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BOUNDEDNESS OF MARCINKIEWICZ INTEGRALS ON HARDY SPACES H^p OVER NON–HOMOGENEOUS METRIC MEASURE SPACES

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Abstract. Under the assumption that $(\mathscr{X}, d, \mu)$ is a non-homogeneous metric measure space, the authors prove that the Marcinkiewicz integral operator is bounded from the molecular Hardy space $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ (or the atomic Hardy space $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$) into the Lebesgue space $L^p(\mu)$. To this end, some boundedness criteria on these Hardy spaces are established.

1. Introduction

It is known that the Marcinkiewicz integral, introduced by Marcinkiewicz [19] on the one-dimensional Euclidean space $\mathbb{R}$ and by Stein [24] on the higher-dimensional Euclidean space $\mathbb{R}^n$, plays an important role in harmonic analysis and PDE. In the classical Euclidean space equipped with the Lebesgue measure, there are lots of papers focus on the boundedness of the Marcinkiewicz integral operator on varieties of function spaces; see, for example, [34, 31, 8, 18] and the references therein.

On the other hand, many theories of harmonic analysis on the classical Euclidean spaces have been generalized into the metric measure spaces. One of them is the *space of homogeneous type* in the sense of Coifman and Weiss [6, 7], that is, a metric space $(\mathscr{X}, d)$ equipped with a non-negative measure μ satisfying the following *measure doubling condition*: there exists a positive constant $C_{(\mu)}$ such that, for all balls $B(x,r) := \{y \in \mathscr{X} : d(x,y) < r\}$ with $x \in \mathscr{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,2r)) \leqslant C_{(\mu)}\mu(B(x,r)). \tag{1.1}$$

Another generalized setting is the metric measure space with *non-doubling measure*. To be precise, if μ is a non-negative Radon measure on $\mathbb{R}^n$ satisfying the *polynomial growth condition* that there exist some positive constants C_0 and $\kappa \in (0,n]$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \leqslant C_0 r^\kappa, \tag{1.2}$$

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then it may not satisfy the doubling condition (1.1). The analysis on such non-doubling context has proved to play a striking role in solving the long-standing open Vitushkin's conjecture and Painlevé's problem; see [29, 30]. Moreover, many classical results concerning the usual operators (such as the Calderón-Zygmund operators) and function spaces have been proved still valid for this setting; see, for example, [20, 27, 28, 29, 21, 22, 23, 5, 4, 12, 32] and the references therein.

However, the measure μ as in (1.2) is different from, not general than the doubling measure as in (1.1); see Hytönen [13]. In [13], Hytönen introduced a new class of metric measure spaces, which include both spaces of homogeneous type and metric spaces with non-doubling measures as special cases. These new metric measure spaces are assumed to satisfy both the so-called upper doubling condition and the geometrically doubling condition (see, respectively, Definitions 1.1 and 1.2 below) and are called as *non-homogeneous metric measure spaces*. In this new setting, Lin and Yang [17] established the equivalent boundedness of Marcinkiewicz integral operators. Recently, Fu et al. [9] introduced the Hardy spaces H^p and obtained the boundedness of Calderón-Zygmund operators on these spaces. More research on function spaces and the boundedness of various operators in this setting can be found in [14, 15, 10, 26, 25, 2, 3] and the references therein. We refer the reader to the monograph [33] for more developments on harmonic analysis in this new context.

In this paper, we establish the boundedness of the Marcinkiewicz integral operator on the Hardy space H^p over non-homogeneous metric measure spaces.

In order to state our main results, we first recall the following notion of upper doubling metric measure spaces originally introduced by Hytönen [13].

DEFINITION 1.1. A metric measure space $(\mathscr{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathscr{X}$ and there exist a *dominating function* $\lambda : \mathscr{X} \times (0,\infty) \to (0,\infty)$ and a positive constant $C_{(\lambda)}$, depending on λ, such that, for each $x \in \mathscr{X}$, $r \to \lambda(x,r)$ is non-decreasing and, for all $x \in \mathscr{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \leqslant \lambda(x,r) \leqslant C_{(\lambda)}\lambda(x,r/2). \tag{1.3}$$

REMARK 1.1. (i) If we take $\lambda(x,r) := \mu(B(x,r))$ for all $x \in \mathscr{X}$ and $r \in (0,\infty)$, then the upper doubling space goes back to the space of homogeneous type. Moreover, let $(\mathscr{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mu)$ with μ be as in (1.2). By taking $\lambda(x,r) := C_0 r^\kappa$ for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, we see that it is also an upper doubling space.

(ii) Hytönen et al. in [16] proved that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \leqslant \lambda$, $C_{(\widetilde{\lambda})} \leqslant C_{(\lambda)}$ and, for all $x, y \in \mathscr{X}$ with $d(x,y) \leqslant r$,

$$\widetilde{\lambda}(x,r) \leqslant C_{(\widetilde{\lambda})}\widetilde{\lambda}(y,r). \tag{1.4}$$

Based on this, in the whole paper, we *always assume* that λ satisfies (1.4).

The following notion of geometrically doubling can be found in [6, pp. 66–67] and is also known as *metrically doubling* (see [11, p. 81]). It should be pointed out that Coifman and Weiss in [6, pp. 66–68] proved that spaces of homogeneous type are geometrically doubling.

DEFINITION 1.2. A metric space $(\mathscr{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N} := \{1, 2, \ldots\}$ such that, for any ball $B(x, r) \subset \mathscr{X}$ with $x \in \mathscr{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Now we recall the definition of the discrete coefficient $\widetilde{K}_{B,S}^{(\rho),p}$ first introduced by Bui and Duong [1] when $p = 1$. Before this we first give an assumption that, for any two balls $B, S \subset \mathscr{X}$, *if* $B = S$, *then* $c_B = c_S$ *and* $r_B = r_S$, here and hereafter, for any ball B, we denote its *center* and *radius*, respectively, by c_B and r_B; see [11, pp. 1–2]. This shows that if $B \subset S$, then $r_B \leqslant 2r_S$, which guarantees that the definition of $\widetilde{K}_{B,S}^{(\rho),p}$ makes sense; see [9, pp. 314–315] for the details.

DEFINITION 1.3. For any $\rho \in (1, \infty)$, $p \in (0, 1]$ and any two balls $B \subset S \subset \mathscr{X}$, let

$$\widetilde{K}_{B,S}^{(\rho),p} := \left\{ 1 + \sum_{k=-\lfloor \log_\rho 2 \rfloor}^{N_{B,S}^{(\rho)}} \left[\frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)} \right]^p \right\}^{1/p},$$

here and hereafter, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the *biggest integer which is not bigger than* a, and $N_{B,S}^{(\rho)}$ is the *smallest integer* satisfying $\rho^{N_{B,S}^{(\rho)}} r_B \geqslant r_S$.

REMARK 1.2. (i) Obviously,

$$\widetilde{K}_{B,S}^{(\rho),p} \sim \left\{ 1 + \sum_{k=1}^{N_{B,S}^{(\rho)} + \lfloor \log_\rho 2 \rfloor + 1} \left[\frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)} \right]^p \right\}^{1/p}.$$

(ii) Hytönen in [13] introduced the following coefficient $K_{B,S}$, which can be seen as a continuous version of the $\widetilde{K}_{B,S}^{(\rho),p}$ when $p = 1$: for any two balls $B \subset S \subset \mathscr{X}$,

$$K_{B,S} := 1 + \int_{(2S)\setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \, d\mu(x). \tag{1.5}$$

On $(\mathbb{R}^n, |\cdot|, \mu)$ with μ as in (1.2), $K_{B,S} \sim \widetilde{K}_{B,S}^{(\rho),1}$, but $K_{B,S}$ and $\widetilde{K}_{B,S}^{(\rho),1}$ are usually not equivalent on non-homogeneous metric measure spaces; see [10] for the details.

DEFINITION 1.4. Let $\rho \in (1, \infty)$, $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$, and $\gamma \in [1, \infty)$. A function b in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$ is called a $(p, q, \gamma, \rho)_\lambda$-*atomic* block if

(i) there exists a ball B such that $\operatorname{supp}(b) \subset B$;

(ii) $\int_{\mathscr{X}} b(x) d\mu(x) = 0$;

(iii) for any $j \in \{1, 2\}$, there exist a function a_j supported on a ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$ and

$$\|a_j\|_{L^q(\mu)} \leqslant [\mu(\rho B_j)]^{1/q-1}[\lambda(c_B, r_B)]^{1-1/p}[\widetilde{K}_{B_j,B}^{(\rho),p}]^{-\gamma}.$$

Moreover, let $|b|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} := |\lambda_1| + |\lambda_2|$.

A function f is said to belong to the *space* $\widetilde{\mathbb{H}}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ if there exists a sequence of $(p,q,\gamma,\rho)_\lambda$-atomic blocks, $\{b_i\}_{i=1}^\infty$, such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$, and

$$\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} < \infty.$$

Moreover, define

$$\|f\|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} := \inf\left\{\left[\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

The *atomic Hardy space* $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ is then defined as the completion of $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ with respect to the p-quasi-norm $\|\cdot\|^p_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}$.

DEFINITION 1.5. Let $\rho \in (1,\infty)$, $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$, $\gamma \in [1,\infty)$ and $\varepsilon \in (0,\infty)$. A function b in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$ is called a $(p,q,\gamma,\varepsilon,\rho)_\lambda$-*molecular* block if

(i) $\int_{\mathscr{X}} b(x)\,d\mu(x) = 0$;

(ii) there exist some ball $B := B(c_B, r_B)$, with $c_B \in \mathscr{X}$ and $r_B \in (0,\infty)$, and some constants $\widetilde{M}, M \in \mathbb{N}$ such that, for all $k \in \mathbb{Z}_+$ and $j \in \{1,\cdots,M_k\}$ with $M_k = \widetilde{M}$ if $k = 0$ and $M_k = M$ if $k \in \mathbb{N}$, there exist functions $m_{k,j}$ supported on some balls $B_{k,j} \subset U_k(B)$ for all $k \in \mathbb{Z}_+$, where $U_0(B) := \rho^2 B$ and $U_k(B) := \rho^{k+2}B \setminus \rho^{k-2}B$ with $k \in \mathbb{N}$, and $\lambda_{k,j} \in \mathbb{C}$ such that $b = \sum_{k=0}^\infty \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j}$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$,

$$\|m_{k,j}\|_{L^q(\mu)} \leqslant \rho^{-k\varepsilon}[\mu(\rho B_{k,j})]^{1/q-1}[\lambda(c_B, \rho^{k+2}r_B)]^{1-1/p}[\widetilde{K}^{(\rho),p}_{B_{k,j},\rho^{k+2}B}]^{-\gamma} \tag{1.6}$$

and

$$|b|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} := \sum_{k=0}^\infty \sum_{j=1}^{M_k} |\lambda_{k,j}|^p < \infty.$$

A function f is said to belong to the *space* $\widetilde{\mathbb{H}}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ if there exists a sequence of $(p,q,\gamma,\varepsilon,\rho)_\lambda$-molecular blocks, $\{b_i\}_{i=1}^\infty$, such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$, and

$$\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} < \infty.$$

Moreover, define

$$\|f\|_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} := \inf\left\{\left[\sum_{i=1}^{\infty} |b_i|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

The *molecular Hardy space* $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ is then defined as the completion of $\widetilde{\mathbb{H}}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ with respect to the p-quasi-norm $\|\cdot\|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}$.

REMARK 1.3. (i) When $p=1$, the atomic Hardy space $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and the molecular Hardy space $\widetilde{H}^{1,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ were introduced by Fu et al. [10]. It was proved in [10] that $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu) = \widetilde{H}^{1,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ and they are independent of the choices of ρ, q, γ and ε. Thus, in what follows, we denote $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ simply by $\widetilde{H}^1(\mu)$.

(ii) When $p \in (0,1)$, it is unclear whether the atomic Hardy space $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and the molecular Hardy space $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ are independent of the choices of ρ, q, γ and ε. Moreover, $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu) \subset \widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ in the sense that there exists a map T from $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ to $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ such that, for any $f \in \widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$, there is a unique element $\widetilde{f} \in \widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ satisfying $T(f) = \widetilde{f}$ and $\|\widetilde{f}\|_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} \lesssim \|f\|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}$; see [9].

Let K be a locally integrable function on $(\mathscr{X}\times\mathscr{X})\setminus\{(x,x): x\in\mathscr{X}\}$. Assume that there exists a positive constant C such that, for all $x, y \in \mathscr{X}$ with $x \neq y$,

$$|K(x,y)| \leqslant C\frac{d(x,y)}{\lambda(x, d(x,y))}; \tag{1.7}$$

and there exist positive constants $\delta \in (0,1]$ and $c_{(K)}$, depending on K, such that, for all $x, \widetilde{x}, y \in \mathscr{X}$ with $d(x,y) \geqslant c_{(K)} d(x,\widetilde{x})$,

$$|K(x,y) - K(\widetilde{x},y)| + |K(y,x) - K(y,\widetilde{x})| \leqslant C\frac{[d(x,\widetilde{x})]^{\delta}[d(x,y)]^{1-\delta}}{\lambda(x, d(x,y))}. \tag{1.8}$$

The Marcinkiewicz integral operator $\mathscr{M}$ with kernel K satisfying (1.7) and (1.8) is defined by setting, for all $x \in \mathscr{X}$,

$$\mathscr{M}(f)(x) := \left[\int_0^{\infty}\left|\int_{d(x,y)<t} K(x,y)f(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right]^{1/2}. \tag{1.9}$$

REMARK 1.4. (i) In the classical Euclidean space $\mathbb{R}^n$, where $\lambda(x,r) := Cr^n$, let

$$K(x,y) := \frac{1}{|x-y|^{n-1}}\Omega(x-y)$$

with Ω being homogeneous of degree zero and $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ for some $\alpha \in (0,1]$, then K satisfies (1.7) and (1.8), and $\mathscr{M}(f)$ in (1.9) is just the Marcinkiewicz integral introduced by Stein [24].

(ii) If K satisfies (1.8), then it also satisfies the following *Hörmander type condition* that there exists a positive constant C such that, for all $y, y' \in \mathscr{X}$,

$$\int_{d(x,y)\geqslant 2d(y,y')} \left[|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)|\right]\frac{1}{d(x,y)}\,d\mu(x) \leqslant C. \quad (1.10)$$

The Marcinkiewicz integral $\mathscr{M}(f)$ associated to K satisfying (1.7) and (1.10) is just the Marcinkiewicz integral on non-homogeneous metric measure spaces in [17].

In what follows, let $\nu := \log_2 C_{(\lambda)}$ and δ be as in (1.8). The main results of this paper are stated as follows.

THEOREM 1.1. *Let* $\rho \in (1,\infty)$, $\dfrac{\nu}{\nu+\min\{\frac14,\delta\}} < p \leqslant 1 < q < \infty$, $\varepsilon \geqslant \max\{\frac14,\delta\}$ *and* $\gamma \in [1,\infty)$. *Assume that the Marcinkiewicz integral operator* $\mathscr{M}$, *defined by* (1.9), *associated with kernel K satisfying* (1.7) *and* (1.8) *is bounded on* $L^2(\mu)$. *Then* $\mathscr{M}$ *is bounded from* $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ *into* $L^p(\mu)$.

THEOREM 1.2. *Let* $\rho \in (1,\infty)$, $\dfrac{\nu}{\nu+\min\{\frac14,\delta\}} < p \leqslant 1 < q < \infty$ *and* $\gamma \in [1,\infty)$. *Assume that the Marcinkiewicz integral operator* $\mathscr{M}$, *defined by* (1.9), *associated with kernel K satisfying* (1.7) *and* (1.8) *is bounded on* $L^2(\mu)$. *Then* $\mathscr{M}$ *is bounded from* $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ *into* $L^p(\mu)$.

The paper is organized as follows. In Section 2, our major job is to establish the boundedness criteria of the operator T on the Hardy spaces $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ with the assumption that T is sublinear when $p=1$ and is non-negative sublinear when $p\in(0,1)$ (see Theorems 2.1 and 2.2 below). Section 3 is devoted to proving Theorems 1.1 and 1.2. To this end, we first recall the properties of the discrete coefficient $\widetilde{K}^{(\rho),p}_{B,S}$ and show the boundedness of $\mathscr{M}$ from the Hardy space $\widetilde{H}^1(\mu)$ to the Lebesgue space $L^1(\mu)$ (see Lemma 3.2 below), and then we prove Theorem 1.1 by using some ideas from [9, Theorem 4.8] with much more complicated demonstrations. Theorem 1.2 can be seen as a corollary of Theorem 1.1.

Throughout this paper, C denotes a *positive constant* that is independent of the main parameters, but whose value may vary from line to line. We denote by $C_{(\alpha)}$ a positive constant depending on the parameter α. The expression $Y \lesssim Z$ means that there exists a positive constant C such that $Y \leqslant CZ$. The expression $A \sim B$ means that $A \lesssim B \lesssim A$. Given any $q \in (0,\infty)$, its *conjugate index* is denoted by $q' := q/(q-1)$.

2. Boundedness criteria

THEOREM 2.1. *Let $\rho, q \in (1,\infty)$, $\gamma \in [1,\infty)$ and T be a sublinear operator bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. If there exists a positive constant C such that, for all $(1,q,\gamma,\rho)_\lambda$-atomic blocks b,*

$$\|Tb\|_{L^1(\mu)} \leqslant C|b|_{\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)}, \tag{2.1}$$

then T is extended to be a bounded sublinear operator from $\widetilde{H}^1(\mu)$ to $L^1(\mu)$.

Proof. The argument is almost the same as the one used in the proof of [32, Theorem 1.13]. We will repeat it for the sake of completeness. Let $f \in \widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $f = \sum_{i=1}^\infty b_i$, where, for each $i \in \mathbb{N}$, b_i is a $(1,q,\gamma,\rho)_\lambda$-atomic block. By the boundedness of T from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, we have that, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} \mu\left(\left\{x \in \mathscr{X} : \left|T\left(\sum_{i=N+1}^\infty b_i\right)(x)\right| > \varepsilon\right\}\right) \lesssim \lim_{N\to\infty} \frac{1}{\varepsilon}\sum_{i=N+1}^\infty \|b_i\|_{L^1(\mu)} = 0.$$

This, via the Riesz theorem, shows that there exists a subsequence $\{T(\sum_{i=1}^{j_k} b_i)\}_k$ of $\{T(\sum_{i=1}^{j} b_i)\}_j$ such that, for μ-a.e. $x \in \mathscr{X}$,

$$\begin{aligned}|T(f)(x)| &\leqslant \left|T\left(\sum_{i=1}^{j_k-1} b_i\right)(x)\right| + \left|T\left(\sum_{i=j_k}^{\infty} b_i\right)(x)\right| \\ &\leqslant \sum_{i=1}^{j_k-1}|T(b_i)(x)| + \left|T\left(\sum_{i=j_k}^{\infty} b_i\right)(x)\right| \to \sum_{i=1}^\infty |T(b_i)(x)|, \quad j_k \to \infty.\end{aligned}$$

By the sublinearity of T, we have that, for μ-a.e. $x \in \mathscr{X}$, $|T(f)(x)| \lesssim \sum_{i=1}^\infty |T(b_i)(x)|$, which together with (2.1), implies that

$$\|T(f)\|_{L^1(\mu)} \lesssim \sum_{i=1}^\infty \|T(b_i)\|_{L^1(\mu)} \lesssim \sum_{i=1}^\infty |b_i|_{\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)}.$$

From this, we conclude that $T(f) \in L^1(\mu)$ and $\|T(f)\|_{L^1(\mu)} \lesssim \|f\|_{\widetilde{H}^1(\mu)}$, which complete the proof of Theorem 2.1. $\square$

THEOREM 2.2. *Let $\rho \in (1,\infty)$, $0 < p < 1 \leqslant q \leqslant \infty$, $\gamma \in [1,\infty)$ and $\varepsilon \in (0,\infty)$. Let T be a non-negative sublinear operator. Assume that T is bounded on $L^2(\mu)$.*

(i) *If there exists a positive constant C such that, for all $(p,q,\gamma,\varepsilon,\rho)_\lambda$-molecular blocks b,*

$$\|T(b)\|_{L^p(\mu)} \leqslant C|b|_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}, \tag{2.2}$$

then T is extended to be a bounded operator from $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ to $L^p(\mu)$.

(ii) *If there exists a positive constant C such that, for all $(p,q,\gamma,\rho)_\lambda$-atomic blocks b,*

$$\|T(b)\|_{L^p(\mu)} \leqslant C|b|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}, \tag{2.3}$$

then T is extended to be a bounded operator from $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ to $L^p(\mu)$.

Proof. To prove (i), we first claim that, for any $f \in \widetilde{\mathbb{H}}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ with $p \in (0,1)$,

$$\|T(f)\|^p_{L^p(\mu)} \lesssim \|f\|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}.$$

In fact, for any $f \in \widetilde{\mathbb{H}}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ with $p \in (0,1)$, there exists a sequence $\{b_i\}_{i\in\mathbb{N}}$ of $(p,q,\gamma,\varepsilon,\rho)_\lambda$-molecular blocks such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ and

$$\sum_{i=1}^\infty |b_i|^p_{\widetilde{\mathbb{H}}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} \sim \|f\|^p_{\widetilde{\mathbb{H}}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}.$$

By the fact that T is a non-negative sublinear operator and is bounded on $L^2(\mu)$, we see that, for any $N \in \mathbb{N}$,

$$\begin{aligned}\left\|T\left(\sum_{i=1}^N b_i\right) - T(f)\right\|_{L^2(\mu)} &\leqslant \left\|T\left(\sum_{i=1}^N b_i - f\right)\right\|_{L^2(\mu)}\\ &\lesssim \left\|\sum_{i=1}^N b_i - f\right\|_{L^2(\mu)} \to 0, \quad \text{as} \quad N\to\infty,\end{aligned}$$

which further implies that, for all $\eta \in (0,\infty)$,

$$\mu\left(\left\{x\in\mathscr{X} : \left|T\left(\sum_{i=1}^N b_i\right)(x) - T(f)(x)\right| > \eta\right\}\right) \to 0, \quad \text{as} \quad N\to\infty. \tag{2.4}$$

From the Riesz theorem, we conclude that there exists a subsequence $\{T(\sum_{i=1}^{N_k} b_i)\}_k$ of $\{T(\sum_{i=1}^N b_i)\}_N$ such that, for μ-a.e. $x \in \mathscr{X}$,

$$T(f)(x) = \lim_{k\to\infty} T\left(\sum_{i=1}^{N_k} b_i\right)(x),$$

which, together with the sublinearity of T, the assumption that $p\in(0,1)$, the Fatou lemma and (2.2), implies that

$$\begin{aligned}\|T(f)\|^p_{L^p(\mu)} &= \left\|\lim_{k\to\infty} T\left(\sum_{i=1}^{N_k} b_i\right)\right\|^p_{L^p(\mu)} \leqslant \int_{\mathscr{X}} \limsup_{k\to\infty} \sum_{i=1}^{N_k} [T(b_i)(x)]^p \, d\mu(x)\\ &\leqslant \liminf_{k\to\infty} \sum_{i=1}^{N_k} \int_{\mathscr{X}} [T(b_i)(x)]^p \, d\mu(x) \leqslant \liminf_{k\to\infty} \sum_{i=1}^{N_k} |b_i|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}\\ &\lesssim \sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)} \sim \|f\|^p_{\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)}.\end{aligned}$$

This finishes the proof of our claim. By a standard density argument, we extend T to be a bounded sublinear integral from $\widetilde{H}^{p,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ into $L^p(\mu)$, which completes the proof of (i).

An argument similar to that used in the proof of (i) leads to (ii), which completes the proof of Theorem 2.2. □

3. Proof of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we first recall the following useful properties of $\widetilde{K}^{(\rho),p}_{B,S}$ proved in [9].

LEMMA 3.1. *Let* $p \in (0,1]$ *and* $\rho \in (1,\infty)$.
(i) *For all balls* $B \subset R \subset S$,

$$[\widetilde{K}^{(\rho),p}_{B,R}]^p \leqslant C_{(\rho)}[\widetilde{K}^{(\rho),p}_{B,S}]^p, \quad [\widetilde{K}^{(\rho),p}_{R,S}]^p \leqslant \widetilde{c}_{(\rho,p,\nu)}[\widetilde{K}^{(\rho),p}_{B,S}]^p$$

and

$$[\widetilde{K}^{(\rho),p}_{B,S}]^p \leqslant [\widetilde{K}^{(\rho),p}_{B,R}]^p + c_{(\rho,p,\nu)}[\widetilde{K}^{(\rho),p}_{R,S}]^p,$$

where $C_{(\rho)}$ *is a positive constant depending on* ρ, $c_{(\rho,p,\nu)}$ *and* $\widetilde{c}_{(\rho,p,\nu)}$ *are positive constants depending on* ρ, p *and* ν.

(ii) *Let* $\alpha \in [1,\infty)$. *For all balls* $B \subset S$ *with* $r_S \leqslant \alpha r_B$, $[\widetilde{K}^{(\rho),p}_{B,S}]^p \leqslant C_{(\alpha,\rho)}$, *where* $C_{(\alpha,\rho)}$ *is a positive constant depending on* α *and* ρ.

Recall that when $p=1$, the Hardy spaces $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\widetilde{H}^{1,q,\gamma,\varepsilon}_{\mathrm{mb},\rho}(\mu)$ coincide and are simply denoted by $\widetilde{H}^1(\mu)$. By Theorem 2.1 and an argument similar to that used in the proof of [17, Theorem 2.3], we obtain the following boundedness of $\mathscr{M}$ on the Hardy space $\widetilde{H}^1(\mu)$ and the Lebesgue space $L^p(\mu)$ with $p \in [1,\infty)$. We omit the details here.

LEMMA 3.2. *Let* K *satisfy* (1.7) *and* (1.8), *and* $\mathscr{M}$ *be as in* (1.9). *If* $\mathscr{M}$ *is bounded on* $L^2(\mu)$, *then* $\mathscr{M}$ *is bounded from* $L^1(\mu)$ *to* $L^{1,\infty}(\mu)$ *and from* $\widetilde{H}^1(\mu)$ *to* $L^1(\mu)$, *and is bounded on* $L^p(\mu)$ *for all* $p \in (1,\infty)$.

Now we prove Theorem 1.1.

Proof of Theorem 1.1. The case of $p=1$ has been showed in Lemma 3.2. It remains to be proved the case of $p \in (0,1)$. Let ρ, p, q, γ and ε be as in the assumptions of Theorem 1.2. For the sake of simplicity, we take $\rho=2$ and $\gamma=1$. With some slight modifications, the arguments here are still valid for general cases. By Theorem 2.2(i), we only need to show that, for all $(p,q,1,\varepsilon,2)_\lambda$-molecular blocks b,

$$\|\mathscr{M}(b)\|_{L^p(\mu)} \lesssim |b|_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}. \tag{3.1}$$

Now let $b=\sum_{k=0}^{\infty}\sum_{j=1}^{M_k}\lambda_{k,j}m_{k,j}$ be a $(p,q,1,\varepsilon,2)_\lambda-$molecular block, where, for any $k \in \mathbb{Z}_+$ and $j \in \{1,\cdots,M_k\}$, $\mathrm{supp}\,(m_{k,j}) \subset B_{k,j} \subset U_k(B)$ for some balls B and $B_{k,j}$ as in Definition 1.5. Without loss of generality, we may assume that $\widetilde{M} = M$ in Definition 1.5. Let $\ell^* := \min\{\ell - 5, \lfloor \frac{\ell}{2} \rfloor\}$. Since $\mathscr{M}$ is sublinear, we have

$$\begin{aligned}
\|\mathscr{M}(b)\|_{L^p(\mu)}^p \leqslant& \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left|\mathscr{M}\left(\sum_{k=0}^{\ell^*}\sum_{j=1}^{M} \lambda_{k,j} m_{k,j}\right)(x)\right|^p d\mu(x) \\
&+ \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left|\mathscr{M}\left(\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M} \lambda_{k,j} m_{k,j}\right)(x)\right|^p d\mu(x) \\
&+ \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left|\mathscr{M}\left(\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M} \lambda_{k,j} m_{k,j}\right)(x)\right|^p d\mu(x) \\
&+ \sum_{\ell=0}^{4} \int_{U_\ell(B)} |\mathscr{M}(b)(x)|^p \, d\mu(x) \\
=:& \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

We first estimate III. For $x \in U_\ell(B)$ and $y \in B_{k,j} \subset U_k(B)$ with $k \geqslant \ell + 5$, we have

$$d(x, c_B) < 2^{\ell+2} r_B = \frac{1}{2}\left(2^{\ell+3} r_B\right) \leqslant \frac{1}{2}\left(2^{k-2} r_B\right) \leqslant \frac{1}{2} d(y, c_B), \tag{3.2}$$

which further implies that $d(x,y) \sim d(y, c_B)$. It then follows from (1.4) that

$$\lambda(x, d(x,y)) \sim \lambda(x, d(y, c_B)) \geqslant \lambda(x, d(x, c_B)) \sim \lambda(c_B, d(x, c_B)).$$

From this, together with Minkowski's inequality, (1.7), Hölder's inequality, (1.3) and (1.6), we deduce that

$$\begin{aligned}
\mathrm{III} =& \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} \left|\int_0^{\infty} \left|\int_{d(x,y)<t} \sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(y) K(x,y) d\mu(y)\right|^2 \frac{dt}{t^3}\right|^{p/2} d\mu(x) \\
\lesssim& \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B)} \left[\int_{\mathscr{X}} |K(x,y)||m_{k,j}(y)| \frac{1}{d(x,y)} d\mu(y)\right]^p d\mu(x) \\
\lesssim& \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B)} \left[\int_{B_{k,j}} \frac{|m_{k,j}(y)|}{[\lambda(x, d(x,y))]} d\mu(y)\right]^p d\mu(x) \\
\lesssim& \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B)} \frac{1}{[\lambda(c_B, d(x, c_B))]^p} d\mu(x) \, [\mu(B_{k,j})]^{p/q'} \|m_{k,j}\|_{L^q(\mu)}^p \\
\lesssim& \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M} |\lambda_{k,j}|^p \frac{\mu(2^{\ell+2}B)}{[\lambda(c_B, 2^{\ell-2} r_B)]^p} [\mu(B_{k,j})]^{p/q'} \|m_{k,j}\|_{L^q(\mu)}^p \\
\lesssim& \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M} |\lambda_{k,j}|^p \mu(2^{\ell+2}B)^{1-p} [\mu(B_{k,j})]^{p/q'} \\
&\times 2^{-k\varepsilon p} [\mu(2B_{k,j})]^{-p/q'} [\lambda(c_B, 2^{k+2} r_B)]^{p-1}
\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell+5}^{\infty}\sum_{j=1}^{M}2^{-k\varepsilon p}|\lambda_{k,\,j}|^p \lesssim \sum_{j=1}^{M}\sum_{k=10}^{\infty}\sum_{\ell=5}^{k-5}2^{-k\varepsilon p}|\lambda_{k,\,j}|^p\\ &\lesssim \sum_{j=1}^{M}\sum_{k=10}^{\infty}k2^{-k\varepsilon p}|\lambda_{k,\,j}|^p \lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p \sim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}.\end{aligned}$$

To estimate the term I, write

$$\begin{aligned}\mathrm{I} &\lesssim \sum_{\ell=5}^{\infty}\int_{U_\ell(B)}\left|\int_0^{d(x,c_B)+2^{\ell^*+2}r_B}\left|\int_{d(x,y)<t}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}\lambda_{k,\,j}m_{k,\,j}(y)K(x,\,y)d\mu(y)\right|^2\frac{dt}{t^3}\right|^{p/2}d\mu(x)\\ &\quad+\sum_{\ell=5}^{\infty}\int_{U_\ell(B)}\left|\int_{d(x,c_B)+2^{\ell^*+2}r_B}^{\infty}\left|\int_{d(x,y)<t}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}\lambda_{k,\,j}m_{k,\,j}(y)K(x,\,y)d\mu(y)\right|^2\frac{dt}{t^3}\right|^{p/2}d\mu(x)\\ &=: \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

Similar to (3.2), if $x\in U_\ell(B)$ and $y\in B_{k,\,j}\subset U_k(B)$ with $0\leqslant k\leqslant \ell^*$, we then have $d(y,\,c_B)\leqslant \frac{1}{2}d(x,\,c_B)$, which implies that

$$d(y,\,c_B)\leqslant \frac{1}{2}d(x,\,c_B)\leqslant d(x,y)\leqslant \frac{3}{2}d(x,\,c_B). \tag{3.3}$$

This, together with a trivial computation, leads to that, for all $x\in U_\ell(B)$ and $y\in B_{k,\,j}\subset U_k(B)$ with $0\leqslant k\leqslant \ell^*$,

$$\left|\frac{1}{[d(x,\,y)]^2}-\frac{1}{[d(x,\,c_B)+2^{\ell^*+2}r_B]^2}\right|\lesssim \frac{2^{\ell/2}r_B}{[d(x,\,c_B)]^3}.$$

On the other hand, for $p\in(\frac{\nu}{\nu+\min\{\frac{1}{4},\,\delta\}},1)$ and $\varepsilon\geqslant \max\{\frac{1}{4},\,\delta\}$, we have $\nu(1-p)-p/4<0$ and $-\varepsilon p-\nu(1-p)<0$. From the above estimates, together with Minkowski's inequality, (1.7), (3.3), (1.4), Hölder's inequality, (1.6) and (1.3), we deduce that

$$\begin{aligned}\mathrm{I}_1 &\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p\\ &\quad\times\int_{U_\ell(B)}\left|\int_{B_{k,\,j}}|m_{k,\,j}(y)||K(x,y)|\left(\int_{d(x,y)}^{d(x,c_B)+2^{\ell^*+2}r_B}\frac{1}{t^3}dt\right)^{1/2}d\mu(y)\right|^p d\mu(x)\\ &\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left|\int_{B_{k,j}}|m_{k,j}(y)||K(x,\,y)|\left(\frac{2^{\ell/2}r_B}{[d(x,\,c_B)]^3}\right)^{1/2}d\mu(y)\right|^p d\mu(x)\\ &\leqslant \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,\,j}|^p\int_{U_\ell(B)}\frac{2^{-\ell p/4}}{[\lambda(c_B,\,d(x,\,c_B))]^p}d\mu(x)\left|\int_{B_{k,\,j}}|m_{k,\,j}(y)|d\mu(y)\right|^p\end{aligned}$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-\ell p/4}\int_{U_\ell(B)}\frac{1}{[\lambda(c_B,2^{\ell-2}r_B)]^p}d\mu(x)[\mu(B_{k,j})]^{p/q'}\left\|m_{k,j}\right\|_{L^q(\mu)}^p$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-\ell p/4}\frac{\mu(2^{\ell+2}B)}{[\lambda(c_B,2^{\ell-2}r_B)]^p}[\mu(B_{k,j})]^{p/q'}$$

$$\times 2^{-k\varepsilon p}[\mu(2B_{k,j})]^{-p/q'}[\lambda(c_B,2^{k+2}r_B)]^{p-1}$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p C_{(\lambda)}^{(\ell-k)(1-p)}2^{-k\varepsilon p}2^{-\ell p/4}$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{[\nu(1-p)-p/4]\ell}2^{[-\varepsilon p-\nu(1-p)]k}$$

$$\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p \sim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}.$$

To estimate I_2, write

$$\mathrm{I}_2 \lesssim \sum_{\ell=5}^{\infty}\int_{U_\ell(B)}\left|\int_{d(x,c_B)+2^{\ell^*+2}r_B}^{\infty}\left|\int_{d(x,y)<t}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right.\right.$$

$$\left.\left.\times\ [K(x,y)-K(x,c_B)]d\mu(y)\right|^2\frac{dt}{t^3}\right|^{p/2}d\mu(x)$$

$$+\sum_{\ell=5}^{\infty}\int_{U_\ell(B)}\left|\int_{d(x,c_B)+2^{\ell^*+2}r_B}^{\infty}\right.$$

$$\left.\times\left|\int_{d(x,y)<t}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)K(x,c_B)d\mu(y)\right|^2\frac{dt}{t^3}\right|^{p/2}d\mu(x)$$

$$=: \mathrm{I}_{2,1}+\mathrm{I}_{2,2}$$

Now we deal with the term $\mathrm{I}_{2,1}$. Notice that, for $p\in(\frac{\nu}{\nu+\min\{\frac14,\delta\}},1)$ and $\varepsilon\geqslant\max\{\frac14,\delta\}$, $\nu(1-p)-\delta p<0$ and $\delta p-\varepsilon p-\nu(1-p)<0$. It then follows from Minkowski's inequality, (1.8), (1.4), (3.3), Hölder's inequality, (1.6) and (1.3), that

$$\mathrm{I}_{2,1}\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left|\int_{B_{k,j}}|K(x,y)-K(x,c_B)||m_{k,j}(y)|\frac{1}{d(x,y)}d\mu(y)\right|^p d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left|\int_{B_{k,j}}\frac{[d(y,c_B)]^\delta[d(x,y)]^{1-\delta}}{\lambda(y,d(x,y))}|m_{k,j}(y)|\frac{1}{d(x,y)}d\mu(y)\right|^p d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left|\int_{B_{k,j}}\frac{|m_{k,j}(y)|2^{(k+2)\delta}r_B^\delta}{\lambda(c_B,d(x,c_B))2^{(\ell-2)\delta}r_B^\delta}d\mu(y)\right|^p d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\frac{2^{(k-\ell)\delta p}}{[\lambda(c_B,d(x,c_B))]^p}d\mu(x)\left[\int_{B_{k,j}}|m_{k,j}(y)|d\mu(y)\right]^p$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p \int_{U_\ell(B)} \frac{2^{(k-\ell)\delta p}}{[\lambda(c_B, d(x,c_B))]^p} d\mu(x)[\mu(B_{k,j})]^{p/q'} \|m_{k,j}\|_{L^q(\mu)}^p$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p \frac{\mu(2^{\ell+2}B)2^{(k-\ell)\delta p}}{[\lambda(c_B, 2^{\ell-2}r_B)]^p}[\mu(B_{k,j})]^{p/q'}$$

$$\times 2^{-k\varepsilon p}[\mu(2B_{k,j})]^{-p/q'}[\lambda(c_B, 2^{k+2}r_B)]^{p-1}$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{(k-\ell)\delta p}2^{-k\varepsilon p}[\lambda(c_B, 2^{\ell+2}r_B)]^{1-p}[\lambda(c_B, 2^{k+2}r_B)]^{p-1}$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{(k-\ell)\delta p}2^{-k\varepsilon p}C_{(\lambda)}^{(\ell-k)(1-p)}$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{[\nu(1-p)-\delta p]\ell}2^{k[\delta p-\varepsilon p-\nu(1-p)]}$$

$$\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p \sim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}.$$

Then we turn to estimate $\mathrm{I}_{2,2}$. If we fix $x\in U_\ell(B)$, then, for all $y\in B_{k,j}\subset U_k(B)$ with $0\leqslant k\leqslant \ell^*$, we have

$$d(x,y)\leqslant d(x,c_B)+d(y,c_B)<d(x,c_B)+2^{\ell^*+2}r_B.$$

From this, together with the vanishing moment of b, Minkowski's inequality, (1.7), (1.4), Hölder's inequality, (1.6) and (1.3), we deduce that

$$\mathrm{I}_{2,2}=\sum_{\ell=5}^{\infty}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left|\int_{d(x,c_B)+2^{\ell^*+2}r_B}^{\infty}\left|\int_{\mathscr{X}}\sum_{k=0}^{\ell^*}\sum_{j=1}^{M}m_{k,j}(y)K(x,c_B)d\mu(y)\right|^2\frac{dt}{t^3}\right|^{p/2}d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p$$

$$\times\int_{U_\ell(B)}\left|\int_{\mathscr{X}}|m_{k,j}(y)||K(x,c_B)|\left(\int_{d(x,c_B)+2^{\ell^*+2}r_B}^{\infty}\frac{1}{t^3}dt\right)^{1/2}d\mu(y)\right|^p d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left|\int_{\mathscr{X}}|m_{k,j}(y)||K(x,c_B)|\frac{1}{d(x,c_B)}d\mu(y)\right|^p d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\left|\int_{\mathscr{X}}|m_{k,j}(y)|\frac{1}{\lambda(x,d(x,c_B))}d\mu(y)\right|^p d\mu(x)$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\frac{1}{[\lambda(c_B,d(x,c_B))]^p}d\mu(x)\left|\int_{B_{k,j}}|m_{k,j}(y)|d\mu(y)\right|^p$$

$$\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)}\frac{1}{[\lambda(c_B,2^{\ell-2}r_B]^p}d\mu(x)\,\|m_{k,j}\|^p_{L^q(\mu)}\,[\mu(B_{k,j})]^{p/q'}$$

$$\begin{aligned}
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p \frac{\mu(2^{\ell+2}B)}{[\lambda(c_B,2^{\ell-2}r_B)]^p}\\
&\quad\times 2^{-k\varepsilon p}[\mu(2B_{k,j})]^{-p/q'}[\lambda(c_B,2^{k+2}r_B)]^{p-1}[\mu(B_{k,j})]^{p/q'}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-k\varepsilon p}\lesssim \sum_{k=1}^{\infty}\sum_{\ell=0}^{\max\{k+4,2k\}}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-k\varepsilon p}\\
&\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}k2^{-k\varepsilon p}|\lambda_{k,j}|^p\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\sim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)},
\end{aligned}$$

which, together with the estimates for I_1 and $\mathrm{I}_{2,1}$, implies that $\mathrm{I}\lesssim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}$.

Now we estimate II. Write

$$\begin{aligned}
\mathrm{II}&\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{2B_{k,j}}\left|\mathscr{M}(m_{k,j})(x)\right|^p d\mu(x)\\
&\quad+\sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)\setminus 2B_{k,j}}\left|\mathscr{M}(m_{k,j})(x)\right|^p d\mu(x)\\
&=: \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}$$

By Lemma 3.1, we see that $\mathscr{M}$ is bounded on $L^q(\mu)$, which, together with Hölder's inequality, (1.6) and (1.3), implies that

$$\begin{aligned}
\mathrm{II}_1&\leqslant \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left\|\mathscr{M}(m_{k,j})\right\|^p_{L^q(\mu)}[\mu(2B_{k,j})]^{1-p/q}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left\|m_{k,j}\right\|^p_{L^q(\mu)}[\mu(2B_{k,j})]^{1-p/q}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-k\varepsilon p}[\mu(2B_{k,j})]^{-p/q'}[\lambda(c_B,2^{k+2}r_B)]^{p-1}[\mu(2B_{k,j})]^{1-p/q}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}2^{-k\varepsilon p}|\lambda_{k,j}|^p\lesssim \sum_{k=1}^{\infty}\sum_{\ell=k-4}^{\max\{k+4,2k\}}\sum_{j=1}^{M}2^{-k\varepsilon p}|\lambda_{k,j}|^p\\
&\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}k2^{-k\varepsilon p}|\lambda_{k,j}|^p\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\sim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}.
\end{aligned}$$

To estimate II_2, we notice that, for $x\notin 2B_{k,j}$ and $y\in B_{k,j}$,

$$d(x,y)\geqslant d(x,c_{B_{k,j}})-d(y,c_{B_{k,j}})\geqslant \frac{1}{2}d(x,c_{B_{k,j}}).$$

On the other hand, from Lemma 3.1, we deduce that, for all $k\in\mathbb{N}$,

$$\left[\widetilde{K}^{(2),p}_{B_{k,j},2^{2k+1}B}\right]^p\lesssim k\left[\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}\right]^p.$$

The above estimates, together with Minkowski's inequality, (1.7), (1.4), (1.3), Hölder's inequality and (1.6), show that

$$\begin{aligned}
\mathrm{II}_2 &\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}\int_{U_\ell(B)\setminus 2B_{k,j}}\left[\int_0^\infty\left|\int_{d(x,y)<t}K(x,y)m_{k,j}(y)d\mu(y)\right|^2\frac{dt}{t^3}\right]^{p/2}d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)\setminus 2B_{k,j}}\left[\int_{\mathscr{X}}|K(x,y)||m_{k,j}(y)|\frac{1}{d(x,y)}d\mu(y)\right]^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_\ell(B)\setminus 2B_{k,j}}\frac{1}{[\lambda(c_{B_{k,j}},d(x,c_{B_{k,j}}))]^p}\left[\int_{B_{k,j}}|m_{k,j}(y)|d\mu(y)\right]^p d\mu(x)\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{2^{\max\{k+6,2k+2\}}B\setminus 2B_{k,j}}\frac{1}{[\lambda(c_{B_{k,j}},d(x,c_{B_{k,j}}))]^p}d\mu(x)\,\|m_{k,j}\|_{L^1(\mu)}^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\,\|m_{k,j}\|_{L^1(\mu)}^p\sum_{i=0}^{N^{(2)}_{B_{k,j},2^{\max\{k+5,2k+1\}}B}+1}\frac{\mu(2^{i+1}B_{k,j})}{[\lambda(c_{B_{k,j}},2^i r_{B_{k,j}})]^p}\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\,\|m_{k,j}\|_{L^1(\mu)}^p\\
&\quad\times\left[\mu\left(2^{N^{(2)}_{B_{k,j},2^{\max\{k+5,2k+1\}}B}+2}B_{k,j}\right)\right]^{1-p}\left[\widetilde{K}^{(2),p}_{B_{k,j},2^{\max\{k+5,2k+1\}}B}\right]^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\,\|m_{k,j}\|_{L^q(\mu)}^p\,[\mu(B_{k,j})]^{p/q'}\\
&\quad\times[\mu(2^{\max\{k+9,2k+5\}}B)]^{1-p}\left[\widetilde{K}^{(2),p}_{B_{k,j},2^{\max\{k+5,2k+1\}}B}\right]^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\times 2^{-k\varepsilon p}[\mu(2B_{k,j})]^{-p/q'}[\lambda(c_B,2^{k+2}r_B)]^{p-1}\left[\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}\right]^{-p}\\
&\quad\times[\mu(B_{k,j})]^{p/q'}[\mu(2^{\max\{k+9,2k+5\}}B)]^{1-p}\left[\widetilde{K}^{(2),p}_{B_{k,j},2^{\max\{k+5,2k+1\}}B}\right]^p\\
&\lesssim \sum_{\ell=5}^{\infty}\sum_{k=\ell^*+1}^{\ell+4}\sum_{j=1}^{M}kC_{(\lambda)}^{k(1-p)}2^{-k\varepsilon p}|\lambda_{k,j}|^p\\
&\lesssim \sum_{k=1}^{\infty}\sum_{\ell=k-4}^{\max\{k+4,2k\}}\sum_{j=1}^{M}k2^{[\nu(1-p)-\varepsilon p]k}|\lambda_{k,j}|^p\\
&\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}k^2 2^{[\nu(1-p)-\varepsilon p]k}|\lambda_{k,j}|^p\lesssim \sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\sim|b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)},
\end{aligned}$$

where, in the second to the last inequality, we use the fact that, for $p\in(\frac{\nu}{\nu+\min\{\frac14,\delta\}},1)$ and $\varepsilon\geqslant\max\{\frac14,\delta\}$, $\nu(1-p)-\varepsilon p<0$.

Combining the estimates for II_1 and II_2, we conclude that $\mathrm{II} \lesssim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}$.

Finally, we deal with IV. We further write

$$\begin{aligned}\mathrm{IV} &\leqslant \sum_{\ell=0}^{4} \int_{U_\ell(B)} \left| \mathscr{M}\left(\sum_{k=0}^{\ell+4} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(x) \right) \right|^p d\mu(x) \\ &\quad + \sum_{\ell=0}^{4} \int_{U_\ell(B)} \left| \mathscr{M}\left(\sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(x) \right) \right|^p d\mu(x) \\ &=: \mathrm{IV}_1 + \mathrm{IV}_2.\end{aligned}$$

By some arguments similar to that used in the estimates for II and III, we respectively obtain $\mathrm{IV}_1 \lesssim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}$ and $\mathrm{IV}_2 \lesssim |b|^p_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)}$. We omit the details here.

Combining the estimates for I to IV, we obtain the desired estimate (3.1), which completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Let ρ, p, q and γ be as in assumptions of Theorem 1.2. For the sake of simplicity, we take $\rho = 2$ and $\gamma = 1$. By Theorem 2.2(ii), it suffices to show that, for any $(p,q,1,2)_\lambda$-atomic block b,

$$\|Tb\|_{L^p(\mu)} \lesssim |b|_{\widetilde{H}^{p,q,1}_{\mathrm{atb},2}(\mu)},$$

which is an easy consequence of the facts that b is also a $(p,q,1,\varepsilon,2)$-molecular block and $|b|_{\widetilde{H}^{p,q,1,\varepsilon}_{\mathrm{mb},2}(\mu)} \lesssim |b|_{\widetilde{H}^{p,q,1}_{\mathrm{atb},2}(\mu)}$ (see [9, (4.3)]), together with (3.1). We then finish the proof of Theorem 1.2. $\square$

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