OPTIMAL BOUNDS FOR THE FIRST SEIFFERT MEAN IN TERMS OF THE CONVEX COMBINATION OF THE LOGARITHMIC AND NEUMAN–SÁNDOR MEAN

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Abstract. In this paper, we find the least value $\alpha$ and the greatest value $\beta$ such that the double inequality

$$\alpha L(a,b) + (1 - \alpha)M(a,b) < P(a,b) < \beta L(a,b) + (1 - \beta)M(a,b)$$

holds for all $a,b > 0$ with $a \neq b$, where $L(a,b), M(a,b)$ and $P(a,b)$ are the logarithmic, the Neuman-Sándor, and the first Seiffert means of two positive numbers $a$ and $b$, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Neuman–Sándor mean $M(a,b)$, the first Seiffert mean $P(a,b)$, and the logarithmic mean $L(a,b)$ are defined by

$$M(a,b) = \frac{a - b}{2 \sinh^{-1}((a - b)/(a + b))},$$
$$P(a,b) = \frac{a - b}{4 \tan^{-1}(\sqrt{a/b}) - \pi},$$
$$L(a,b) = \frac{b - a}{\log b - \log a},$$

respectively. It can be observed that the first Seiffert mean $P(a,b)$ and the logarithmic mean can be rewritten as (see as [12])

$$P(a,b) = \frac{a - b}{2 \sin^{-1}((a - b)/(a + b))},$$
$$L(a,b) = \frac{a - b}{2 \tanh^{-1}((a - b)/(a + b))},$$

where $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$, $\tan^{-1}(x) = \frac{1}{2} \log[(1 - x)/(1 + x)]$ and $\sin^{-1}(x) = \arcsin x$ are the inverse hyperbolic sine, inverse hyperbolic tangent and inverse sine functions, respectively.

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Recently, the means $L$, $M$, and $P$ and other means have been the subject of intensive research. Many remarkable inequalities for means can be found in the literature [2, 4, 6, 10, 15-18].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b) = (a + b)/2$, $S(a,b) = \sqrt{(a^2 + b^2)/2}$, $T(a,b) = (a - b)/[2 \tan^{-1}((a - b)/(a + b))]$ and

$$M_p(a,b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases}$$

denote the harmonic, geometric, identric, arithmetic, root-square, second Seiffert and the $p$-th power means of two positive numbers $a$ and $b$ with $a \neq b$ respectively. Then it is well-known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b) < A(a,b) < M(a,b) < T(a,b) < S(a,b)$$

hold for $a,b > 0$ with $a \neq b$.

Neuman and Sándor [12, 13] proved that inequalities

$$\frac{\pi}{4 \log(1 + \sqrt{2})} T(a,b) < M(a,b) < \frac{A(a,b)}{\log(1 + \sqrt{2})},$$

$$\frac{\pi}{4 \sinh^{-1}(1)} T(a,b) < M(a,b) < \frac{\pi}{2 \sinh^{-1}(1)} P(a,b),$$

$$\sqrt{A(a,b)} T(a,b) < M(a,b) < \sqrt{A^2(a,b) + T^2(a,b)},$$

$$\frac{G(x,y)}{G(1-x,1-y)} < \frac{L(x,y)}{L(1-x,1-y)} < \frac{P(x,y)}{P(1-x,1-y)}$$

$$< \frac{A(x,y)}{A(1-x,1-y)} < \frac{M(x,y)}{M(1-x,1-y)} < \frac{T(x,y)}{T(1-x,1-y)}$$

hold for all $a, b > 0$ and $x,y \in (0, 1/2]$ with $a \neq b$ and $x \neq y$.

The following bounds for the Seiffert means $P(a,b)$ and $T(a,b)$ in terms of the power mean were presented by Jagers in [8]

$$M_{\frac{1}{2}}(a,b) < P(a,b) < M_{\frac{3}{2}}(a,b)$$

for all $a,b > 0$ with $a \neq b$. Hästö [7] improved the results of [8] and found the sharp lower power mean bound for the Seiffert mean $P(a,b)$ as follows

$$P(a,b) > M_{\frac{\log 2}{\log \pi}}(a,b)$$

for all $a,b > 0$ with $a \neq b$.

In [1], Alzer and Qiu proved

$$M_c(a,b) \leq \frac{1}{2} L(a,b) + \frac{1}{2} I(a,b)$$
for all $a, b > 0$ with the best possible parameter $c = \frac{\log 2}{1 + \log 2}$,

$$[G(a, b)]^{A(a, b)} < [L(a, b)]^{I(a, b)} < [A(a, b)]^{G(a, b)}$$

for $a, b \geq e$, and

$$[A(a, b)]^{G(a, b)} < [I(a, b)]^{L(a, b)} < [G(a, b)]^{A(a, b)}$$

for $0 < a, b < e$.

In [11] and [9], the authors proved that the double inequalities

$$S^{\alpha_1}(a, b)A^{1-\alpha_1}(a, b) < M(a, b) < S^{\beta_1}(a, b)A^{1-\beta_1}(a, b),$$

$$\alpha_2 A(a, b) + (1 - \alpha_2)G(a, b) < P(a, b) < \beta_2 A(a, b) + (1 - \beta_2)G(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 < 1/3$, $\beta_1 \geq 2(\log(2 + \sqrt{2}) - \log 3)/\log 2$, $\alpha_2 < \pi/2$, $\beta_2 \geq 2/3$, respectively.

In [5], it was shown that

$$H^{\alpha_3}(a, b)L^{1-\alpha_3}(a, b) \geq M_{1-4\alpha_3}(a, b),$$

$$H^{\beta_3}(a, b)L^{1-\beta_3}(a, b) \leq M_{1-4\beta_3}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \in [\frac{1}{4}, 1)$, $\beta_3 \in (0, \frac{3\sqrt{3} - 5}{40}]$.

In [14], the authors proved that

$$\alpha_4 H(a, b) + (1 - \alpha_4)L(a, b) > M_{1-4\alpha_4}(a, b),$$

$$\beta_4 H(a, b) + (1 - \beta_4)L(a, b) < M_{1-4\beta_4}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \in [\frac{1}{4}, 1)$, $\beta_4 \in (0, 3\sqrt{345}/80 - 11/16)$.

The aim of this paper is to find the least value $\alpha$ and the greatest value $\beta$ such that the double inequality

$$\alpha L(a, b) + (1 - \alpha)M(a, b) < P(a, b) < \beta L(a, b) + (1 - \beta)M(a, b)$$

holds for all $a, b > 0$ with $a \neq b$. 
2. Lemmas

To establish our main result, we need several lemmas, which we present in this section.

**Lemma 2.1.** It holds that

\[
x + \frac{2 - \beta}{6\beta} x^3 + \frac{36\beta^2 - 49a + 40}{360\beta^2} x^5 > \tanh^{-1} x, \quad x \in (0, 0.93), \quad (2.1)
\]

\[
x + \frac{1}{3} x^3 + \frac{1}{5} x^5 < \tanh^{-1} x, \quad x \in (0, 0.52), \quad (2.2)
\]

\[
x^2 + \frac{97}{720} x^6 - \frac{1}{96} x^8 + \frac{9}{1600} x^{10} < \left( x - \frac{1}{3} x^3 + \frac{3}{160} x^5 \right) \tanh^{-1} x, \quad x \in (0.52, 0.72), \quad (2.3)
\]

where \( \beta = 1 - \frac{2}{\pi} \sinh^{-1}(1) \approx 0.4389. \)

**Proof.** Let

\[
f(x) = \tanh^{-1}(x) - \left( x + \frac{2 - \beta}{6\beta} x^3 + \frac{36\beta^2 - 49\beta + 40}{360\beta^2} x^5 \right).
\]

Then we can get

\[
f'(x) = \frac{1}{1 - x^2} - \left( 1 + \frac{2 - \beta}{2\beta} x^2 + \frac{36\beta^2 - 49\beta + 40}{72\beta^2} x^4 \right) = \frac{x^2}{1 - x^2} g(x), \quad (2.4)
\]

where

\[
g(x) = \frac{3\beta - 2}{2\beta} + \frac{-72\beta^2 + 121\beta - 40}{72\beta^2} x^2 + \frac{36\beta^2 - 49\beta + 40}{72\beta^2} x^4.
\]

It is easy to verify that there exist \( x_0 \in (0, 0.93), \) \( x_0 \approx 0.8634, \) such that \( g(x) < 0 \) for \( x \in (0, x_0), \) \( g(x_0) = 0, \) and \( g(x) > 0 \) for \( x \in (x_0, 0.93). \) Thus, equation (2.4) implies that \( f(x) \) is decrease on \( (0, x_0) \) and increase on \( (x_0, 0.93). \) Therefore, \( f(x) < 0 \) for \( x \in (0, 0.93) \) follows from the fact that \( f(0) = 0, \) \( f(0.93) < 0 \) and the monotonicity of \( f(x). \) That means inequality (2.1) holds.

Observe that

\[
\tanh^{-1} x = \sum_{n=0}^{\infty} \frac{1}{2n + 1} x^{2n+1}, \quad -1 < x < 1.
\]

So it is obvious that inequality (2.2) holds.

Let

\[
h(x) = \left( x - \frac{1}{3} x^3 + \frac{3}{160} x^5 \right) \tanh^{-1} x - \left( x^2 + \frac{97}{720} x^6 - \frac{1}{96} x^8 + \frac{9}{1600} x^{10} \right).
\]
Then direct computation leads to
\[
 h(x) = \left(x - \frac{1}{3}x^3 + \frac{3}{160}x^5\right) \sum_{n=0}^{\infty} \frac{1}{2n+1}x^{2n+1} - \left(x^2 + \frac{97}{720}x^6 - \frac{1}{96}x^8 + \frac{9}{1600}x^{10}\right)
 = x^6\left(-\frac{13}{480} + \frac{13}{140}x^2 + \frac{157}{2548}x^4 - \frac{1039}{30240}x^6 + \frac{1}{480}x^8\right)
 + \left(x - \frac{1}{3}x^3 + \frac{3}{160}x^5\right) \sum_{n=6}^{\infty} \frac{1}{2n+1}x^{2n+1}.
\]
Noting that \(-\frac{13}{480} + \frac{13}{140}x^2 + \frac{157}{2548}x^4 - \frac{1039}{30240}x^6 + \frac{1}{480}x^8 > 0\) and \(x - \frac{1}{3}x^3 + \frac{3}{160}x^5 > 0\) for \(x \in (0.52, 0.72)\), so \(h(x) > 0\) for \(x \in (0.52, 0.72)\) and the inequality (2.3) holds. \(\square\)

**Lemma 2.2.** The inequalities
\[
 x + \frac{1}{6}x^3 + \frac{3}{40}x^5 < \sin^{-1}x, \quad x \in (0, 0.93), \quad (2.5)
 x + \frac{1}{6}x^3 + \frac{9}{100}x^5 > \sin^{-1}x, \quad x \in (0, 0.52), \quad (2.6)
 x + \frac{1}{6}x^3 + \frac{9}{80}x^5 > \sin^{-1}x, \quad x \in (0.52, 0.72) \quad (2.7)
\]
hold.

**Proof.** It is known that
\[
 \sin^{-1}x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}.
\]
So it’s easy to see that inequality (2.5) holds.

Let \(h_i(x) = f_i(x) - g(x), F_i(x)/x^4 = (\frac{f_i(x)}{g(x)})^2 - 1, i = 1, 2, \) where
\[
 f_1(x) = x + \frac{1}{6}x^3 + \frac{9}{100}x^5,
 f_2(x) = x + \frac{1}{6}x^3 + \frac{9}{80}x^5,
\]
and
\[
 g(x) = \sin^{-1}x.
\]
Then it follows that
\[
 f_1'(x) = 1 + \frac{1}{2}x^2 + \frac{9}{20}x^4,
 f_2'(x) = 1 + \frac{1}{2}x^2 + \frac{9}{16}x^4,
 g'(x) = \frac{1}{\sqrt{1-x^2}}.
\]
respectively. Thus (2.7) follow from the fact that 
and the monotonicity of $F$. It's easy to see that there exists $x_1 \in (0, 0.52)$ and $x_2 \in (0.52, 0.72)$, such that $F_1(x_1) = 0$, $F_2(x_2) = 0$, $F_1(x)$ and $F_2(x)$ are strictly decrease in $(0, 0.52)$ and $(0.52, 0.72)$, respectively. Thus $h_1(x)$ and $h_2(x)$ are increase on $(0, x_1)$ and $(0.52, x_2)$, respectively, and decrease on $(x_1, 0.52)$ and $(x_2, 0.72)$, respectively. Therefore, inequalities (2.6) and (2.7) follow from the fact that $h_1(0) = 0$, $h_1(0.52) > 0$, $h_2(0.52) > 0$, $h_2(0.72) > 0$ and the monotonicity of $h_1(x)$ and $h_2(x)$, respectively. □

**Lemma 2.3.** It holds that

\[
x - \frac{1}{6} x^3 + \frac{1}{16} x^5 < \sinh^{-1} x, \quad x \in (0, 0.52), \quad (2.8)
\]

\[
x - \frac{1}{6} x^3 + \frac{1}{20} x^5 < \sinh^{-1} x, \quad x \in (0.52, 0.72), \quad (2.9)
\]

\[
x - \frac{1}{6} x^3 + \frac{1}{10} x^5 > \sinh^{-1} x, \quad x \in (0, 0.93). \quad (2.10)
\]

**Proof.** Let $h_i(x) = f_i(x) - g(x)$, $F_i(x)/x^4 = \left( \frac{f_i(x)}{g(x)} \right)^2 - 1$, $i = 1, 2, 3$, where

\[
f_1(x) = x - \frac{1}{6} x^3 + \frac{1}{16} x^5,
\]

\[
f_2(x) = x - \frac{1}{6} x^3 + \frac{1}{20} x^5,
\]

\[
f_3(x) = x - \frac{1}{6} x^3 + \frac{1}{10} x^5,
\]

\[
g(x) = \sinh^{-1}(x).
\]

Then direct computation lead to

\[
f'_1(x) = 1 - \frac{1}{2} x^2 + \frac{5}{16} x^4,
\]

\[
f'_2(x) = 1 - \frac{1}{2} x^2 + \frac{1}{4} x^4,
\]

\[
f'_3(x) = 1 - \frac{1}{2} x^2 + \frac{1}{2} x^4,
\]

\[
g'(x) = \frac{1}{\sqrt{1 + x^2}}.
\]
and

\[
\begin{align*}
F_1(x) &= \frac{25}{256}x^6 - \frac{55}{256}x^4 + \frac{9}{16}x^2 - \frac{1}{8}, \\
F_2(x) &= \frac{1}{16}x^6 - \frac{3}{16}x^4 + \frac{1}{2}x^2 - \frac{1}{4}, \\
F_3(x) &= \frac{1}{4}x^6 - \frac{1}{4}x^4 + \frac{3}{4}x^2 + \frac{1}{4}.
\end{align*}
\]

It can be verified that there exists \( x_1 \in (0, 0.52) \) such that \( F_1(x) < 0 \) for \( x \in (0, x_1) \) and \( F_1(x) > 0 \) for \( x \in (x_1, 0.52) \). So \( h_1(x) \) is decrease on \((0, x_1)\) and increase on \((x_1, 0.52)\). Therefore, inequality (2.8) follows from \( h_1(0) = 0, \ h_1(0.52) < 0 \) and the monotonicity of \( h_1(x) \).

Observe that \( F_2(x) < 0 \) and \( F_3(x) > 0 \) for \( x \in (0.52, 0.72) \) and \( x \in (0, 0.93) \), respectively. It imply that \( h_2(x) \) and \( h_3(x) \) are decrease on \((0.52, 0.72)\) and increase on \((0, 0.93)\), respectively. Furthermore, considering \( h_2(0.52) < 0 \) and \( h_3(0) = 0 \), one has inequalities (2.9) and (2.10). □

**Lemma 2.4.** Let

\[
t(x) = (1 - \beta) \sqrt{1 - x^2} \tanh^{-1} x + 2(1 - \beta) \sin^{-1} x - 2 \sinh^{-1} x.
\]

Then \( t(x) > 0 \) for \( x \in (0.93, 1) \), where \( \beta = 1 - \frac{2}{\pi} \sinh^{-1}(1) \).

**Proof.** Direct computation lead to that

\[
t'(x) = \frac{\phi(x)}{\sqrt{1 - x^2}}, \tag{2.11}
\]

where

\[
\phi(x) = (1 - \beta) \sqrt{1 + x^2(3 - x \tanh^{-1} x)} - 2 \sqrt{1 - x^2}.
\]

It follows that

\[
\phi'(x) = -\frac{(1 + \beta)x^2 \tanh^{-1} x}{\sqrt{1 + x^2}} + (1 - \beta) \sqrt{1 + x^2} \alpha(x) + \frac{x}{1 - x^2} \beta(x), \tag{2.12}
\]

where

\[
\alpha(x) = \frac{3x}{(1 + x^2)} - \tanh^{-1} x,
\]

\[
\beta(x) = 2 \sqrt{1 - x^2} - (1 - \beta) \sqrt{1 + x^2}.
\]

Noting that \( \alpha(0.93) < 0, \ \beta(0.93) < 0 \), and both

\[
\alpha'(x) = 2 \frac{1 - 4x^2 + x^4}{(1 + x^2)^2(1 - x^2)} < 0
\]
and
\[ \beta'(x) = \frac{-2x[\sqrt{1+x^2} + (1-\beta)\sqrt{1-x^2}]}{\sqrt{1-x^2}} < 0 \]
for \( x \in (0.93, 1) \), we can get both \( \alpha(x) < 0 \) and \( \beta(x) < 0 \) for \( x \in (0.93, 1) \). Thus, equation (2.12) implies that \( \phi(x) \) is decrease on \( (0.93, 1) \). Considering that \( \phi(0.93) > 0 \) and \( \phi(1^-) < 0 \), it is easy to see that there exist a point \( \lambda \in (0.93, 1) \), such that \( \phi(x) \) is increase on \( (0.93, \lambda) \) and decrease on \( (\lambda, 1) \). Equation (2.11) implies that \( \phi(x) \) and \( \tau(x) \) have same monotonicity on \( (0.93, 1) \). Therefore, \( \tau(x) > 0 \) for \( x \in (0.93, 1) \) follows from \( \tau(0.93) > 0, \tau(1^-) = 0 \) and its monotonicity. □

**Lemma 2.5.** For \( x \in (0.72, 0.9) \), the following inequalities hold:

\[
\frac{1}{\sinh^{-1}x} < a_1x + b_1, \quad (2.13)
\]
\[
\frac{2}{\tanh^{-1}x} < \frac{9}{2}x + \frac{109}{20}, \quad (2.14)
\]
\[
\frac{3}{\sin^{-1}x} > \left(\frac{9}{2} + a_1\right)x + \left(\frac{109}{20} + b_1\right), \quad (2.15)
\]

where \( a_1 = \frac{50}{9}\left(\frac{1}{\sinh^{-1}(0.9)} - \frac{1}{\sinh^{-1}(0.72)}\right) \) and \( b_1 = \frac{5}{\sinh^{-1}(0.72)} - \frac{4}{\sinh^{-1}(0.9)} \).

**Proof.** Simple computation deduce that
\[
\left(\frac{1}{\sinh^{-1}x}\right)'' = \frac{2\sinh^{-1}x + x(\sinh^{-1}x)^2}{(\sinh^{-1}x)^4(1+x^2)} > 0
\]
for any \( x \in (0, 1) \). So \( 1/\sinh^{-1}x \) is convex on \( (0.72, 0.9) \). Observe that the line \( y = a_1x + b_1 \) intersects the curve \( y = 1/\sinh^{-1}x \) at two points which abscissas are 0.72 and 0.9. Thus the geometric property of convex function deduce the inequality (2.13).

Let
\[
f_1(x) = \tanh^{-1}x - \frac{2}{\frac{9}{2}x + \frac{109}{20}},
\]
\[
f_2(x) = \frac{3}{\left(\frac{9}{2} + a_1\right)x + \left(\frac{109}{20} + b_1\right)} - \sin^{-1}x.
\]

It follows that
\[
f_1'(x) = \frac{1}{1-x^2} + \frac{9}{\left(\frac{9}{2}x + \frac{109}{20}\right)^2} > 0,
\]
\[
f_2'(x) = -\frac{3\left(\frac{9}{2} + a_1\right)}{\left[\left(\frac{9}{2} + a_1\right)x + \left(\frac{109}{20} + b_1\right)\right]^2} - \frac{1}{\sqrt{1-x^2}} < 0.
\]
Considering that \( f_1(0.72) > 0 \) and \( f_2(0.9) > 0 \), respectively, we can get inequalities (2.14) and (2.15). □
LEMMA 2.6. For $x \in (0.9, 1)$, the following inequalities hold:

\[
\frac{1}{\sinh^{-1}x} < a_2x + b_2, \quad (2.16)
\]
\[
\frac{3}{\sin^{-1}x} > a_3x + b_3, \quad (2.17)
\]
\[
\frac{2}{\tanh^{-1}x} < a_4x + b_4, \quad (2.18)
\]

where \(a_2 = 10\frac{1}{\sinh^{-1}(1)} - \frac{1}{\sinh^{-1}(0.9)}\), \(b_2 = \frac{10}{\sin^{-1}(0.9)} - \frac{9}{\sin^{-1}(1)}\), \(a_3 = 30\frac{1}{\sin^{-1}(1)} - \frac{1}{\sin^{-1}(0.9)}\), \(b_3 = \frac{30}{\sin^{-1}(0.9)} - \frac{27}{\sin^{-1}(1)}\), \(a_4 = a_3 - 2\), \(b_4 = b_3 - b_2\).

**Proof.** The proof of inequality (2.16) is same as that of inequality (2.13).

Let \(g(x) = \alpha(x) - \beta(x)\) and \(f(x) = \left(\frac{\alpha'(x)}{\beta'(x)}\right)^2 - 1\), where

\[
\alpha(x) = \sin^{-1}x, \\
\beta(x) = \frac{3}{a_3x + b_3}.
\]

Then direct computation lead to

\[
f(x) = \frac{10a_3^2x^2 + 2a_3b_3x + b_3^2 - 9a_3^2}{a_3^2(1-x^2)}.
\]

Observe that \(10a_3^2x^2 + 2a_3b_3x + b_3^2 - 9a_3^2\) is increase on \((0.9, 1)\), \(f(0.9) < 0\), and \(f(1) > 0\). Thus \(g(x)\) is decrease firstly and then increase on \((0.9, 1)\). Furthermore, it is clear that \(g(0.9) = g(1) = 0\). Therefore, inequality (2.17) holds.

Let

\[
h(x) = \tanh^{-1}x - \frac{2}{a_4x + b_4}.
\]

It follows that \(h(0.9) > 0\) and

\[
h'(x) = \frac{1}{1-x^2} - \frac{a_4}{(a_4x + b_4)^2} = \frac{(a_4^2 + a_4)x^2 + 2a_4b_4x + b_4^2 - a_4}{(1-x^2)(a_4x + b_4)^2} > 0.
\]

Thus inequality (2.18) holds. \(\square\)

LEMMA 2.7. Let

\[
g(x) = [(1-\beta)\sin^{-1}x - \sin^{-1}x]\tanh^{-1}x,
\]

where \(\beta = 1 - \frac{2}{\pi}\sin^{-1}(1)\). Then \(g(x)\) is increase on \((0.93, 1)\).
\textbf{Proof.} Direct computation deduce that
\[
g'(x) = \left( \frac{1 - \beta}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 + x^2}} \right) \tanh^{-1} x + \frac{(1 - \beta) \sin^{-1} x - \sinh^{-1} x}{1 - x^2},
\]
\[
g''(x) = \left( \frac{(1 - \beta)x}{\sqrt{(1 - x^2)^3}} - \frac{x}{\sqrt{(1 + x^2)^3}} \right) \tanh^{-1} x
\]
\[
+ \frac{2x}{(1 - x^2)^2} ((1 - \beta) \sin^{-1} x - \sinh^{-1} x)
\]
\[
+ \left( \frac{1 - \beta}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 + x^2}} \right) \frac{2}{1 - x^2}
\]
\[
= x [(1 - \beta) \sqrt{1 - x^2} \tanh^{-1} x + 2 (1 - \beta) \sin^{-1} x - 2 \sinh^{-1} x]
\]
\[
\frac{(1 - x^2)^2}{2(1 - x^2)}
\]
Observe that \( \frac{1 - \beta}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 + x^2}} > 0 \) for \( x \in (0.93, 1) \). Considering Lemma 2.4, we get \( g''(x) > 0 \) for \( x \in (0.93, 1) \). Noting \( g'(0.93) > 0 \), it is easy to see that \( g'(x) > 0 \) for \( x \in (0.93, 1) \). □

\section{3. Main result}

\textbf{THEOREM 3.1.} The double inequality
\[
\alpha L(a,b) + (1 - \alpha) M(a,b) < P(a,b) < \beta L(a,b) + (1 - \beta) M(a,b)
\]
(3.1)
holds for all \( a, b > 0 \) if and only if \( \alpha \geq \frac{2}{3} \) and \( \beta \leq 1 - \frac{2}{\pi} \sinh^{-1}(1) = 0.4389 \ldots \)

\textbf{Proof.} Because \( P(a,b), M(a,b) \) and \( T(a,b) \) are symmetric and homogeneous of degree 1, without loss of generality, we assume that \( a > b \). Let \( p \in (0,1), x = \frac{a-b}{a+b} \in (0,1) \) and \( \lambda = 1 - \frac{2}{\pi} \sinh^{-1}(1) \). Then by (1.1), (1.2) and (1.3), direct computations lead to
\[
\frac{L(a,b)}{A(a,b)} = \frac{x}{\tanh^{-1} x},
\]
\[
\frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1} x},
\]
\[
\frac{P(a,b)}{A(a,b)} = \frac{x}{\sin^{-1} x}.
\]
Then
\[
D_p(x) := \frac{pL(a,b) + (1-p)M(a,b) - P(a,b)}{A(a,b)}
\]
\[
= \frac{x}{\tanh^{-1} x} + (1-p) \frac{x}{\sinh^{-1} x} - \frac{x}{\sin^{-1} x}.
\]
(3.2)
From inequalities (2.2), (2.6) and (2.8), we can get

\[-3D_2^2(x) = \frac{3}{\sin^{-1} x} - \frac{2}{\tanh^{-1} x} - \frac{1}{\sinh^{-1} x}\]

\[> \frac{3}{x + \frac{1}{6}x^3 + \frac{9}{100}x^5} - \frac{2}{x + \frac{1}{5}x^3 + \frac{1}{16}x^5} - \frac{1}{x - \frac{1}{6}x^3 + \frac{1}{16}x^5}\]

\[= \frac{31}{1200} - \frac{11}{720}x^2 + \frac{33}{4000}x^4\]

\[> 0\]  \hspace{1cm} (3.3)

for \(x \in (0, 0.52)\).

From inequalities (2.3), (2.7), and (2.9), we obtain

\[-3D_2^2(x) = \frac{3}{\sin^{-1} x} - \frac{2}{\tanh^{-1} x} - \frac{1}{\sinh^{-1} x}\]

\[> \frac{3}{x + \frac{1}{6}x^3 + \frac{9}{100}x^5} - \frac{2}{\tanh^{-1} x} - \frac{1}{x - \frac{1}{6}x^3 + \frac{1}{20}x^5}\]

\[= \frac{2x - \frac{3}{2}x^3 + \frac{3}{10}x^5}{(x^2 + \frac{97}{720}x^6 - \frac{1}{96}x^8 + \frac{9}{1600}x^{10})\tanh^{-1} x}\]

\[> 0\]  \hspace{1cm} (3.4)

for \(x \in (0.52, 0.72)\).

By Lemma 2.5, we get

\[-3D_2^2(x) = \frac{3}{\sin^{-1} x} - \frac{2}{\tanh^{-1} x} - \frac{1}{\sinh^{-1} x}\]

\[> \left(\frac{9}{2} + a_1\right)x + \frac{109}{20} + b_1 - \left(\frac{9}{2}x + \frac{109}{20}\right) - (a_1 x + b_1) = 0\]  \hspace{1cm} (3.5)

for \(x \in (0.72, 0.9)\) as well as Lemma 2.6 deduce that

\[-3D_2^2(x) = \frac{3}{\sin^{-1} x} - \frac{2}{\tanh^{-1} x} - \frac{1}{\sinh^{-1} x}\]

\[> a_3 x + b_3 - [(a_3 - a_2)x + b_3 - b_2] - (a_2 x + b_2) = 0.\]  \hspace{1cm} (3.6)

for \(x \in (0.9, 1)\).

Therefore, it follows from inequalities (3.3)--(3.6) that

\[\frac{2}{3}L(a, b) + \frac{1}{3}M(a, b) < P(a, b)\]  \hspace{1cm} (3.7)

holds for all \(a, b > 0\) with \(a \neq b\).
From inequalities (2.1), (2.5), and (2.10), we have
\[
D_\lambda(x) = \frac{\lambda}{\tanh^{-1} x} + \frac{1 - \lambda}{\sinh^{-1} x} - \frac{1}{\sin x} > x + \frac{2 - \lambda}{6 \lambda} x^3 + \frac{36 \lambda^2 - 49 \lambda + 40}{360 \lambda^2} x^5 + x - \frac{1 - \lambda}{6 x^3 + \frac{10}{x} x^5} - x + \frac{1}{6 x^3 + \frac{3}{4} x^5} = x^8 \left( \frac{80 - 156 \lambda + 76 \lambda^2}{2160 \lambda^2} + \frac{111 \lambda^2 - 71 \lambda - 40}{14400 \lambda^2} \right) \left( x - \frac{1}{6 x^3 + \frac{10}{x} x^5} \right) \left( x + \frac{1}{6 x^3 + \frac{3}{4} x^5} \right) > 0
\]
(3.8)
for \( x \in (0, 0.93) \).

Simple computation lead to
\[
D_\lambda(x) = \frac{F(x)}{(\tanh^{-1} x)(\sinh^{-1} x)(\sin^{-1} x)},
\]
(3.9)
where
\[
F(x) = \lambda \sin^{-1} x \sinh^{-1} x + [(1 - \lambda) \sin^{-1} x - \sin^{-1} x] \tanh^{-1} x.
\]
It is obvious that \( \lambda \sin^{-1} x \sinh^{-1} x \) is increase on \((0.93, 1)\). Considering Lemma 2.7, we get that \( F(x) \) is increase on \((0.93, 1)\). Noting that \( F(0.93) > 0 \). Thus equation (3.9) implies that
\[
D_\lambda(x) > 0
\]
(3.10)
for \( x \in (0.93, 1) \).

Therefore, it follows from inequalities (3.8) and (3.10) that for \( x \in (0, 1) \)
\[
P(a, b) < \beta L(a, b) + (1 - \beta) M(a, b)
\]
(3.11)
holds for all \( a, b > 0 \) with \( a \neq b \).

Finally, by easy computations, equations (1.1), (1.2) and (1.3) lead to
\[
\frac{P(a, b) - M(a, b)}{L(a, b) - M(a, b)} = \frac{x/\sin^{-1}(x) - x/\sinh^{-1}(x)}{x/\tanh^{-1}(x) - x/\sin^{-1}(x)},
\]
(3.12)
\[
\lim_{x \to 0^+} x/\tanh^{-1}(x) - x/\sin^{-1}(x) = \frac{2}{3}, \quad \lim_{x \to 1^-} x/\tanh^{-1}(x) - x/\sin^{-1}(x) = \lambda.
\]
(3.13)
(3.14)
Thus, we have the following claims.

Claims 1. If \( \alpha < \frac{2}{3} \), then (3.12) and (3.13) imply that there exists \( \sigma \in (0, 1) \) such that \( \alpha L(a, b) + (1 - \alpha) M(a, b) > P(a, b) \) for all \( a, b \) with \( (a - b)/(a + b) \in (0, \sigma) \).

Claims 2. If \( \beta > \lambda \), then (3.12) and (3.14) imply that there exists \( \zeta \in (0, 1) \) such that \( \beta L(a, b) + (1 - \beta) M(a, b) < P(a, b) \) for all \( a, b \) with \( (a - b)/(a + b) \in (1 - \zeta, 1) \).

Inequalities (3.7) and (3.11) in conjunction with the above two claims mean the proof is completed.  \( \square \)
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