_{j=1}^{n} \omega_j A_j^{-1}\right)^{-1}, \ldots, \left(tA_n^{-1} + (1-t)\sum_{j=1}^{n} \omega_j A_j^{-1}\right)^{-1}\right)\\ &\leqslant G_K(\omega; A_1, \ldots, A_n)\end{aligned}$$

for all $t \in [0,1]$.

Secondly, we consider an interpolation between the quasi-arithmetic means. We define the quasi-arithmetic mean of operators:

$$\mathfrak{M}_\varphi(\omega; \mathbf{A}; \Phi) := \varphi^{-1}\left(\sum_{j=1}^{n} \omega_j \Phi_j\left(\varphi(A_j)\right)\right),$$

where $\mathbf{A} = (A_1, \ldots, A_n)$ is an n-tuple of selfadjoint operators in $\mathscr{B}_h(H)$ with spectra in an interval J, $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n)$ is an n-tuple of unital positive linear maps $\Phi_j : \mathscr{B}(H) \to \mathscr{B}(K)$ for all $j = 1, \ldots, n$, $\omega = (\omega_1, \ldots, \omega_n)$ is a weight vector, and $\varphi : J \to \mathbb{R}$ is a strictly monotone function, also see [6, 7]. If φ^{-1} is operator concave on $\varphi(J)$, then

$$\mathfrak{M}_\varphi(\omega; \mathbf{A}; \Phi) \geqslant \mathfrak{M}_1(\omega; \mathbf{A}; \Phi) = \sum_{j=1}^{n} \omega_j \Phi_j(A_j). \tag{3.2}$$

The power mean is a special case of the quasi-arithmetic mean:

$$\mathfrak{M}_r(\omega; \mathbf{A}; \Phi) := \begin{cases} \left(\sum_{j=1}^{n} \omega_j \Phi_j\left(A_j^r\right)\right)^{1/r} & \text{for } r \in \mathbb{R}\backslash\{0\}, \\ \exp\left(\sum_{j=1}^{n} \omega_j \Phi_j\left(\log(A_j)\right)\right) & \text{for } r = 0\,, \end{cases}$$

where $A_1, \ldots, A_n$ are positive invertible operators.

By virtue of Theorem 2.2, we have an interpolation of the quasi-arithmetic mean and the weighted arithmetic mean in (3.2):

THEOREM 3.4. *Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an n-tuple of selfadjoint operators in $\mathscr{B}_h(H)$ with spectra in an interval J, $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n)$ be an n-tuple of unital positive linear maps $\Phi_j : \mathscr{B}(H) \to \mathscr{B}(K)$, $\omega = (\omega_1, \ldots, \omega_n)$ be a weight vector, and $e_j = (0, \ldots, 1, \ldots, 0)$ for $j = 1, \ldots, n$, are the standard basis vector in $\mathbb{R}^n$. If $\varphi : J \to \mathbb{R}$ is a strictly monotone function such that φ^{-1} is operator concave on $\varphi(J)$, then*

$$\mathfrak{M}_\varphi(\omega; \mathbf{A}; \Phi) \geqslant \sum_{j=1}^{n} \omega_j \mathfrak{M}_\varphi\left(te_j + (1-t)\omega; \mathbf{A}; \Phi\right) \geqslant \mathfrak{M}_1(\omega; \mathbf{A}; \Phi) \tag{3.3}$$

for all $t \in [0,1]$. *Moreover, the mapping* $M : [0,1] \mapsto \mathscr{B}(H)$ *defined by*

$$M(t) := \sum_{j=1}^{n} \omega_j \mathfrak{M}_{\varphi}\Big(te_j + (1-t)\omega; \mathbf{A}; \Phi\Big),$$

is monotonically nonincreasing and concave on $[0,1]$.

Moreover, if φ^{-1} *is operator convex on* $\varphi(J)$, *then the reverse inequalities are valid in* (3.3) *and* $M(t)$ *is monotonically nondecreasing and convex on* [0,1].

Proof. Since φ^{-1} is operator concave on $\varphi(I)$, if we replace A_j by $\varphi(A_j)$ in Theorem 2.2, then

$$\begin{aligned}\varphi^{-1}\left(\sum_{j=1}^{n}\omega_j\Phi_j(\varphi(A_j))\right) &\geqslant \sum_{j=1}^{n}\omega_j\varphi^{-1}\Big(t\Phi_j(\varphi(A_j)) + (1-t)\sum_{k=1}^{n}\omega_k\Phi_k(\varphi(A_k))\Big)\\ &\geqslant \sum_{j=1}^{n}\omega_j\Phi_j(A_j),\end{aligned}$$

which give the desired inequalities (3.3). Also, it follows from Theorem 2.2 that $M(t)$ is monotonically nonincreasing and concave on $[0,1]$. $\square$

Applying Theorem 3.4 we obtain the following interpolation of the power-arithmetic mean inequality.

COROLLARY 3.5. *Let the assumptions of Theorem* 3.4 *hold and* $A_1, A_2, \ldots, A_n$ *be positive invertible operators. If* $r \geqslant 1$ *then*

$$\begin{aligned}\left(\sum_{j=1}^{n}\omega_j\Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} &\geqslant \sum_{j=1}^{n}\omega_j\Big(t\Phi_j(A_j^r) + (1-t)\sum_{k=1}^{n}\omega_k\Phi_k(A_k^r)\Big)^{\frac{1}{r}}\\ &\geqslant \sum_{j=1}^{n}\omega_j\Phi_j\left(A_j\right)\end{aligned} \tag{3.4}$$

for all $t \in [0,1]$. *Moreover, the mapping* $M : [0,1] \mapsto \mathscr{B}(H)$ *defined by*

$$M(t) := \sum_{j=1}^{n}\omega_j\Big(t\Phi_j(A_j^r) + (1-t)\sum_{k=1}^{n}\omega_k\Phi_k(A_k^r)\Big)^{\frac{1}{r}},$$

is monotonically nonincreasing and concave on $[0,1]$.

Moreover, if $r \leqslant -1$ *or* $1/2 \leqslant r \leqslant 1$, *then the reverse inequalities are valid in* (3.4) *and* $M(t)$ *is monotonically nondecreasing and convex on* $[0,1]$.

REMARK 1. Combining inequalities in Corollary 3.5, we obtain

$$\begin{aligned}\left(\sum_{j=1}^{n}\omega_j\Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} &\geqslant \sum_{j=1}^{n}\omega_j\Big(t\Phi_j(A_j^r)+(1-t)\sum_{k=1}^{n}\omega_k\Phi_k(A_k^r)\Big)^{\frac{1}{r}}\\ &\geqslant \sum_{j=1}^{n}\omega_j\Phi_j\left(A_j\right) \geqslant \sum_{j=1}^{n}\omega_j\Big(t\Phi_j(A_j^s)+(1-t)\sum_{k=1}^{n}\omega_k\Phi_k(A_k^s)\Big)^{\frac{1}{s}}\\ &\geqslant \left(\sum_{j=1}^{n}\omega_j\Phi_j\left(A_j^s\right)\right)^{\frac{1}{s}}\end{aligned}$$

for all $t\in[0,1]$ and $r\in[1,\infty)$ and $s\in(-\infty,-1]\cup[1/2,1]$.

Combining inequalities in Theorem 3.4 we can obtain some generalized inequalities in Remark 1.

COROLLARY 3.6. *Let $\mathbf{A}=(A_1,\dots,A_n)$ be an n-tuple of selfadjoint operators in $\mathscr{B}_h(H)$ with spectra in an interval J, $\Phi=(\Phi_1,\dots,\Phi_n)$ be an n-tuple of unital positive linear maps $\Phi_j:\mathscr{B}(H)\to\mathscr{B}(K)$, $\omega=(\omega_1,\dots,\omega_n)$ and $\nu=(\nu_1,\dots,\nu_n)$ be weight vectors such that $\omega\geqslant\nu$. If $\varphi,\psi:J\to\mathbb{R}$ are strictly monotone functions such that φ^{-1} is operator concave on $\varphi(J)$ and ψ^{-1} is operator convex on $\psi(J)$, then*

$$\begin{aligned}\mathfrak{M}_\varphi(\omega;\mathbf{A};\Phi) &\geqslant \sum_{j=1}^{n}\omega_j\mathfrak{M}_\varphi\big(te_j+(1-t)\omega;\mathbf{A};\Phi\big)\geqslant\mathfrak{M}_1(\omega;\mathbf{A};\Phi)\\ \geqslant\mathfrak{M}_1(\nu;\mathbf{A};\Phi) &\geqslant \sum_{j=1}^{n}\nu_j\mathfrak{M}_\psi\big(te_j+(1-t)\nu;\mathbf{A};\Phi\big)\geqslant\mathfrak{M}_\psi(\nu;\mathbf{A};\Phi)\end{aligned}$$

holds for all $t\in[0,1]$.

COROLLARY 3.7. *Let $\mathbf{A}=(A_1,\dots,A_n)$ and $\mathbf{B}=(B_1,\dots,B_n)$ be two n-tuples of selfadjoint operators in $\mathscr{B}_h(H)$ with spectra in $[m_A,M_A]$ and $[m_B,M_B]$, respectively, such that $m_A\geqslant M_B$. Let $\Phi=(\Phi_1,\dots,\Phi_n)$ be an n-tuple of unital positive linear maps $\Phi_j:\mathscr{B}(H)\to\mathscr{B}(K)$, $\omega=(\omega_1,\dots,\omega_n)$ and $\nu=(\nu_1,\dots,\nu_n)$ be weight vectors. If $\varphi:[m_A,M_A]\to\mathbb{R}$ and $\psi:[m_B,M_B]\to\mathbb{R}$ are strictly monotone functions such that φ^{-1} is operator concave on $\varphi([m_A,M_A])$ and ψ^{-1} is operator convex on $\psi([m_B,M_B])$, then*

$$\begin{aligned}\mathfrak{M}_\varphi(\omega;\mathbf{A};\Phi) &\geqslant \sum_{j=1}^{n}\omega_j\mathfrak{M}_\varphi\big(te_j+(1-t)\omega;\mathbf{A};\Phi\big)\geqslant\mathfrak{M}_1(\omega;\mathbf{A};\Phi)\\ \geqslant\mathfrak{M}_1(\nu;\mathbf{B};\Phi) &\geqslant \sum_{j=1}^{n}\nu_j\mathfrak{M}_\psi\big(te_j+(1-t)\nu;\mathbf{B};\Phi\big)\geqslant\mathfrak{M}_\psi(\omega;\mathbf{B};\Phi)\end{aligned}$$

holds for all $t\in[0,1]$.

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Jadranka Mićić Hot
Faculty of Mechanical Engiering and Naval Architecture
University of Zagreb
Ivan Lucica 5, 10000 Zagreb, Croatia
e-mail: `jmicic@fsb.hr`

Yuki Seo
Department of Mathematics Education
Osaka Kyoiku University
Asahigaoka, Kashiwara, Osaka582-8582, Japan
e-mail: `yukis@cc.osaka-kyoiku.ac.jp`