

ON REVERSES OF THE GOLDEN–THOMPSON TYPE INEQUALITIES

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Abstract. In this paper we present some reverses of the Golden-Thompson type inequalities: Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and $\alpha \in [0, 1]$. Then for all $p > 0$ and $k = 1, 2, \dots, n$

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}},$$

where $A \#_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}}$ is α -geometric mean, $S(t)$ is the so called Specht ratio and \preceq_{ols} is the so called Olson order. The same inequalities are also provided with other constants. The obtained inequalities improve some known results.

1. Introduction

In what follows, capital letters A, B, H and K stand for $n \times n$ matrices or bounded linear operators on an n -dimensional complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. For a pair A, B of Hermitian matrices, we say $A \leq B$ if $B - A \geq 0$. Let A and B be two positive definite matrices. For each $\alpha \in [0, 1]$, the weighted geometric mean $A \#_{\alpha} B$ of A and B in the sense of Kubo-Ando [9] is defined by

$$A \#_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}}.$$

Also for positive semidefinite matrices A and B , the weak log-majorization $A \prec_{wlog} B$ means that

$$\prod_{j=1}^k \lambda_j(A) \leq \prod_{j=1}^k \lambda_j(B), \quad k = 1, 2, \dots, n,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ are the eigenvalues of A listed in decreasing order. If equality holds when $k = n$, we have the log-majorization $A \prec_{log} B$. It is known that the weak log-majorization $A \prec_{wlog} B$ implies $\|A\|_u \leq \|B\|_u$ for any unitarily invariant norm $\|\cdot\|_u$, i.e. $\|UAV\|_u = \|A\|_u$ for all A and all unitaries U, V . See [2] for theory of majorization.

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In [14], Specht obtained an inequality for the arithmetic and geometric means of positive numbers: Let $x_1 \geq \dots \geq x_n > 0$ and set $t = x_1/x_n$. Then

$$\frac{x_1 + \dots + x_n}{n} \leq S(t)(x_1 \dots x_n)^{\frac{1}{n}},$$

where

$$S(t) = \frac{(t-1)t^{1/(t-1)}}{e \log t} \quad (t \neq 1) \quad \text{and} \quad S(1) = 1 \tag{1}$$

is called the Specht ratio at t . Note that $\lim_{p \rightarrow 0} S(t^p)^{\frac{1}{p}} = 1$, $S(t^{-1}) = S(t) > 1$ for $t \neq 1$, $t > 0$ [5]. Specht’s inequality is a ratio type reverse inequality of the classical arithmetic-geometric mean inequality. Using this nice ratio we can state our main result in Section 2.

The Golden-Thompson trace inequality, which is of importance in statistical mechanics and in the theory of random matrices, states that $Tr e^{H+K} \leq Tr e^H e^K$ for arbitrary Hermitian matrices H and K . This inequality has been complemented in several ways [1, 8]. Ando and Hiai in [1] proved that for every unitarily invariant norm $\|\cdot\|_u$ and $p > 0$

$$\|(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}\|_u \leq \|e^{(1-\alpha)H + \alpha K}\|_u. \tag{2}$$

Seo in [12] found some upper bounds on $\|e^{(1-\alpha)H + \alpha K}\|_u$ in terms of scalar multiples of $\|(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}\|_u$, which show reverse of the Golden-Thompson type inequality (2). In this paper we establish another reverses of this inequality, which improve and refine Seo’s results. In fact the general sandwich condition $sA \leq B \leq tA$ for positive definite matrices, is the key for our statements. Also, the so called Olson order \preceq_{ols} is used. For positive operators, $A \preceq_{ols} B$ if and only if $A^r \leq B^r$ for every $r \geq 1$ [11]. Our results are parallel to eigenvalue inequalities obtained in [3] and [7].

2. Reverse inequalities via Specht ratio

To study the Golden-Thompson inequality, Ando-Hiai in [1] developed the following log-majorizations:

$$A^r \#_{\alpha} B^r \prec_{\log} (A \#_{\alpha} B)^r, \quad r \geq 1,$$

or equivalently

$$(A^p \#_{\alpha} B^p)^{\frac{1}{p}} \prec_{\log} (A^q \#_{\alpha} B^q)^{\frac{1}{q}}, \quad 0 < q \leq p.$$

There are some literatures [13] on the converse of these inequalities in terms of unitarily invariant norm $\|\cdot\|_u$. By the following lemmas, we obtain a new reverse of these inequalities in terms of eigenvalue inequalities.

LEMMA 1. Let A and B be positive definite matrices such that $sA \leq B \leq tA$ for some scalars $0 < s \leq t$, and $\alpha \in [0, 1]$. Then

$$A^r \sharp_{\alpha} B^r \leq (\max\{S(s), S(t)\})^r (A \sharp_{\alpha} B)^r, \quad 0 < r \leq 1, \quad (3)$$

where $S(t)$ is the Specht ratio defined as (1).

Proof. Let f be an operator monotone function on $[0, \infty)$. Then according to the proof of Theorem 1 in [6], we have

$$f(A) \sharp_{\alpha} f(B) \leq f(M(A \sharp_{\alpha} B)),$$

where $M = \max\{S(s), S(t)\}$. Putting $f(t) = t^r$ for $0 < r \leq 1$, we reach inequality (3). \square

LEMMA 2. Let A and B be positive definite matrices such that $sA \preceq_{ols} B \preceq_{ols} tA$ for some scalars $0 < s \leq t$, and $\alpha \in [0, 1]$. Then

$$\lambda_k(A \sharp_{\alpha} B)^r \leq \max\{S(s^r), S(t^r)\} \lambda_k(A^r \sharp_{\alpha} B^r), \quad r \geq 1, \quad (4)$$

and hence,

$$\lambda_k(A^q \sharp_{\alpha} B^q)^{\frac{1}{q}} \leq (\max\{S(s^p), S(t^p)\})^{\frac{1}{p}} \lambda_k(A^p \sharp_{\alpha} B^p)^{\frac{1}{p}}, \quad 0 < q \leq p, \quad (5)$$

where $S(t)$ is the Specht ratio defined as (1) and $k = 1, 2, \dots, n$.

Proof. First note that the condition $sA \preceq_{ols} B \preceq_{ols} tA$ is equivalent to the condition $s^v A^v \leq B^v \leq t^v A^v$ for every $v \geq 1$. In particular, we have $sA \leq B \leq tA$ for $v = 1$. Also, for $r \geq 1$ we have $0 < \frac{1}{r} \leq 1$ and by (3)

$$A^{\frac{1}{r}} \sharp_{\alpha} B^{\frac{1}{r}} \leq (\max\{S(s), S(t)\})^{\frac{1}{r}} (A \sharp_{\alpha} B)^{\frac{1}{r}}. \quad (6)$$

On the other hand, from the condition $s^v A^v \leq B^v \leq t^v A^v$ for every $v \geq 1$ and letting $v = r$, we have

$$s^r A^r \leq B^r \leq t^r A^r.$$

Now if we let $X = A^r$, $Y = B^r$, $w = s^r$ and $z = t^r$, then

$$wX \leq Y \leq zX. \quad (7)$$

Using (6) under the condition (7), we have

$$X^{\frac{1}{r}} \sharp_{\alpha} Y^{\frac{1}{r}} \leq (\max\{S(w), S(z)\})^{\frac{1}{r}} (X \sharp_{\alpha} Y)^{\frac{1}{r}},$$

and this is the same as

$$A \sharp_{\alpha} B \leq (\max\{S(s^r), S(t^r)\})^{\frac{1}{r}} (A^r \sharp_{\alpha} B^r)^{\frac{1}{r}}.$$

Hence

$$\lambda_k(A\sharp_{\alpha}B) \leq (\max\{S(s^r), S(t^r)\})^{\frac{1}{r}} \lambda_k(A^r\sharp_{\alpha}B^r)^{\frac{1}{r}}.$$

By taking r -th power on both sides and using the spectral mapping theorem, we get the desired inequality (4). Note that from the minimax characterization of eigenvalues of a Hermitian matrix [2] it follows immediately that $A \leq B$ implies $\lambda_k(A) \leq \lambda_k(B)$ for each k . Similarly since $p/q \geq 1$, from inequality (4)

$$\lambda_k(A\sharp_{\alpha}B)^{\frac{p}{q}} \leq \max\{S(s^{\frac{p}{q}}), S(t^{\frac{p}{q}})\} \lambda_k(A^{\frac{p}{q}}\sharp_{\alpha}B^{\frac{p}{q}}). \tag{8}$$

Replacing A and B by A^q and B^q in (8), and using the sandwich condition $s^q A^q \leq B^q \leq t^q A^q$, we have

$$\lambda_k(A^q\sharp_{\alpha}B^q)^{\frac{p}{q}} \leq \max\{S(s^p), S(t^p)\} \lambda_k(A^p\sharp_{\alpha}B^p).$$

This completes the proof. \square

Note that eigenvalue inequalities immediately imply log-majorization and unitarily invariant norm inequalities.

COROLLARY 1. *Let A and B be positive definite matrices such that $mI \leq A, B \leq MI$ for some scalars $0 < m \leq M$ with $h = M/m$, and let $\alpha \in [0, 1]$. Then*

$$A^r\sharp_{\alpha}B^r \leq S(h)^r (A\sharp_{\alpha}B)^r, \quad 0 < r \leq 1, \tag{9}$$

and hence

$$\lambda_k(A\sharp_{\alpha}B)^r \leq S(h^r) \lambda_k(A^r\sharp_{\alpha}B^r), \quad r \geq 1, \tag{10}$$

$$\lambda_k(A^q\sharp_{\alpha}B^q)^{\frac{1}{q}} \leq S(h^p)^{\frac{1}{p}} \lambda_k(A^p\sharp_{\alpha}B^p)^{\frac{1}{p}}, \quad 0 < q \leq p, \tag{11}$$

where $S(t)$ is the Specht ratio defined as (1) and $k = 1, 2, \dots, n$.

Proof. Since $mI \leq A, B \leq MI$ implies $\frac{m}{M}A \leq B \leq \frac{M}{m}A$, the inequality (9) is obtained by letting $s = m/M$, $t = M/m$ in Lemma 1. Also from $mI \leq A, B \leq MI$, we have $m^{\nu}I \leq A^{\nu}, B^{\nu} \leq M^{\nu}I$ for every $\nu \geq 1$, and so

$$\left(\frac{m}{M}\right)^{\nu} A^{\nu} \leq B^{\nu} \leq \left(\frac{M}{m}\right)^{\nu} A^{\nu}. \tag{12}$$

Using Lemma 2 under the condition (12), we reach inequalities (10) and (11). Note that $S(h) = S(\frac{1}{h})$ for every $h > 0$. \square

REMARK 1. We remark that the matrix inequality (9) is more stronger than corresponding norm inequality obtained by Seo in [12, Corollary 3.2]. Also, inequality (11) is presented in [12, Lemma 3.1].

In the sequel we show a reverse of the Golden-Thompson type inequality (2), which is our main result.

THEOREM 1. *Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and let $\alpha \in [0, 1]$. Then for all $p > 0$,*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}},$$

where $S(t)$ is the Specht ratio defined as (1) and $k = 1, 2, \dots, n$.

Proof. Replacing A and B by e^H and e^K in the inequality (5) of Lemma 2, we can write

$$\lambda_k(e^{qH} \#_{\alpha} e^{qK})^{\frac{1}{q}} \leq (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}, \quad 0 < q \leq p.$$

By [8, Lemma 3.3], we have

$$e^{(1-\alpha)H+\alpha K} = \lim_{q \rightarrow 0} (e^{qH} \#_{\alpha} e^{qK})^{\frac{1}{q}},$$

and hence it follows that for each $p > 0$,

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}. \quad \square$$

COROLLARY 2. *Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and let $\alpha \in [0, 1]$. Then for every unitarily invariant norm $\|\cdot\|_u$ and all $p > 0$,*

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leq (\max\{S(e^{sp}), S(e^{tp})\})^{\frac{1}{p}} \|(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}\|_u, \quad (13)$$

and the right-hand side of (13) converges to the left-hand side as $p \downarrow 0$. In particular,

$$\|e^{H+K}\|_u \leq \max\{S(e^{2s}), S(e^{2t})\} \|(e^{2H} \#_{\alpha} e^{2K})\|_u.$$

COROLLARY 3. [12, Theorem 3.3–Theorem 3.4] *Let H and K be Hermitian matrices such that $mI \leq H, K \leq MI$ for some scalars $m \leq M$, and let $\alpha \in [0, 1]$. Then for all $p > 0$,*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq S(e^{(M-m)p})^{\frac{1}{p}} \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}, \quad k = 1, 2, \dots, n.$$

So, for every unitarily invariant norm $\|\cdot\|_u$

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leq S(e^{(M-m)p})^{\frac{1}{p}} \|(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}\|_u,$$

and the right-hand side of these inequalities converges to the left-hand side as $p \downarrow 0$.

Proof. From $mI \leq H, K \leq MI$, we have $e^{vm} \leq e^{vH}, e^{vK} \leq e^{vM}$ for every $v \geq 1$ and so we can derive $e^{m-M} e^H \preceq_{ols} e^K \preceq_{ols} e^{M-m} e^H$. Now the assertion is obtained by applying Theorem 1 and the fact that for every $t > 0$, $S(t) = S(\frac{1}{t})$. \square

3. Reverse inequalities via Kantorovich constant

A well-known matrix version of the Kantorovich inequality [10] asserts that if A and U are two matrices such that $0 < mI \leq A \leq MI$ and $UU^* = I$, then

$$UA^{-1}U^* \leq \frac{(m+M)^2}{4mM}(UAU^*)^{-1}. \tag{14}$$

Let $w > 0$. The generalized Kantorovich constant $K(w, \alpha)$ is defined by

$$K(w, \alpha) := \frac{(w^\alpha - w)}{(\alpha - 1)(w - 1)} \left(\frac{\alpha - 1}{\alpha} \frac{w^\alpha - 1}{w^\alpha - w} \right)^\alpha, \tag{15}$$

for any real number $\alpha \in \mathbb{R}$ [5]. In fact, $K(\frac{M}{m}, -1) = K(\frac{M}{m}, 2)$ is the constant occurring in (14).

Now as a result of the following statement, we have another reverse Golden-Thompson type inequality which refines corresponding inequality in [12].

PROPOSITION 1. [7, Theorem 3] Let H and K be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and let $\alpha \in [0, 1]$. Then

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq K(e^{p(t-s)}, \alpha)^{-\frac{1}{p}} \lambda_k(e^{pH} \#_\alpha e^{pK})^{\frac{1}{p}}, \quad p > 0, \tag{16}$$

where $K(w, \alpha)$ is the generalized Kantorovich constant defined as (15).

THEOREM 2. Let H and K be Hermitian matrices such that $mI \leq K, H \leq MI$ for some scalars $m \leq M$ and let $\alpha \in [0, 1]$. Then for every $p > 0$,

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \lambda_k(e^{pH} \#_\alpha e^{pK})^{\frac{1}{p}}, \quad k = 1, 2, \dots, n,$$

and the right-hand side of this inequality converges to the left-hand side as $p \downarrow 0$. In particular,

$$\lambda_k(e^{H+K}) \leq \frac{e^{2M} + e^{2m}}{2e^M e^m} \lambda_k(e^{2H} \# e^{2K}), \quad k = 1, 2, \dots, n.$$

Proof. Since $mI \leq K, H \leq MI$ implies $e^{m-M} e^H \preceq_{ols} e^K \preceq_{ols} e^{M-m} e^H$, desired inequalities are obtained by letting $s = m - M$ and $t = M - m$ in Proposition 1. For the convergence, we know that $\frac{2w^{\frac{1}{4}}}{w^{\frac{1}{2}} + 1} \leq K(w, \alpha) \leq 1$, for every $\alpha \in [0, 1]$. So, for every $p > 0$

$$1 \leq K(w^p, \alpha)^{-\frac{1}{p}} \leq \left(\frac{2w^{\frac{p}{4}}}{w^{\frac{p}{2}} + 1} \right)^{-\frac{1}{p}}.$$

A simple calculation shows that

$$\lim_{p \rightarrow 0} -\frac{1}{p} \log \left(\frac{2w^{\frac{p}{4}}}{w^{\frac{p}{2}} + 1} \right) = \lim_{p \rightarrow 0} \frac{(w^{\frac{p}{2}} - 1) \log(w)}{4(w^{\frac{p}{2}} + 1)} = 0,$$

and hence $\lim_{p \rightarrow 0} \left(\frac{2w^{\frac{p}{4}}}{w^{\frac{p}{2}} + 1} \right)^{-\frac{1}{p}} = 1$. Now by using the sandwich condition and letting $w = e^{2(M-m)}$, we have $\lim_{p \rightarrow 0} K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} = 1$. \square

REMARK 2. Under the assumptions of Theorem 2, Seo in [12, Theorem 4.2] proved that

$$\|e^{(1-\alpha)H + \alpha K}\|_u \leq K(e^{(M-m)}, p)^{-\frac{\alpha}{p}} K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \|(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}\|_u, \quad 0 < p \leq 1,$$

and

$$\|e^{(1-\alpha)H + \alpha K}\|_u \leq K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \|(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}\|_u, \quad p \geq 1.$$

But the first inequality of Theorem 2 shows that the sharper constant for all $p > 0$ is $K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}}$. Since for $0 < p \leq 1$, $K(e^{(M-m)}, p)^{-\frac{\alpha}{p}} \geq 1$ and hence

$$K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \leq K(e^{(M-m)}, p)^{-\frac{\alpha}{p}} K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}}.$$

4. Some related results

It has been shown [6] that if $f : [0, \infty) \rightarrow [0, \infty)$ is operator monotone function and $0 < ml \leq A \leq B \leq MI \leq I$ with $h = \frac{M}{m}$, then for all $\alpha \in [0, 1]$

$$f(A) \#_{\alpha} f(B) \leq \exp \left(\alpha(1 - \alpha) \left(1 - \frac{1}{h} \right)^2 \right) f(A \#_{\alpha} B). \tag{17}$$

This new ratio has been introduced by Furuichi and Minculete in [4], which is different from Specht ratio and Kantorovich constant. By applying (17) for $f(t) = t^r$, $0 < r \leq 1$ we have the following results similar to Lemma 1 and Lemma 2.

LEMMA 3. Let A and B be positive definite matrices such that $0 < ml \leq A \leq B \leq MI \leq I$ with $h = M/m$, and let $\alpha \in [0, 1]$. Then

$$A^r \#_{\alpha} B^r \leq \exp \left(r\alpha(1 - \alpha) \left(1 - \frac{1}{h} \right)^2 \right) (A \#_{\alpha} B)^r, \quad 0 < r \leq 1.$$

LEMMA 4. Let A and B be positive definite matrices such that $0 < ml \preceq_{ots} A \preceq_{ots} B \preceq_{ots} MI \preceq_{ots} I$ with $h = M/m$, and let $\alpha \in [0, 1]$. Then for all $k = 1, 2, \dots, n$,

$$\lambda_k(A \#_{\alpha} B)^r \leq \exp \left(\alpha(1 - \alpha) \left(1 - \frac{1}{h^r} \right)^2 \right) \lambda_k(A^r \#_{\alpha} B^r), \quad r \geq 1,$$

$$\lambda_k(A^q \#_{\alpha} B^q)^{\frac{1}{q}} \leq \exp \left(\frac{1}{p} \alpha(1 - \alpha) \left(1 - \frac{1}{h^p} \right)^2 \right) \lambda_k(A^p \#_{\alpha} B^p)^{\frac{1}{p}}, \quad 0 < q \leq p. \tag{18}$$

THEOREM 3. *Let H and K be Hermitian matrices such that $e^m I \preceq_{ols} e^H \preceq_{ols} e^K \preceq_{ols} e^M I \preceq_{ols} I$ for some scalars $m \leq M$, and let $\alpha \in [0, 1]$. Then for all $p > 0$ and $k = 1, 2, \dots, n$*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1 - \frac{1}{e^{p(M-m)}}\right)^2\right) \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}},$$

and so, for every unitarily invariant norm $\|\cdot\|_u$

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leq \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1 - \frac{1}{e^{p(M-m)}}\right)^2\right) \|(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}\|_u.$$

Proof. The proof is similar to that of Theorem 1, by replacing A and B with e^H and e^K , and $h = e^{M-m}$ in the inequality (18). \square

REMARK 3. Under the different conditions, the different coefficients are not comparable. But it is known that if we have a certain statement under the sandwich condition $0 < sA \leq B \leq tA$, then the same statement is also true under the condition $0 < ml \leq A, B \leq MI$ and $0 < ml \leq A \leq B \leq MI \leq I$. Hence, we can compare the following special cases:

(1) Comparison of the constants in Theorem 3 and in Theorem 2:

Let $e^m I \preceq_{ols} e^H \preceq_{ols} e^K \preceq_{ols} e^M I \preceq_{ols} I$. Operator monotony of $\log(t)$ leads to $ml \leq H \leq K \leq MI \leq I$, and so $ml \leq H, K \leq MI$. Now by applying Theorem 2 we have

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}, \quad p > 0.$$

Also, by Theorem 3

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1 - \frac{1}{e^{p(M-m)}}\right)^2\right) \lambda_k(e^{pH} \#_{\alpha} e^{pK})^{\frac{1}{p}}, \quad p > 0.$$

Letting $h = e^{M-m} \geq 1$, the following numerical examples show that there is no ordering between these inequalities.

(i) Take $\alpha = \frac{1}{2}$, $p = \frac{1}{2}$ and $h = 2$, then we have

$$K(h^{2p}, \alpha)^{-\frac{1}{p}} - \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1 - \frac{1}{h^p}\right)^2\right) \simeq -0.0134963.$$

(ii) Take $\alpha = \frac{1}{2}$, $p = \frac{1}{2}$ and $h = 8$, then we have

$$K(h^{2p}, \alpha)^{-\frac{1}{p}} - \exp\left(\frac{1}{p}\alpha(1-\alpha)\left(1 - \frac{1}{h^p}\right)^2\right) \simeq 0.0631159.$$

(2) Comparison of the constants in Lemma 3 and in Lemma 1:

Let $0 < ml \leq A \leq B \leq MI \leq I$. Then the following sandwich condition is obtained

$$m \leq \frac{m}{M} \leq 1 \leq A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq \frac{M}{m} \leq \frac{1}{m}.$$

Now by letting $s = 1$ and $t = \frac{M}{m} = h$ in Lemma 1, we get

$$A^r \sharp_{\alpha} B^r \leq S(h)^r (A \sharp_{\alpha} B)^r, \quad 0 < r \leq 1. \quad (19)$$

Also, by Lemma 3

$$A^r \sharp_{\alpha} B^r \leq \exp\left(r\alpha(1-\alpha)\left(1-\frac{1}{h}\right)^2\right) (A \sharp_{\alpha} B)^r, \quad 0 < r \leq 1. \quad (20)$$

It is shown in [4, Remark 2.4] that there is no ordering between coefficients of (19) and (20). Therefore, we may conclude evaluation of Lemma 3 and Lemma 1 are different.

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