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SOME INEQUALITIES RELATED TO p–SCHATTEN NORM

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Abstract. In this paper, we investigate the known operator inequalities for the p-Schatten norm and obtain some refinements of these inequalities when parameters taking values in different regions. Let $A_1,\cdots,A_n,B_1,\cdots,B_n \in B_p(H)$ such that $\Sigma_{i,j=1}^n A_i^* B_j = 0$. Then for $0 < p \leqslant 2$, $p \geqslant \lambda > 0$ and $\mu \geqslant 2$,

$$n^{2(\frac{1}{p}-\frac{1}{\lambda})}(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^{\lambda})^{\frac{1}{\lambda}} \leqslant n^{\frac{2}{p}-\frac{1}{2}}(\sum_{i=1}^n \|A_i\|_p^p + \sum_{i=1}^n \|B_i\|_p^p)^{\frac{1}{p}}$$
$$\leqslant 2^{\frac{1}{p}-\frac{2}{p\mu}} n^{\frac{3}{p}-\frac{2}{p\mu}-\frac{1}{2}}(\sum_{i=1}^n \|A_i\|_p^{\frac{p\mu}{2}} + \sum_{i=1}^n \|B_i\|_p^{\frac{p\mu}{2}})^{\frac{2}{p\mu}}.$$

For $p \geqslant 2$, $p \leqslant \lambda$ and $0 < \mu \leqslant 2$, the inequalities are reversed. Moreover, we get some applications of our results.

1. Introduction

Let $B(H)$ be the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert space H. $|A| = (X^*X)^{\frac{1}{2}}$ denotes the absolute value of an operator $A \in B(H)$. If $A \in B(H)$ is compact, let $\{s_j(A)\}_{j=1}^{\infty}$ be the sequence of decreasingly ordered singular values of A. For $0 < p < \infty$, let $\|A\|_p = (tr|A|^p)^{\frac{1}{p}} = (\sum_{j=1}^{\infty} s_j^p(A))^{\frac{1}{p}}$, where tr is the usual trace function. This defines the Schatten p-norm (quasi-norm, resp.) for $1 \leqslant p < \infty$ ($0 < p < 1$, resp.) on the set

$$B_p(H) = \{X \in B(H) : \|X\|_p < \infty\},$$

which is called the p-Schatten class of $B(H)$ (see [5]). The Schatten p-norms are unitarily invariant and when $p = 1$, $\|A\|_1 = tr|A|$ is called the trace norm of A.

There are some classical Clarkson's inequalities for the Schatten p-norms of operators in $B_p(H)$ (See [3]). If A, $B \in B_p(H)$, then

$$2^{p-1}(\|A\|_p^p + \|B\|_p^p) \leqslant \|A-B\|_p^p + \|A+B\|_p^p \leqslant 2(\|A\|_p^p + \|B\|_p^p) \tag{1.1}$$

for $0 < p \leqslant 2$ and

$$2(\|A\|_p^p + \|B\|_p^p) \leqslant \|A-B\|_p^p + \|A+B\|_p^p \leqslant 2^{p-1}(\|A\|_p^p + \|B\|_p^p) \tag{1.2}$$

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for $2 \leqslant p < \infty$. For $p = 2$, by (1.1) and (1.2), we have

$$\|A-B\|_2^2 + \|A+B\|_2^2 = 2(\|A\|_2^2 + \|B\|_2^2),$$

which is called parallelogram law. When $p \neq 2$, the equality $2(\|A\|_p^p + \|B\|_p^p) = \|A - B\|_p^p + \|A+B\|_p^p$ holds if and only if $A^*B = AB^* = 0$, or equivalently $R(A)$ and $R(B)$ are orthogonal. (See [3]).

Hirzallah, Kittaneh and Moslehian etc. have obtained some generalizations of (1.1) to n-tuples of operators and many different conclusions by using various methods such as complex interpolation method, concavity and convexity of certain functions, etc. (See $[1,6,7,8]$).

Recently, some refinements of some p-Schatten inequalities have been given by Conde and Moslehian in $[4]$.

THEOREM 1.1. ([4]) *Let $A_1,\cdots,A_n,B_1,\cdots,B_n \in B_p(H)$ such that $\Sigma_{i,j=1}^n A_i^* B_j = 0$, then for $0 < p \leqslant 2$, $p \leqslant \lambda$ and $0 < \mu \leqslant 2$,*

$$\begin{aligned} 2^{\frac{1}{2}-\frac{1}{\mu}} n^{1-\frac{1}{\mu}} \Big(\sum_{i=1}^n \|A_i\|_p^\mu + \sum_{i=1}^n \|B_i\|_p^\mu\Big)^{\frac{1}{\mu}} &\leqslant n^{\frac{1}{2}} \Big(\sum_{i=1}^n \|A_i\|_p^2 + \sum_{i=1}^n \|B_i\|_p^2\Big)^{\frac{1}{2}} \\ &\leqslant n^{2(\frac{1}{p}-\frac{1}{\lambda})} \Big(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda\Big)^{\frac{1}{\lambda}}. \end{aligned} \tag{1.3}$$

For $2 \leqslant p$, $0 < \lambda \leqslant p$ and $2 \leqslant \mu$, the inequalities are reversed.

THEOREM 1.2. ([4]) *Let $A_1,\cdots,A_n,B_1,\cdots,B_n \in B_p(H)$ such that $\Sigma_{i,j=1}^n A_i^* B_j = 0$, then for $0 < p \leqslant 2$, $p \leqslant \lambda$ and $0 < \mu \leqslant 2$,*

$$n\Big(\frac{1}{n^2}\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\mu\Big)^{\frac{1}{\mu}} \leqslant n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{\lambda}} \Big(\sum_{i=1}^n \|(|A_i|^2 + |B_i|^2)^{\frac{1}{2}}\|_p^\lambda\Big)^{\frac{1}{\lambda}}. \tag{1.4}$$

For $2 \leqslant p$, $0 < \lambda \leqslant p$ and $2 \leqslant \mu$, the inequality is reversed.

In this paper, motivated by the above conclusions, we consider some refinements of p-Schatten norm inequalities when p, λ and μ taking values in different regions.

2. Main results

In this section we consider the p-Schatten norm inequalities of (1.3) and (1.4) when parameters taking values in different regions. We start our works with the following lemmas that we will use along the paper.

FACT 1. $M_s(\overline{x}) \leqslant M_{s'}(\overline{x})$ for $0 < s \leqslant s'$, where $M_s(\overline{x}) = (\frac{1}{n}\sum_{i=1}^n x_i^s)^{\frac{1}{s}}$, $\overline{x} = (x_1,\cdots,x_n)$ is an n-tuples of non-negative numbers.

FACT 2. $\|T\|_p^2 = \||T|^2\|_{\frac{p}{2}}$ for any $T \in B_p(H)$ with $p > 0$.

Proof. By the definition of Schatten p-norm, we have $\|T\|_p^2 = (tr|T|^p)^{\frac{2}{p}}$ and $\||T|^2\|_{\frac{p}{2}} = (tr(|T|^2)^{\frac{p}{2}})^{\frac{2}{p}} = (tr|T|^p)^{\frac{2}{p}}$. $\square$

LEMMA 2.1. ([4]) *Let* $A_1,\cdots,A_n,B_1,\cdots,B_n \in B(H)$ *such that* $\Sigma_{i,j=1}^n A_i^*B_j = 0$, *then*

$$\sum_{i,j=1}^n |A_i \pm B_j|^2 = \sum_{i,j=1}^n |A_i|^2 + |B_j|^2 \pm \sum_{i,j=1}^n A_i^*B_j + B_j^*A_i$$
$$= \sum_{i,j=1}^n |A_i|^2 + |B_j|^2. \tag{2.1}$$

LEMMA 2.2. ([2–3]) *If* $A_1,\cdots,A_n \in B_p(H)$ *for some* $p > 0$, *and* $A_1,\cdots,A_n$ *are positive, then for* $0 < p \leqslant 1$,

$$n^{p-1}\sum_{i=1}^n \|A_i\|_p^p \leqslant (\sum_{i=1}^n \|A_i\|_p)^p \leqslant \|\sum_{i=1}^n A_i\|_p^p \leqslant \sum_{i=1}^n \|A_i\|_p^p \tag{2.2}$$

and for $1 \leqslant p < \infty$ *the inequalities are reversed.*

They are a refinement of Lemma 2.1 in [7]. A commutative version of the previous lemma for scalars is the following:

Let $\overline{x} = (x_1,\ldots,x_n)$ be an n-tuples of non-negative numbers, then

$$n^{p-1}\sum_{i=1}^n x_i^p \leqslant (\sum_{i=1}^n x_i)^p \leqslant \sum_{i=1}^n x_i^p \tag{2.3}$$

for $0 < p \leqslant 1$ and

$$\sum_{i=1}^n x_i^p \leqslant (\sum_{i=1}^n x_i)^p \leqslant n^{p-1}\sum_{i=1}^n x_i^p \tag{2.4}$$

for $1 \leqslant p < \infty$.

LEMMA 2.3. ([2]) *If* $T_1,\cdots,T_n$ *are positive operators in* $B_p(H)$ *then*

$$\|\sum_{i=1}^n T_i\|_p \geqslant \sum_{i=1}^n \|T_i\|_p \tag{2.5}$$

for $0 < p < 1$.

THEOREM 2.4. *Let* $A_1,\cdots,A_n,B_1,\cdots,B_n \in B_p(H)$ *such that* $\Sigma_{i,j=1}^n A_i^*B_j = 0$. *Then for* $0 < p \leqslant 2$, $p \geqslant \lambda > 0$ *and* $\mu \geqslant 2$,

$$n^{2(\frac{1}{p}-\frac{1}{\lambda})}(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda)^{\frac{1}{\lambda}} \leqslant n^{\frac{2}{p}-\frac{1}{2}}(\sum_{i=1}^n \|A_i\|_p^p + \sum_{i=1}^n \|B_i\|_p^p)^{\frac{1}{p}}$$
$$\leqslant 2^{\frac{1}{p}-\frac{2}{p\mu}} n^{\frac{3}{p}-\frac{2}{p\mu}-\frac{1}{2}}(\sum_{i=1}^n \|A_i\|_p^{\frac{p\mu}{2}} + \sum_{i=1}^n \|B_i\|_p^{\frac{p\mu}{2}})^{\frac{2}{p\mu}}.$$

For $p \geqslant 2$, $p \leqslant \lambda$ *and* $0 < \mu \leqslant 2$, *the inequalities are reversed.*

Proof. Let $0 < p \leqslant 2,\ p \geqslant \lambda > 0,\ \mu \geqslant 2$. It follows from $M_\lambda(\overline{x}) \leqslant M_p(\overline{x})$ that

$$n^{2(\frac{1}{p}-\frac{1}{\lambda})}(\sum_{i,j=1}^{n}\|A_i \pm B_j\|_p^\lambda)^{\frac{1}{\lambda}} = n^{\frac{2}{p}}(\frac{1}{n^2}\sum_{i,j=1}^{n}\||A_i \pm B_j|\|_p^\lambda)^{\frac{1}{\lambda}}$$
$$\leqslant (\sum_{i,j=1}^{n}\||A_i \pm B_j|\|_p^p)^{\frac{1}{p}}.$$

Applying the Fact 2, formula (2.3), Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned}
(\sum_{i,j=1}^{n}\||A_i \pm B_j|\|_p^p)^{\frac{1}{p}} &= (\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{\frac{p}{2}}^{\frac{p}{2}})^{\frac{1}{p}} \\
&\leqslant [(n^2)^{1-\frac{p}{2}}(\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{\frac{p}{2}})^{\frac{p}{2}}]^{\frac{1}{p}} \\
&\leqslant ((n^2)^{1-\frac{p}{2}}\|\sum_{i,j=1}^{n}|A_i \pm B_j|^2\|_{\frac{p}{2}}^{\frac{p}{2}})^{\frac{1}{p}} \\
&= n^{\frac{2}{p}-1}\|\sum_{i,j=1}^{n}|A_i \pm B_j|^2\|_{\frac{p}{2}}^{\frac{1}{2}} \\
&= n^{\frac{2}{p}-1}\|\sum_{i,j=1}^{n}|A_i|^2 + |B_j|^2\|_{\frac{p}{2}}^{\frac{1}{2}} \\
&= n^{\frac{2}{p}-1}n^{\frac{1}{2}}\|\sum_{i=1}^{n}|A_i|^2 + \sum_{i=1}^{n}|B_i|^2\|_{\frac{p}{2}}^{\frac{1}{2}} \\
&= n^{\frac{2}{p}-\frac{1}{2}}\|\sum_{i=1}^{n}|A_i|^2 + \sum_{i=1}^{n}|B_i|^2\|_{\frac{p}{2}}^{\frac{1}{2}}.
\end{aligned}$$

Using Lemma 2.2 and the convexity of the function $f(x) = x^\alpha$ on $[0,+\infty)$ for $1 \leqslant \alpha < \infty$, we obtain

$$\begin{aligned}
n^{\frac{2}{p}-\frac{1}{2}}\|\sum_{i=1}^{n}|A_i|^2 + \sum_{i=1}^{n}|B_i|^2\|_{\frac{p}{2}}^{\frac{1}{2}} &\leqslant n^{\frac{2}{p}-\frac{1}{2}}[(\sum_{i=1}^{n}(\||A_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}} + \||B_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}}))^{\frac{2}{p}}]^{\frac{1}{2}} \\
&= n^{\frac{2}{p}-\frac{1}{2}}(\sum_{i=1}^{n}\||A_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}} + \sum_{i=1}^{n}\||B_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}})^{\frac{1}{p}} \\
&= n^{\frac{2}{p}-\frac{1}{2}}[(\sum_{i=1}^{n}\||A_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}} + \sum_{i=1}^{n}\||B_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}})^{\frac{\mu}{2}}]^{\frac{2}{p\mu}} \\
&= n^{\frac{2}{p}-\frac{1}{2}}[(2n)^{\frac{\mu}{2}}(\frac{1}{2n}(\sum_{i=1}^{n}\||A_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}} + \sum_{i=1}^{n}\||B_i|^2\|_{\frac{p}{2}}^{\frac{p}{2}}))^{\frac{\mu}{2}}]^{\frac{2}{p\mu}} \\
&\leqslant n^{\frac{2}{p}-\frac{1}{2}}[(2n)^{\frac{\mu}{2}}\frac{1}{2n}(\sum_{i=1}^{n}\||A_i|^2\|_{\frac{p}{2}}^{\frac{p\mu}{4}} + \sum_{i=1}^{n}\||B_i|^2\|_{\frac{p}{2}}^{\frac{p\mu}{4}})]^{\frac{2}{p\mu}}
\end{aligned}$$

$$= n^{\frac{2}{p}-\frac{1}{2}}(2n)^{\frac{1}{p}-\frac{2}{p\mu}}(\sum_{i=1}^{n}\|A_i\|_p^{\frac{p\mu}{2}}+\sum_{i=1}^{n}\|B_i\|_p^{\frac{p\mu}{2}})^{\frac{2}{p\mu}}$$
$$= 2^{\frac{1}{p}-\frac{2}{p\mu}}n^{\frac{3}{p}-\frac{2}{p\mu}-\frac{1}{2}}(\sum_{i=1}^{n}\|A_i\|_p^{\frac{p\mu}{2}}+\sum_{i=1}^{n}\|B_i\|_p^{\frac{p\mu}{2}})^{\frac{2}{p\mu}}.$$

When $p \geqslant 2,\ p \leqslant \lambda$ and $0 < \mu \leqslant 2$, a similar argument shows that the equalities are reversed. □

COROLLARY 2.5. *Let $A_1,\cdots,A_n,B_1,\cdots,B_n \in B_p(H)$ such that $\sum_{i,j=1}^{n} A_i^* B_j = 0$. Then*

$$\sum_{i,j=1}^{n}\|A_i \pm B_j\|_2^2 = n(\sum_{i=1}^{n}\|A_i\|_2^2+\sum_{i=1}^{n}\|B_i\|_2^2).$$

Proof. Motivated by Theorem 2.4, let $\lambda = \mu = p = 2$. □

COROLLARY 2.6. *Let $A_1,\cdots,A_n \in B_p(H)$ such that $\sum_{i=1}^{n} A_i = 0$. Then*

$$\sum_{i,j=1}^{n}\|A_i \pm A_j\|_2^2 = 2n\sum_{i=1}^{n}\|A_i\|_2^2.$$

Proof. $\sum_{i=1}^{n} A_i = 0$ implies that $\sum_{i,j=1}^{n} A_i^* A_j = 0$. The statement is a consequence of Corollary 2.5. □

THEOREM 2.7. *Let $A_1,\cdots,A_n,B_1,\cdots,B_n \in B_p(H)$ such that $\sum_{i,j=1}^{n} A_i^* B_j = 0$. Then for $0 < p \leqslant 2,\ p \geqslant \lambda > 0$ and $\mu \geqslant 2$,*

$$n(\frac{1}{n^2}\sum_{i,j=1}^{n}\|A_i \pm B_j\|_p^{\mu})^{\frac{1}{\mu}} \geqslant n^{2-\frac{2}{p}-\frac{1}{\lambda}}(\sum_{i=1}^{n}\|(|A_i|^2+|B_i|^2)^{\frac{1}{2}}\|_p^{\lambda})^{\frac{1}{\lambda}}.$$

For $p \geqslant 2,\ p \leqslant \lambda$ and $0 < \mu \leqslant 2$, the inequality is reversed.

Proof. We suppose that $0 < p \leqslant 2,\ p \geqslant \lambda > 0$ and $\mu \geqslant 2$. Using Fact 2, formula (2.3), (2.5)and Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
n\Big[\frac{1}{n^2}\sum_{i,j=1}^{n}\|A_i \pm B_j\|_p^{\mu}\Big]^{\frac{1}{\mu}} &= n\Big[\frac{1}{n^2}\sum_{i,j=1}^{n}(\|A_i \pm B_j\|_p^2)^{\frac{\mu}{2}}\Big]^{\frac{1}{\mu}} \\
&= n\Big(\frac{1}{n^2}\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{\frac{p}{2}}^{\frac{\mu}{2}}\Big)^{\frac{1}{\mu}} \\
&\geqslant n[n^{-\mu}(\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{\frac{p}{2}})^{\frac{\mu}{2}}]^{\frac{1}{\mu}} \\
&= (\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{\frac{p}{2}})^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant [(n^2)^{\frac{p}{2}-1}\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{\frac{p}{2}}^{\frac{p}{2}}]^{\frac{1}{p}}\\
&= [n^{p-2}\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{\frac{p}{2}}^{\frac{p}{2}}]^{\frac{1}{p}}\\
&\geqslant [n^{p-2}\|\sum_{i,j=1}^{n}|A_i \pm B_j|^2\|_{\frac{p}{2}}^{\frac{p}{2}}]^{\frac{1}{p}}\\
&= [n^{\frac{3p}{2}-2}\|\sum_{i=1}^{n}(|A_i|^2+|B_i|^2)\|_{\frac{p}{2}}^{\frac{p}{2}}]^{\frac{1}{p}}\\
&\geqslant [n^{\frac{3p}{2}-2}(\sum_{i=1}^{n}\||A_i|^2+|B_i|^2\|_{\frac{p}{2}})^{\frac{p}{2}}]^{\frac{1}{p}}\\
&= n^{\frac{3}{2}-\frac{2}{p}}[\sum_{i=1}^{n}\||A_i|^2+|B_i|^2\|_{\frac{p}{2}}]^{\frac{1}{2}}\\
&= n^{\frac{3}{2}-\frac{2}{p}}[\sum_{i=1}^{n}\|(|A_i|^2+|B_i|^2)^{\frac{1}{2}}\|_p^2]^{\frac{1}{2}}\\
&= n^{\frac{3}{2}-\frac{2}{p}}[\sum_{i=1}^{n}(\|(|A_i|^2+|B_i|^2)^{\frac{1}{2}}\|_p^{\lambda})^{\frac{2}{\lambda}}]^{\frac{1}{2}}\\
&\geqslant n^{\frac{3}{2}-\frac{2}{p}}[n^{1-\frac{2}{\lambda}}(\sum_{i=1}^{n}\|(|A_i|^2+|B_i|^2)^{\frac{1}{2}}\|_p^{\lambda})^{\frac{2}{\lambda}}]^{\frac{1}{2}}\\
&= n^{2-\frac{2}{p}-\frac{1}{\lambda}}(\sum_{i=1}^{n}\|(|A_i|^2+|B_i|^2)^{\frac{1}{2}}\|_p^{\lambda})^{\frac{1}{\lambda}}.
\end{aligned}
$$

When $p \geqslant 2,\ p \leqslant \lambda$ and $0 < \mu \leqslant 2$, a similar argument shows that the inequalities are reversed. □

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