

NEW REFINEMENTS OF JENSEN'S INEQUALITY AND ENTROPY UPPER BOUNDS

GUOXIANG LU

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Abstract. In this paper some new refinements of the discrete Jensen's inequality with variables are presented. The new refinements are better than the inequalities given by Simic(2009), Dragomir(2010) and Țăpuș, Popescu (2012). As the applications of the refinements, stronger upper bounds for the Shannon's entropy are obtained as well.

1. Introduction

Let $I = [a, b]$ be a closed interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ is a convex function on I . If $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is an n -tuple in I^n , and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ is a positive n -tuple such that $\sum_{i=1}^n p_i = 1$, then the well-known Jensen's inequality

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i) \quad (1)$$

holds [7, 10]. If f is concave, then the preceding inequality is reversed.

Jensen's inequality is probably the most significant of all inequalities. It is applied widely in mathematics, statistics, information theory and many other inequalities are its particular cases (such as Cauchy inequality, Hölder inequality, Minkowski inequality, Ky Fan inequality, etc.). There are a considerable number of refinements and applications related to Jensen's inequality recently (cf. Dragomir [2, 3], Horváth [4, 5, 6], Popescu [8], Simic [9, 10, 11], Țăpuș and Popescu [12], Pečarić and Perić [13]).

In [11], Simic obtained new bounds of Jensen's inequality and Shannon's entropy.

LEMMA 1. *If f , \mathbf{x} , \mathbf{p} are defined as above, then*

$$\begin{aligned} 0 &\leq \max_{1 \leq \mu < \nu \leq n} \left[p_\mu f(x_\mu) + p_\nu f(x_\nu) - (p_\mu + p_\nu) f\left(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu}\right) \right] \\ &\leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right). \end{aligned} \quad (2)$$

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And in [13], Pečarić and Perić indicate that Lemma 1 is a special case of former and more general results they obtain.

In information theory [1], if the probability distribution F is given by $P(X = i) = p_i, p_i > 0, i = 1, 2, \dots, n, s.t. \sum_{i=1}^n p_i = 1$, then the Shannon’s entropy is defined as $H(X) := \sum_{i=1}^n p_i \log \frac{1}{p_i}$. The corresponding result in [11] can be rewritten as

LEMMA 2. Define $\phi := \min_{1 \leq i \leq n} \{p_i\}; \psi := \max_{1 \leq i \leq n} \{p_i\}$. Then

$$H(X) \leq \log n - \left[\phi \log \left(\frac{2\phi}{\phi + \psi} \right) + \psi \log \left(\frac{2\psi}{\phi + \psi} \right) \right]. \tag{3}$$

In [2], Dragomir obtained a different refinement of Jensen’s inequality.

LEMMA 3. If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then

$$\begin{aligned} f \left(\sum_{i=1}^n p_i x_i \right) &\leq \min_{\mu \in \{1, 2, \dots, n\}} \left[(1 - p_\mu) f \left(\frac{\sum_{i=1}^n p_i x_i - p_\mu x_\mu}{1 - p_\mu} \right) + p_\mu f(x_\mu) \right] \\ &\leq \frac{1}{n} \left[\sum_{\mu=1}^n (1 - p_\mu) f \left(\frac{\sum_{i=1}^n p_i x_i - p_\mu x_\mu}{1 - p_\mu} \right) + \sum_{\mu=1}^n p_\mu f(x_\mu) \right] \\ &\leq \max_{\mu \in \{1, 2, \dots, n\}} \left[(1 - p_\mu) f \left(\frac{\sum_{i=1}^n p_i x_i - p_\mu x_\mu}{1 - p_\mu} \right) + p_\mu f(x_\mu) \right] \\ &\leq \sum_{i=1}^n p_i f(x_i) \end{aligned} \tag{4}$$

In [12], Țăpuș and Popescu obtained sharper bounds of Jensen’s inequality and Shannon’s entropy based on the work [11].

LEMMA 4. If $f, \mathbf{x}, \mathbf{p}$ are defined as above and some notations are established as follows

$$\begin{aligned} S_2 &= \max_{1 \leq \mu_1 < \mu_2 \leq n} \left[p_{\mu_1} f(x_{\mu_1}) + p_{\mu_2} f(x_{\mu_2}) - (p_{\mu_1} + p_{\mu_2}) f \left(\frac{p_{\mu_1} x_{\mu_1} + p_{\mu_2} x_{\mu_2}}{p_{\mu_1} + p_{\mu_2}} \right) \right], \\ &\dots \\ S_{n-1} &= \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \left[\sum_{i=1}^{n-1} p_{\mu_i} f(x_{\mu_i}) - \left(\sum_{i=1}^{n-1} p_{\mu_i} \right) f \left(\frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}} \right) \right], \end{aligned}$$

where $\mu_1, \mu_2, \dots, \mu_{n-1} \in \{1, 2, \dots, n\}$ are different items, then

$$0 \leq S_2 \leq S_3 \leq \dots \leq S_{n-1} \leq \sum_{i=1}^n p_i f(x_i) - f \left(\sum_{i=1}^n p_i x_i \right). \tag{5}$$

LEMMA 5.

$$H(X) \leq \log n - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \log \left[\left(\frac{n-1}{\sum_{i=1}^{n-1} p_{\mu_i}} \right)^{\sum_{i=1}^{n-1} p_{\mu_i}} \left(\prod_{i=1}^{n-1} p_{\mu_i}^{p_{\mu_i}} \right) \right]. \tag{6}$$

In this paper, we present some new refinements of Jensen's equality and the entropy upper bounds associated with them. The results generalize the inequalities in the above lemmas and show stronger upper bounds compared with the bounds given by [2, 11, 12].

2. Simple refinements of Jensen's inequality

THEOREM 1. *If f , \mathbf{x} , \mathbf{p} are defined as above, then*

$$\begin{aligned} & f\left(\sum_{i=1}^n p_i x_i\right) \\ & \leq \max_{1 \leq \mu < \nu \leq n} \left[p_\mu f(x_\mu) + p_\nu f(x_\nu) + (1 - p_\mu - p_\nu) f\left(\frac{\sum_{i=1}^n p_i x_i - (p_\mu x_\mu + p_\nu x_\nu)}{1 - p_\mu - p_\nu}\right) \right] \\ & \leq \max_{1 \leq \mu < \nu < \eta \leq n} \left[p_\mu f(x_\mu) + p_\nu f(x_\nu) + p_\eta f(x_\eta) \right. \\ & \quad \left. + (1 - p_\mu - p_\nu - p_\eta) f\left(\frac{\sum_{i=1}^n p_i x_i - (p_\mu x_\mu + p_\nu x_\nu + p_\eta x_\eta)}{1 - p_\mu - p_\nu - p_\eta}\right) \right] \\ & \leq \sum_{i=1}^n p_i f(x_i) \end{aligned} \tag{7}$$

The above inequalities are immediate consequences of the Theorem 1 in [5]. Compared with the previous inequalities in Lemma 3, the new inequalities are better improved.

In order to present further generalization and comparison, we start by introducing some notations, as follows:

$$\begin{aligned} T_2 = & \max_{1 \leq \mu_1 < \mu_2 \leq n} \left[p_{\mu_1} f(x_{\mu_1}) + p_{\mu_2} f(x_{\mu_2}) + \right. \\ & \left. (1 - p_{\mu_1} - p_{\mu_2}) f\left(\frac{\sum_{i=1}^n p_i x_i - (p_{\mu_1} x_{\mu_1} + p_{\mu_2} x_{\mu_2})}{1 - p_{\mu_1} - p_{\mu_2}}\right) \right], \end{aligned}$$

...

$$T_{n-1} = \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \left[\sum_{i=1}^{n-1} p_{\mu_i} f(x_{\mu_i}) + \left(1 - \sum_{i=1}^{n-1} p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{1 - \sum_{i=1}^{n-1} p_{\mu_i}}\right) \right],$$

where $\mu_1, \mu_2, \dots, \mu_{n-1} \in \{1, 2, \dots, n\}$ are different items. By using the above notations we can rewrite the inequalities in Theorem 1 as this form

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq T_2 \leq T_3 \leq \sum_{i=1}^n p_i f(x_i).$$

On this basis we give the following generalization of Theorem 1.

THEOREM 2. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then*

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq T_2 \leq T_3 \leq \dots \leq T_{n-1} \leq \sum_{i=1}^n p_i f(x_i). \tag{8}$$

The above inequalities are also immediate consequences of the Theorem 1 in [5]. To the mentioned notations S_j and T_j , we have the following result:

THEOREM 3. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then*

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq S_j + f\left(\sum_{i=1}^n p_i x_i\right) \leq T_j \leq \sum_{i=1}^n p_i f(x_i), \quad 2 \leq j \leq n-1. \tag{9}$$

Proof. The first part and third part of the inequalities can be obtained from Lemma 4 and Theorem 2. So we only discuss the second part. To a determined $j \in \{2, 3, \dots, n-1\}$, we observe the expression

$$\sum_{i=1}^n p_i x_i = \left(\sum_{i=1}^j p_{\mu_i}\right) \frac{\sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^j p_{\mu_i}} + \left(1 - \sum_{i=1}^j p_{\mu_i}\right) \frac{\sum_{i=1}^n p_i x_i - \sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{1 - \sum_{i=1}^j p_{\mu_i}}.$$

By using the Jensen’s inequality we have

$$\begin{aligned} f\left(\sum_{i=1}^n p_i x_i\right) &\leq \left(\sum_{i=1}^j p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^j p_{\mu_i}}\right) \\ &\quad + \left(1 - \sum_{i=1}^j p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{1 - \sum_{i=1}^j p_{\mu_i}}\right). \end{aligned} \tag{10}$$

Adding $\sum_{i=1}^j p_{\mu_i} f(x_{\mu_i}) - \left(\sum_{i=1}^j p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^j p_{\mu_i}}\right)$ to both sides of the expression (10), we obtain

$$\begin{aligned} &\sum_{i=1}^j p_{\mu_i} f(x_{\mu_i}) - \left(\sum_{i=1}^j p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^j p_{\mu_i}}\right) + f\left(\sum_{i=1}^n p_i x_i\right) \\ &\leq \sum_{i=1}^j p_{\mu_i} f(x_{\mu_i}) + \left(1 - \sum_{i=1}^j p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{1 - \sum_{i=1}^j p_{\mu_i}}\right). \end{aligned}$$

So

$$\begin{aligned} &\max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_j \leq n} \left[\sum_{i=1}^j p_{\mu_i} f(x_{\mu_i}) - \left(\sum_{i=1}^j p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^j p_{\mu_i}}\right) \right] + f\left(\sum_{i=1}^n p_i x_i\right) \\ &\leq \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_j \leq n} \left[\sum_{i=1}^j p_{\mu_i} f(x_{\mu_i}) + \left(1 - \sum_{i=1}^j p_{\mu_i}\right) f\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{i=1}^j p_{\mu_i} x_{\mu_i}}{1 - \sum_{i=1}^j p_{\mu_i}}\right) \right]. \end{aligned} \tag{11}$$

By using the expression (11) we deduce the second part of the equalities

$$S_j + f \left(\sum_{i=1}^n p_i x_i \right) \leq T_j. \quad \square$$

So compared with the previous inequalities in Lemma 1 and Lemma 4, the new inequalities in Theorem 3 are better improved.

3. More refinements of Jensen's inequality by four functionals

For further discussion, some new notations are introduced. Let

$$P_i^j = \begin{cases} \sum_{k=i}^j p_k, & \text{for } i \leq j, \\ 0, & \text{for } i > j, \end{cases}$$

and $P_i^j f \left(\frac{1}{P_i^j} \sum_{k=i}^j p_k \right) = 0$ for $i > j$. If $1 \leq m \leq n$, $\mu_1 \leq \dots \leq \mu_m \in \{0, 1, \dots, n, n+1\}$, then we define the following functionals:

$$\begin{aligned} F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= P_1^{\mu_1} f \left(\frac{1}{P_1^{\mu_1}} \sum_{i=1}^{\mu_1} p_i x_i \right) + P_{\mu_1+1}^{\mu_2} f \left(\frac{1}{P_{\mu_1+1}^{\mu_2}} \sum_{i=\mu_1+1}^{\mu_2} p_i x_i \right) + \dots \\ &\quad + P_{\mu_{m-1}+1}^{\mu_m} f \left(\frac{1}{P_{\mu_{m-1}+1}^{\mu_m}} \sum_{i=\mu_{m-1}+1}^{\mu_m} p_i x_i \right) - P_1^{\mu_m} f \left(\frac{1}{P_1^{\mu_m}} \sum_{i=1}^{\mu_m} p_i x_i \right) \\ F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= P_{\mu_1}^{\mu_2-1} f \left(\frac{1}{P_{\mu_1}^{\mu_2-1}} \sum_{i=\mu_1}^{\mu_2-1} p_i x_i \right) + P_{\mu_2}^{\mu_3-1} f \left(\frac{1}{P_{\mu_2}^{\mu_3-1}} \sum_{i=\mu_2}^{\mu_3-1} p_i x_i \right) + \dots \\ &\quad + P_{\mu_m}^n f \left(\frac{1}{P_{\mu_m}^n} \sum_{i=\mu_m}^n p_i x_i \right) - P_{\mu_1}^n f \left(\frac{1}{P_{\mu_1}^n} \sum_{i=\mu_1}^n p_i x_i \right) \\ G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= P_1^{\mu_1} f \left(\frac{1}{P_1^{\mu_1}} \sum_{i=1}^{\mu_1} p_i x_i \right) + P_{\mu_1+1}^{\mu_2} f \left(\frac{1}{P_{\mu_1+1}^{\mu_2}} \sum_{i=\mu_1+1}^{\mu_2} p_i x_i \right) + \dots \\ &\quad + P_{\mu_m+1}^n f \left(\frac{1}{P_{\mu_m+1}^n} \sum_{i=\mu_m+1}^n p_i x_i \right) \\ G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= P_1^{\mu_1-1} f \left(\frac{1}{P_1^{\mu_1-1}} \sum_{i=1}^{\mu_1-1} p_i x_i \right) + P_{\mu_1}^{\mu_2-1} f \left(\frac{1}{P_{\mu_1}^{\mu_2-1}} \sum_{i=\mu_1}^{\mu_2-1} p_i x_i \right) + \dots \\ &\quad + P_{\mu_m}^n f \left(\frac{1}{P_{\mu_m}^n} \sum_{i=\mu_m}^n p_i x_i \right) \end{aligned}$$

Let $\mu_0 = 0, \mu_{m+1} = n + 1$, then for simplicity these functionals can be written as:

$$\begin{aligned}
 F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= \sum_{i=1}^m P_{\mu_{i-1}+1}^{\mu_i} f \left(\frac{1}{P_{\mu_{i-1}+1}^{\mu_i}} \sum_{j=\mu_{i-1}+1}^{\mu_i} p_j x_j \right) - P_1^{\mu_m} f \left(\frac{1}{P_1^{\mu_m}} \sum_{i=1}^m p_i x_i \right) \\
 F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= \sum_{i=1}^m P_{\mu_i}^{\mu_{i+1}-1} f \left(\frac{1}{P_{\mu_i}^{\mu_{i+1}-1}} \sum_{j=\mu_i}^{\mu_{i+1}-1} p_j x_j \right) - P_{\mu_1}^n f \left(\frac{1}{P_{\mu_1}^n} \sum_{i=\mu_1}^n p_i x_i \right) \\
 G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= \sum_{i=1}^m P_{\mu_{i-1}+1}^{\mu_i} f \left(\frac{1}{P_{\mu_{i-1}+1}^{\mu_i}} \sum_{j=\mu_{i-1}+1}^{\mu_i} p_j x_j \right) + P_{\mu_m+1}^n f \left(\frac{1}{P_{\mu_m+1}^n} \sum_{i=\mu_m+1}^n p_i x_i \right) \\
 G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) &= \sum_{i=1}^m P_{\mu_i}^{\mu_{i+1}-1} f \left(\frac{1}{P_{\mu_i}^{\mu_{i+1}-1}} \sum_{j=\mu_i}^{\mu_{i+1}-1} p_j x_j \right) + P_1^{\mu_1-1} f \left(\frac{1}{P_1^{\mu_1-1}} \sum_{i=1}^{\mu_1-1} p_i x_i \right)
 \end{aligned}$$

REMARK 1. For any μ_1 , we note that

$$F_1(f, \mathbf{x}, \mathbf{p}; \mu_1) = F_2(f, \mathbf{x}, \mathbf{p}; \mu_1) = 0; \tag{12}$$

For $\mu_i = i, 1 \leq i \leq n$, we note that

$$F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = \sum_{i=1}^n p_i f(x_i) - f \left(\sum_{i=1}^n p_i x_i \right); \tag{13}$$

For $\mu_i = i, 1 \leq i \leq n$, we note that

$$G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = \sum_{i=1}^n p_i f(x_i). \tag{14}$$

Next we prove the following results.

LEMMA 6. If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then

$$F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) \geq 0, \quad F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) \geq 0.$$

Proof. Let $\mu_0 = 0, \mu_{m+1} = n + 1$. We observe the expression $\sum_{i=1}^m P_{\mu_{i-1}+1}^{\mu_i} = P_1^{\mu_m}$, $\sum_{i=1}^m P_{\mu_i}^{\mu_{i+1}-1} = P_{\mu_1}^n$. That is $\sum_{i=1}^m P_{\mu_{i-1}+1}^{\mu_i} / P_1^{\mu_m} = 1, \sum_{i=1}^m P_{\mu_i}^{\mu_{i+1}-1} / P_{\mu_1}^n = 1$. So applying the Jensen's inequality we have

$$\begin{aligned}
 \sum_{i=1}^m \frac{P_{\mu_{i-1}+1}^{\mu_i}}{P_1^{\mu_m}} f \left(\frac{1}{P_{\mu_{i-1}+1}^{\mu_i}} \sum_{j=\mu_{i-1}+1}^{\mu_i} p_j x_j \right) &\geq f \left(\sum_{i=1}^m \frac{P_{\mu_{i-1}+1}^{\mu_i}}{P_1^{\mu_m}} \left(\frac{1}{P_{\mu_{i-1}+1}^{\mu_i}} \sum_{j=\mu_{i-1}+1}^{\mu_i} p_j x_j \right) \right) \\
 &= f \left(\frac{1}{P_1^{\mu_m}} \sum_{i=1}^m p_i x_i \right), \\
 \sum_{i=1}^m \frac{P_{\mu_i}^{\mu_{i+1}-1}}{P_{\mu_1}^n} f \left(\frac{1}{P_{\mu_i}^{\mu_{i+1}-1}} \sum_{j=\mu_i}^{\mu_{i+1}-1} p_j x_j \right) &\geq f \left(\sum_{i=1}^m \frac{P_{\mu_i}^{\mu_{i+1}-1}}{P_{\mu_1}^n} \left(\frac{1}{P_{\mu_i}^{\mu_{i+1}-1}} \sum_{j=\mu_i}^{\mu_{i+1}-1} p_j x_j \right) \right) \\
 &= f \left(\frac{1}{P_{\mu_1}^n} \sum_{i=\mu_1}^n p_i x_i \right).
 \end{aligned}$$

These yield

$$\begin{aligned} \sum_{i=1}^m P_{\mu_{i-1}+1}^{\mu_i} f\left(\frac{1}{P_{\mu_{i-1}+1}^{\mu_i}} \sum_{j=\mu_{i-1}+1}^{\mu_i} p_j x_j\right) &\geq P_1^{\mu_m} f\left(\frac{1}{P_1^{\mu_m}} \sum_{i=1}^{\mu_m} p_i x_i\right), \\ \sum_{i=1}^m P_{\mu_i}^{\mu_{i+1}-1} f\left(\frac{1}{P_{\mu_i}^{\mu_{i+1}-1}} \sum_{j=\mu_i}^{\mu_{i+1}-1} p_j x_j\right) &\geq P_{\mu_1}^n f\left(\frac{1}{P_{\mu_1}^n} \sum_{i=\mu_1}^n p_i x_i\right). \end{aligned}$$

By the expressions of $F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m)$, $F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m)$, the conclusion follows. \square

THEOREM 4. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then*

$$\begin{aligned} 0 &\leq \max_{1 \leq \mu_1 < \dots < \mu_m \leq n} F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) \\ &\leq \max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1}) \\ &\leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right), \end{aligned} \tag{15}$$

where $1 \leq m \leq n - 1$.

Proof. The first part of the inequalities (15) can be received from Lemma 6. Now in order to prove the second part, we assume the maximum value of $F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m)$ is obtained for $\mu_i = a_i$, $1 \leq i \leq m$ and $1 \leq a_i < a_j \leq n$, $1 \leq i < j \leq m$. Let $a_0 = 0$. Then for any $q \in \{1, 2, \dots, n\} - \{a_1, \dots, a_m\}$, we consider the cases $q > a_m$ and $q < a_m$.

When $q > a_m$,

$$\begin{aligned} &F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_m, q) - F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_m) \\ &= P_{a_{m+1}}^q f\left(\frac{1}{P_{a_{m+1}}^q} \sum_{i=a_{m+1}}^q p_i x_i\right) + P_1^{a_m} f\left(\frac{1}{P_1^{a_m}} \sum_{i=1}^{a_m} p_i x_i\right) - P_1^q f\left(\frac{1}{P_1^q} \sum_{i=1}^q p_i x_i\right). \end{aligned}$$

Observing $P_1^{a_m} + P_{a_{m+1}}^q = P_1^q$, or equivalently, $P_1^{a_m}/P_1^q + P_{a_{m+1}}^q/P_1^q = 1$, by using the Jensen's inequality we have

$$\begin{aligned} &\frac{P_{a_{m+1}}^q}{P_1^q} f\left(\frac{1}{P_{a_{m+1}}^q} \sum_{i=a_{m+1}}^q p_i x_i\right) + \frac{P_1^{a_m}}{P_1^q} f\left(\frac{1}{P_1^{a_m}} \sum_{i=1}^{a_m} p_i x_i\right) \\ &\geq f\left(\frac{P_{a_{m+1}}^q}{P_1^q} \left(\frac{1}{P_{a_{m+1}}^q} \sum_{i=a_{m+1}}^q p_i x_i\right) + \frac{P_1^{a_m}}{P_1^q} \left(\frac{1}{P_1^{a_m}} \sum_{i=1}^{a_m} p_i x_i\right)\right) = f\left(\frac{1}{P_1^q} \sum_{i=1}^q p_i x_i\right). \end{aligned}$$

These yield $F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_m, q) - F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_m) \geq 0$.

When $q < a_m$, without loss of generality we assume $q \in (a_{j-1}, a_j)$, $1 \leq j \leq m$.

$$\begin{aligned} & F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_{j-1}, q, a_j, \dots, a_m) - F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_m) \\ &= P_{a_{j-1}+1}^q f \left(\frac{1}{P_{a_{j-1}+1}^q} \sum_{i=a_{j-1}+1}^q p_i x_i \right) + P_{q+1}^{a_j} f \left(\frac{1}{P_{q+1}^{a_j}} \sum_{i=q+1}^{a_j} p_i x_i \right) \\ &\quad - P_{a_{j-1}+1}^{a_j} f \left(\frac{1}{P_{a_{j-1}+1}^{a_j}} \sum_{i=a_{j-1}+1}^{a_j} p_i x_i \right). \end{aligned}$$

Observing $P_{a_{j-1}+1}^q + P_{q+1}^{a_j} = P_{a_{j-1}+1}^{a_j}$, or equivalently, $P_{a_{j-1}+1}^q / P_{a_{j-1}+1}^{a_j} + P_{q+1}^{a_j} / P_{a_{j-1}+1}^{a_j} = 1$, by using the Jensen's inequality we have

$$\begin{aligned} & \frac{P_{a_{j-1}+1}^q}{P_{a_{j-1}+1}^{a_j}} f \left(\frac{1}{P_{a_{j-1}+1}^q} \sum_{i=a_{j-1}+1}^q p_i x_i \right) + \frac{P_{q+1}^{a_j}}{P_{a_{j-1}+1}^{a_j}} f \left(\frac{1}{P_{q+1}^{a_j}} \sum_{i=q+1}^{a_j} p_i x_i \right) \\ & \geq f \left(\frac{P_{a_{j-1}+1}^q}{P_{a_{j-1}+1}^{a_j}} \left(\frac{1}{P_{a_{j-1}+1}^q} \sum_{i=a_{j-1}+1}^q p_i x_i \right) + \frac{P_{q+1}^{a_j}}{P_{a_{j-1}+1}^{a_j}} \left(\frac{1}{P_{q+1}^{a_j}} \sum_{i=q+1}^{a_j} p_i x_i \right) \right) \\ & = f \left(\frac{1}{P_{a_{j-1}+1}^{a_j}} \sum_{i=a_{j-1}+1}^{a_j} p_i x_i \right). \end{aligned}$$

These yield $F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_{j-1}, q, a_j, \dots, a_m) - F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_m) \geq 0$.

Now because the expression $\max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1})$ is greater than or equal to $F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_m, q)$ or $F_1(f, \mathbf{x}, \mathbf{p}; a_1, \dots, a_{j-1}, q, a_j, \dots, a_m)$, we deduce the second part of the inequalities (15).

Finally, obviously the maximum value of $m + 1$ is n , and for $\mu_i = i$, $1 \leq i \leq n$, $\max_{1 \leq \mu_1 < \dots < \mu_n \leq n} F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = \sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i)$ from (13). Following the steps above exactly, the third part of the inequalities (15) holds. \square

THEOREM 5. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then*

$$\begin{aligned} 0 & \leq \max_{1 \leq \mu_1 < \dots < \mu_m \leq n} F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) \\ & \leq \max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1}) \\ & \leq \sum_{i=1}^n p_i f(x_i) - f \left(\sum_{i=1}^n p_i x_i \right), \end{aligned} \tag{16}$$

where $1 \leq m \leq n - 1$.

Proof. The first part of the inequalities (16) can be received from Lemma 6. Now in order to prove the second part, we assume the maximum value of $F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m)$ is obtained for $\mu_i = b_i$, $1 \leq i \leq m$ and $1 \leq b_i < b_j \leq n$, $1 \leq i < j \leq m$. Let $b_{m+1} = n$. Then for any $r \in \{1, 2, \dots, n\} - \{b_1, \dots, b_m\}$, we consider the cases $r > b_1$ and $r < b_1$.

When $r < b_1$,

$$F_2(f, \mathbf{x}, \mathbf{p}; r, b_1, \dots, b_m) - F_2(f, \mathbf{x}, \mathbf{p}; b_1, \dots, b_m) \\ = P_r^{b_1-1} f \left(\frac{1}{P_r^{b_1-1}} \sum_{i=r}^{b_1-1} p_i x_i \right) + P_{b_1}^n f \left(\frac{1}{P_{b_1}^n} \sum_{i=b_1}^n p_i x_i \right) - P_r^n f \left(\frac{1}{P_r^n} \sum_{i=r}^n p_i x_i \right).$$

Observing $P_r^{b_1-1} + P_{b_1}^n = P_r^n$, or equivalently, $P_r^{b_1-1}/P_r^n + P_{b_1}^n/P_r^n = 1$, by using the Jensen's inequality we have

$$\frac{P_r^{b_1-1}}{P_r^n} f \left(\frac{1}{P_r^{b_1-1}} \sum_{i=r}^{b_1-1} p_i x_i \right) + \frac{P_{b_1}^n}{P_r^n} f \left(\frac{1}{P_{b_1}^n} \sum_{i=b_1}^n p_i x_i \right) \\ \geq f \left(\frac{P_r^{b_1-1}}{P_r^n} \left(\frac{1}{P_r^{b_1-1}} \sum_{i=r}^{b_1-1} p_i x_i \right) + \frac{P_{b_1}^n}{P_r^n} \left(\frac{1}{P_{b_1}^n} \sum_{i=b_1}^n p_i x_i \right) \right) = f \left(\frac{1}{P_r^n} \sum_{i=r}^n p_i x_i \right).$$

These yield $F_2(f, \mathbf{x}, \mathbf{p}; r, b_1, \dots, b_m) - F_2(f, \mathbf{x}, \mathbf{p}; b_1, \dots, b_m) \geq 0$.

When $r > b_1$, without loss of generality we assume $r \in (b_j, b_{j+1})$, $1 \leq j \leq m$.

$$F_2(f, \mathbf{x}, \mathbf{p}; b_1, \dots, b_j, r, b_{j+1}, \dots, b_m) - F_2(f, \mathbf{x}, \mathbf{p}; b_1, \dots, b_m) \\ = P_{b_j}^{r-1} f \left(\frac{1}{P_{b_j}^{r-1}} \sum_{i=b_j}^{r-1} p_i x_i \right) + P_r^{b_{j+1}-1} f \left(\frac{1}{P_r^{b_{j+1}-1}} \sum_{i=r}^{b_{j+1}-1} p_i x_i \right) \\ - P_{b_j}^{b_{j+1}-1} f \left(\frac{1}{P_{b_j}^{b_{j+1}-1}} \sum_{i=b_j}^{b_{j+1}-1} p_i x_i \right).$$

Observing $P_{b_j}^{r-1} + P_r^{b_{j+1}-1} = P_{b_j}^{b_{j+1}-1}$, or equivalently, $P_{b_j}^{r-1}/P_{b_j}^{b_{j+1}-1} + P_r^{b_{j+1}-1}/P_{b_j}^{b_{j+1}-1} = 1$, by using the Jensen's inequality we have

$$\frac{P_{b_j}^{r-1}}{P_{b_j}^{b_{j+1}-1}} f \left(\frac{1}{P_{b_j}^{r-1}} \sum_{i=b_j}^{r-1} p_i x_i \right) + \frac{P_r^{b_{j+1}-1}}{P_{b_j}^{b_{j+1}-1}} f \left(\frac{1}{P_r^{b_{j+1}-1}} \sum_{i=r}^{b_{j+1}-1} p_i x_i \right) \\ \geq f \left(\frac{P_{b_j}^{r-1}}{P_{b_j}^{b_{j+1}-1}} \left(\frac{1}{P_{b_j}^{r-1}} \sum_{i=b_j}^{r-1} p_i x_i \right) + \frac{P_r^{b_{j+1}-1}}{P_{b_j}^{b_{j+1}-1}} \left(\frac{1}{P_r^{b_{j+1}-1}} \sum_{i=r}^{b_{j+1}-1} p_i x_i \right) \right) \\ = f \left(\frac{1}{P_{b_j}^{b_{j+1}-1}} \sum_{i=b_j}^{b_{j+1}-1} p_i x_i \right).$$

These yield $F_2(f, \mathbf{x}, \mathbf{p}; b_1, \dots, b_j, r, b_{j+1}, \dots, b_m) - F_2(f, \mathbf{x}, \mathbf{p}; b_1, \dots, b_m) \geq 0$.

Now because the expression $\max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1})$ is greater than or equal to $F_2(f, \mathbf{x}, \mathbf{p}; r, b_1, \dots, b_m)$ or $F_2(f, \mathbf{x}, \mathbf{p}; b_1, \dots, b_j, r, b_{j+1}, \dots, b_m)$, we deduce the second part of the inequalities (16).

Finally, obviously the maximum value of $m + 1$ is n , and for $\mu_i = i, 1 \leq i \leq n$, $\max_{1 \leq \mu_1 < \dots < \mu_n \leq n} F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = \sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i)$ from (13). Following the steps above exactly, the third part of the inequalities (16) holds. \square

THEOREM 6. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then*

$$\begin{aligned} f\left(\sum_{i=1}^n p_i x_i\right) &\leq \max_{1 \leq \mu_1 < \dots < \mu_m \leq n} G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) \\ &\leq \max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1}) \\ &\leq \sum_{i=1}^n p_i f(x_i), \end{aligned} \tag{17}$$

where $1 \leq m \leq n - 1$.

Proof. we assume the maximum value of $G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m)$ is obtained for $\mu_i = c_i, 1 \leq i \leq m$ and $1 \leq c_i < c_j \leq n, 1 \leq i < j \leq m$. Let $c_0 = 0, c_{m+1} = n$.

Since $\sum_{i=1}^{m+1} P_{c_{i-1}+1}^{c_i} = 1$, by using the Jensen’s inequality we have

$$\begin{aligned} f\left(\sum_{i=1}^n p_i x_i\right) &= f\left(\sum_{i=1}^{m+1} \sum_{j=c_{i-1}+1}^{c_i} p_j x_j\right) \\ &\leq \sum_{i=1}^{m+1} P_{c_{i-1}+1}^{c_i} f\left(\frac{1}{P_{c_{i-1}+1}^{c_i}} \sum_{j=c_{i-1}+1}^{c_i} p_j x_j\right) = G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m). \end{aligned}$$

From the above the first part of the inequalities (17) holds.

Now we prove the second part. For any $s \in \{1, 2, \dots, n\} - \{c_1, \dots, c_m\}$, we consider the case $s > c_m$ and $s < c_m$.

When $s > c_m$

$$\begin{aligned} &G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m, s) - G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m) \\ &= P_{c_m+1}^s f\left(\frac{1}{P_{c_m+1}^s} \sum_{i=c_m+1}^s p_i x_i\right) + P_{s+1}^n f\left(\frac{1}{P_{s+1}^n} \sum_{i=s+1}^n p_i x_i\right) - P_{c_m+1}^n f\left(\frac{1}{P_{c_m+1}^n} \sum_{i=c_m+1}^n p_i x_i\right). \end{aligned}$$

Observing $P_{c_m+1}^s + P_{s+1}^n = P_{c_m+1}^n$, or equivalently, $P_{c_m+1}^s / P_{c_m+1}^n + P_{s+1}^n / P_{c_m+1}^n = 1$, by using the Jensen’s inequality we have

$$\begin{aligned} &\frac{P_{c_m+1}^s}{P_{c_m+1}^n} f\left(\frac{1}{P_{c_m+1}^s} \sum_{i=c_m+1}^s p_i x_i\right) + \frac{P_{s+1}^n}{P_{c_m+1}^n} f\left(\frac{1}{P_{s+1}^n} \sum_{i=s+1}^n p_i x_i\right) \\ &\geq f\left(\frac{P_{c_m+1}^s}{P_{c_m+1}^n} \left(\frac{1}{P_{c_m+1}^s} \sum_{i=c_m+1}^s p_i x_i\right) + \frac{P_{s+1}^n}{P_{c_m+1}^n} \left(\frac{1}{P_{s+1}^n} \sum_{i=s+1}^n p_i x_i\right)\right) \\ &= f\left(\frac{1}{P_{c_m+1}^n} \sum_{i=c_m+1}^n p_i x_i\right). \end{aligned}$$

These yield $G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m, s) - G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m) \geq 0$.

When $s < c_m$, without loss of generality we assume $s \in (c_k, c_{k+1})$, $0 \leq k \leq m - 1$.

$$G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_k, s, c_{k+1}, \dots, c_m) - G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m) \\ = P_{c_k+1}^s f \left(\frac{1}{P_{c_k+1}^s} \sum_{i=c_k+1}^s p_i x_i \right) + P_{s+1}^{c_{k+1}} f \left(\frac{1}{P_{s+1}^{c_{k+1}}} \sum_{i=s+1}^{c_{k+1}} p_i x_i \right) - P_{c_k+1}^{c_{k+1}} f \left(\frac{1}{P_{c_k+1}^{c_{k+1}}} \sum_{i=c_k+1}^{c_{k+1}} p_i x_i \right).$$

Observing $P_{c_k+1}^s + P_{s+1}^{c_{k+1}} = P_{c_k+1}^{c_{k+1}}$, or equivalently, $P_{c_k+1}^s / P_{c_k+1}^{c_{k+1}} + P_{s+1}^{c_{k+1}} / P_{c_k+1}^{c_{k+1}} = 1$, by using the Jensen’s inequality we have

$$\frac{P_{c_k+1}^s}{P_{c_k+1}^{c_{k+1}}} f \left(\frac{1}{P_{c_k+1}^s} \sum_{i=c_k+1}^s p_i x_i \right) + \frac{P_{s+1}^{c_{k+1}}}{P_{c_k+1}^{c_{k+1}}} f \left(\frac{1}{P_{s+1}^{c_{k+1}}} \sum_{i=s+1}^{c_{k+1}} p_i x_i \right) \\ \geq f \left(\frac{P_{c_k+1}^s}{P_{c_k+1}^{c_{k+1}}} \left(\frac{1}{P_{c_k+1}^s} \sum_{i=c_k+1}^s p_i x_i \right) + \frac{P_{s+1}^{c_{k+1}}}{P_{c_k+1}^{c_{k+1}}} \left(\frac{1}{P_{s+1}^{c_{k+1}}} \sum_{i=s+1}^{c_{k+1}} p_i x_i \right) \right) \\ = f \left(\frac{1}{P_{c_k+1}^{c_{k+1}}} \sum_{i=c_k+1}^{c_{k+1}} p_i x_i \right).$$

These yield $G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_k, s, c_{k+1}, \dots, c_m) - G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m) \geq 0$.

Because the expression $\max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1})$ is greater than or equal to $G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_m, s)$ or $G_1(f, \mathbf{x}, \mathbf{p}; c_1, \dots, c_k, s, c_{k+1}, \dots, c_m)$, we deduce the second part of the inequalities (17).

Finally, obviously the maximum value of $m + 1$ is n , and for $\mu_i = i$, $1 \leq i \leq n$, $\max_{1 \leq \mu_1 < \dots < \mu_n \leq n} G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = \sum_{i=1}^n p_i f(x_i)$ from (14). Following the steps above exactly, the third part of the inequalities (17) holds. \square

THEOREM 7. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then*

$$f \left(\sum_{i=1}^n p_i x_i \right) \leq \max_{1 \leq \mu_1 < \dots < \mu_m \leq n} G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m) \\ \leq \max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1}) \quad (18) \\ \leq \sum_{i=1}^n p_i f(x_i),$$

where $1 \leq m \leq n - 1$.

Proof. we assume the maximum value of $G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m)$ is obtained for $\mu_i = d_i$, $1 \leq i \leq m$ and $1 \leq d_i < d_j \leq n$, $1 \leq i < j \leq m$. Let $d_0 = 1$, $d_{m+1} = n + 1$.

Since $\sum_{i=1}^{m+1} P_{d_{i-1}}^{d_i-1} = 1$, by using the Jensen's inequality we have

$$\begin{aligned} f\left(\sum_{i=1}^n p_i x_i\right) &= f\left(\sum_{i=1}^{m+1} \sum_{j=d_{i-1}}^{d_i-1} p_j x_j\right) \\ &\leq \sum_{i=1}^{m+1} P_{d_{i-1}}^{d_i-1} f\left(\frac{1}{P_{d_{i-1}}^{d_i-1}} \sum_{j=d_{i-1}}^{d_i-1} p_j x_j\right) = G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_m). \end{aligned}$$

From the above the first part of the inequalities (18) holds.

Now we prove the second part. For any $t \in \{1, 2, \dots, n\} - \{d_1, \dots, d_m\}$, we consider the case $t < d_1$ and $t > d_1$.

When $t < d_1$,

$$\begin{aligned} &G_2(f, \mathbf{x}, \mathbf{p}; t, d_1, \dots, d_m) - G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_m) \\ &= P_1^{t-1} f\left(\frac{1}{P_1^{t-1}} \sum_{i=1}^{t-1} p_i x_i\right) + P_t^{d_1-1} f\left(\frac{1}{P_t^{d_1-1}} \sum_{i=t}^{d_1-1} p_i x_i\right) - P_1^{d_1-1} f\left(\frac{1}{P_1^{d_1-1}} \sum_{i=1}^{d_1-1} p_i x_i\right). \end{aligned}$$

Observing $P_1^{t-1} + P_t^{d_1-1} = P_1^{d_1-1}$, or equivalently, $P_1^{t-1}/P_1^{d_1-1} + P_t^{d_1-1}/P_1^{d_1-1} = 1$, by using the Jensen's inequality we have

$$\begin{aligned} &\frac{P_1^{t-1}}{P_1^{d_1-1}} f\left(\frac{1}{P_1^{t-1}} \sum_{i=1}^{t-1} p_i x_i\right) + \frac{P_t^{d_1-1}}{P_1^{d_1-1}} f\left(\frac{1}{P_t^{d_1-1}} \sum_{i=t}^{d_1-1} p_i x_i\right) \\ &\geq f\left(\frac{P_1^{t-1}}{P_1^{d_1-1}} \left(\frac{1}{P_1^{t-1}} \sum_{i=1}^{t-1} p_i x_i\right) + \frac{P_t^{d_1-1}}{P_1^{d_1-1}} \left(\frac{1}{P_t^{d_1-1}} \sum_{i=t}^{d_1-1} p_i x_i\right)\right) \\ &= f\left(\frac{1}{P_1^{d_1-1}} \sum_{i=1}^{d_1-1} p_i x_i\right). \end{aligned}$$

These yield $G_2(f, \mathbf{x}, \mathbf{p}; t, d_1, \dots, d_m) - G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_m) \geq 0$.

When $t > d_1$, without loss of generality we assume $t \in (d_k, d_{k+1})$, $1 \leq k \leq m$.

$$\begin{aligned} &G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_k, t, d_{k+1}, \dots, d_m) - G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_m) \\ &= P_{d_k}^{t-1} f\left(\frac{1}{P_{d_k}^{t-1}} \sum_{i=d_k}^{t-1} p_i x_i\right) + P_t^{d_{k+1}-1} f\left(\frac{1}{P_t^{d_{k+1}-1}} \sum_{i=t}^{d_{k+1}-1} p_i x_i\right) \\ &\quad - P_{d_k}^{d_{k+1}-1} f\left(\frac{1}{P_{d_k}^{d_{k+1}-1}} \sum_{i=d_k}^{d_{k+1}-1} p_i x_i\right). \end{aligned}$$

Observing $P_{d_k}^{t-1} + P_t^{d_{k+1}-1} = P_{d_k}^{d_{k+1}-1}$, or equivalently, $P_{d_k}^{t-1}/P_{d_k}^{d_{k+1}-1} + P_t^{d_{k+1}-1}/P_{d_k}^{d_{k+1}-1} = 1$

= 1, by using the Jensen’s inequality we have

$$\begin{aligned} & \frac{P_{d_k}^{t-1}}{P_{d_k}^{d_{k+1}-1}} f\left(\frac{1}{P_{d_k}^{t-1}} \sum_{i=d_k}^{t-1} p_i x_i\right) + \frac{P_t^{d_{k+1}-1}}{P_{d_k}^{d_{k+1}-1}} f\left(\frac{1}{P_t^{d_{k+1}-1}} \sum_{i=t}^{d_{k+1}-1} p_i x_i\right) \\ & \geq f\left(\frac{P_{d_k}^{t-1}}{P_{d_k}^{d_{k+1}-1}} \left(\frac{1}{P_{d_k}^{t-1}} \sum_{i=d_k}^{t-1} p_i x_i\right) + \frac{P_t^{d_{k+1}-1}}{P_{d_k}^{d_{k+1}-1}} \left(\frac{1}{P_t^{d_{k+1}-1}} \sum_{i=t}^{d_{k+1}-1} p_i x_i\right)\right) \\ & = f\left(\frac{1}{P_{d_k}^{d_{k+1}-1}} \sum_{i=d_k}^{d_{k+1}-1} p_i x_i\right). \end{aligned}$$

These yield $G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_k, t, d_{k+1}, \dots, d_m) - G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_m) \geq 0$.

Because the expression $\max_{1 \leq \mu_1 < \dots < \mu_m < \mu_{m+1} \leq n} G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_m, \mu_{m+1})$ is greater than or equal to $G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_m)$ or $G_2(f, \mathbf{x}, \mathbf{p}; d_1, \dots, d_k, t, d_{k+1}, \dots, d_m)$, we deduce the second part of the inequalities (18).

Finally, obviously the maximum value of $m + 1$ is n , and for $\mu_i = i, 1 \leq i \leq n$, $\max_{1 \leq \mu_1 < \dots < \mu_n \leq n} G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_n) = \sum_{i=1}^n p_i f(x_i)$ from (14). Following the steps above exactly, the third part of the inequalities (18) holds. \square

Similar to the previous representations, we establish new notations as follows:

$$\begin{aligned} S_{1,j} &= \max_{1 \leq \mu_1 < \dots < \mu_j \leq n} F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j), \\ S_{2,j} &= \max_{1 \leq \mu_1 < \dots < \mu_j \leq n} F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j), \\ T_{1,j} &= \max_{1 \leq \mu_1 < \dots < \mu_j \leq n} G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j), \\ T_{2,j} &= \max_{1 \leq \mu_1 < \dots < \mu_j \leq n} G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j), \end{aligned}$$

where $1 \leq j \leq n, \mu_1, \dots, \mu_j \in \{1, 2, \dots, n\}$. Then the previous four theorems can be rewritten as these forms:

$$0 = S_{1,1} \leq S_{1,2} \leq \dots \leq S_{1,n-1} \leq S_{1,n} = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right), \tag{19}$$

$$0 = S_{2,1} \leq S_{2,2} \leq \dots \leq S_{2,n-1} \leq S_{2,n} = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right), \tag{20}$$

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq T_{1,1} \leq \dots \leq T_{1,n-1} \leq T_{1,n} = \sum_{i=1}^n p_i f(x_i), \tag{21}$$

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq T_{2,1} \leq \dots \leq T_{2,n-1} \leq T_{2,n} = \sum_{i=1}^n p_i f(x_i). \tag{22}$$

To the mentioned notations $S_{1,j}, S_{2,j}, T_{1,j}, T_{2,j}$, we have the following results:

THEOREM 8. If f , \mathbf{x} , \mathbf{p} are defined as above, then

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq S_{1,j} + f\left(\sum_{i=1}^n p_i x_i\right) \leq T_{1,j} \leq \sum_{i=1}^n p_i f(x_i), \quad 1 \leq j \leq n. \quad (23)$$

Proof. The first part and the third part of inequalities (23) can be obtained from expressions (19) and (21). So we only discuss the second part by the functionals $F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j)$ and $G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j)$.

If $\mu_j = n$, then

$$F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) + f\left(\sum_{i=1}^n p_i x_i\right) = G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j).$$

If $\mu_j < n$, then

$$\begin{aligned} & G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) - F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) \\ &= P_1^{\mu_j} f\left(\frac{1}{P_1^{\mu_j}} \sum_{i=1}^{\mu_j} p_i x_i\right) + P_{\mu_j+1}^n f\left(\frac{1}{P_{\mu_j+1}^n} \sum_{i=\mu_j+1}^n p_i x_i\right) \\ &\geq f\left(P_1^{\mu_j} \left(\frac{1}{P_1^{\mu_j}} \sum_{i=1}^{\mu_j} p_i x_i\right) + P_{\mu_j+1}^n \left(\frac{1}{P_{\mu_j+1}^n} \sum_{i=\mu_j+1}^n p_i x_i\right)\right) = f\left(\sum_{i=1}^n p_i x_i\right). \end{aligned}$$

The above inequality follows by Jensen's inequality. To sum up, we can find

$$F_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) + f\left(\sum_{i=1}^n p_i x_i\right) \leq G_1(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j).$$

Maximize both sides for μ_1, \dots, μ_j , then the second part of inequalities (23) holds. \square

THEOREM 9. If f , \mathbf{x} , \mathbf{p} are defined as above, then

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq S_{2,j} + f\left(\sum_{i=1}^n p_i x_i\right) \leq T_{2,j} \leq \sum_{i=1}^n p_i f(x_i), \quad 1 \leq j \leq n. \quad (24)$$

Proof. The first part and the third part of inequalities (24) can be obtained from expressions (20) and (22). So we only discuss the second part by the functionals $F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j)$ and $G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j)$.

If $\mu_1 = 1$, then

$$F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) + f\left(\sum_{i=1}^n p_i x_i\right) = G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j).$$

If $\mu_1 > 1$, then

$$\begin{aligned} & G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) - F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) \\ &= P_1^{\mu_1-1} f \left(\frac{1}{P_1^{\mu_1-1}} \sum_{i=1}^{\mu_1-1} p_i x_i \right) + P_{\mu_1}^n f \left(\frac{1}{P_{\mu_1}^n} \sum_{i=\mu_1}^n p_i x_i \right) \\ &\geq f \left(P_1^{\mu_1-1} \left(\frac{1}{P_1^{\mu_1-1}} \sum_{i=1}^{\mu_1-1} p_i x_i \right) + P_{\mu_1}^n \left(\frac{1}{P_{\mu_1}^n} \sum_{i=\mu_1}^n p_i x_i \right) \right) = f \left(\sum_{i=1}^n p_i x_i \right). \end{aligned}$$

The above inequality follows by Jensen’s inequality. To sum up, we can find

$$F_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j) + f \left(\sum_{i=1}^n p_i x_i \right) \leq G_2(f, \mathbf{x}, \mathbf{p}; \mu_1, \dots, \mu_j).$$

Maximize both sides for μ_1, \dots, μ_j , then the second part of inequalities (24) holds. \square

THEOREM 10. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then for any $S_j, 1 \leq j \leq n$, there exists a positive integer $h \geq j$ such that $S_{1,h} \geq S_j$.*

Proof. we assume the maximum value of S_j is obtained for $\mu_i = y_i, 1 \leq i \leq j$ and $1 \leq y_i < y_k \leq n, 1 \leq i < k \leq j$. Let $y_0 = 0$ and $\{y'_1, \dots, y'_h\} = \{y_1 - 1, y_1, \dots, y_j - 1, y_j\}$, where $0 \leq y'_1 < \dots < y'_h \leq n$. Thus we have $h \geq j$ and

$$\begin{aligned} & F_1(f, \mathbf{x}, \mathbf{p}; y'_1, \dots, y'_h) = F_1(f, \mathbf{x}, \mathbf{p}; y_1 - 1, y_1, \dots, y_j - 1, y_j) \\ &= \sum_{i=1}^j P_{y_{i-1}+1}^{y_i-1} f \left(\frac{1}{P_{y_{i-1}+1}^{y_i-1}} \sum_{k=y_{i-1}+1}^{y_i-1} p_i x_i \right) + \sum_{i=1}^j p_{y_i} f(x_{y_i}) - P_1^{y_j} f \left(\frac{1}{P_1^{y_j}} \sum_{i=1}^{y_j} p_i x_i \right). \end{aligned}$$

Next,

$$\begin{aligned} & F_1(f, \mathbf{x}, \mathbf{p}; y'_1, \dots, y'_h) - \left[\sum_{i=1}^j p_{y_i} f(x_{y_i}) - \left(\sum_{i=1}^j p_{y_i} \right) f \left(\frac{\sum_{i=1}^j p_{y_i} x_{y_i}}{\sum_{i=1}^j p_{y_i}} \right) \right] \\ &= \sum_{i=1}^j P_{y_{i-1}+1}^{y_i-1} f \left(\frac{1}{P_{y_{i-1}+1}^{y_i-1}} \sum_{k=y_{i-1}+1}^{y_i-1} p_i x_i \right) + \left(\sum_{i=1}^j p_{y_i} \right) f \left(\frac{\sum_{i=1}^j p_{y_i} x_{y_i}}{\sum_{i=1}^j p_{y_i}} \right) \\ &\quad - P_1^{y_j} f \left(\frac{1}{P_1^{y_j}} \sum_{i=1}^{y_j} p_i x_i \right). \end{aligned}$$

Observing $\sum_{i=1}^j P_{y_{i-1}+1}^{y_i-1} + \sum_{i=1}^j p_{y_i} = P_1^{y_j}$, or equivalently, $\sum_{i=1}^j P_{y_{i-1}+1}^{y_i-1} / P_1^{y_j} + \sum_{i=1}^j p_{y_i} / P_1^{y_j}$

= 1, by using the Jensen's inequality we have

$$\begin{aligned} & \sum_{i=1}^j \frac{P^{y_{i-1}+1}}{P_1^{y_j}} f\left(\frac{1}{P^{y_{i-1}+1}} \sum_{k=y_{i-1}+1}^{y_i-1} p_i x_i\right) + \frac{\sum_{i=1}^j P y_i}{P_1^{y_j}} f\left(\frac{\sum_{i=1}^j P y_i x_{y_i}}{\sum_{i=1}^j P y_i}\right) \\ & \geq f\left(\sum_{i=1}^j \frac{P^{y_{i-1}+1}}{P_1^{y_j}} \left(\frac{1}{P^{y_{i-1}+1}} \sum_{k=y_{i-1}+1}^{y_i-1} p_i x_i\right) + \frac{\sum_{i=1}^j P y_i}{P_1^{y_j}} \left(\frac{\sum_{i=1}^j P y_i x_{y_i}}{\sum_{i=1}^j P y_i}\right)\right) \\ & = f\left(\frac{1}{P_1^{y_j}} \sum_{i=1}^{y_j} p_i x_i\right). \end{aligned}$$

This derives

$$F_1(f, \mathbf{x}, \mathbf{p}; y'_1, \dots, y'_h) - \left[\sum_{i=1}^j p_{y_i} f(x_{y_i}) - \left(\sum_{i=1}^j p_{y_i}\right) f\left(\frac{\sum_{i=1}^j P y_i x_{y_i}}{\sum_{i=1}^j P y_i}\right) \right] \geq 0,$$

or

$$F_1(f, \mathbf{x}, \mathbf{p}; y'_1, \dots, y'_h) \geq \sum_{i=1}^j p_{y_i} f(x_{y_i}) - \left(\sum_{i=1}^j p_{y_i}\right) f\left(\frac{\sum_{i=1}^j P y_i x_{y_i}}{\sum_{i=1}^j P y_i}\right).$$

From the definition, we have $S_{1,h} \geq F_1(f, \mathbf{x}, \mathbf{p}; y'_1, \dots, y'_h)$. Therefore, the inequality $S_{1,h} \geq S_j$ holds. \square

THEOREM 11. *If $f, \mathbf{x}, \mathbf{p}$ are defined as above, then for any $S_j, 1 \leq j \leq n$, there exists a positive integer $l \geq j$ such that $S_{2,l} \geq S_j$.*

Proof. we assume the maximum value of S_j is obtained for $\mu_i = z_i, 1 \leq i \leq j$ and $1 \leq z_i < z_k \leq n, 1 \leq i < k \leq j$. Let $z_{j+1} = n + 1$ and $\{z'_1, \dots, z'_l\} = \{z_1, z_1 + 1, \dots, z_j, z_j + 1\}$, where $1 \leq z'_1 < \dots < z'_l \leq n + 1$. Thus we have $l \geq j$ and

$$\begin{aligned} & F_2(f, \mathbf{x}, \mathbf{p}; z'_1, \dots, z'_l) = F_2(f, \mathbf{x}, \mathbf{p}; z_1, z_1 + 1, \dots, z_j, z_j + 1) \\ & = \sum_{i=1}^j P^{z_{i+1}-1} f\left(\frac{1}{P^{z_{i+1}-1}} \sum_{k=z_i+1}^{z_{i+1}-1} p_i x_i\right) + \sum_{i=1}^j p_{z_i} f(x_{z_i}) - P_{z_1}^n f\left(\frac{1}{P_{z_1}^n} \sum_{i=z_1}^n p_i x_i\right). \end{aligned}$$

Next,

$$\begin{aligned} & F_2(f, \mathbf{x}, \mathbf{p}; z'_1, \dots, z'_l) - \left[\sum_{i=1}^j p_{z_i} f(x_{z_i}) - \left(\sum_{i=1}^j p_{z_i}\right) f\left(\frac{\sum_{i=1}^j P z_i x_{z_i}}{\sum_{i=1}^j P z_i}\right) \right] \\ & = \sum_{i=1}^j P^{z_{i+1}-1} f\left(\frac{1}{P^{z_{i+1}-1}} \sum_{k=z_i+1}^{z_{i+1}-1} p_i x_i\right) + \left(\sum_{i=1}^j p_{z_i}\right) f\left(\frac{\sum_{i=1}^j P z_i x_{z_i}}{\sum_{i=1}^j P z_i}\right) \\ & \quad - P_{z_1}^n f\left(\frac{1}{P_{z_1}^n} \sum_{i=z_1}^n p_i x_i\right). \end{aligned}$$

Observing $\sum_{i=1}^j P_{z_i+1}^{z_i+1} + \sum_{i=1}^j p_{z_i} = P_{z_1}^n$, or equivalently, $\sum_{i=1}^j P_{z_i+1}^{z_i+1} / P_{z_1}^n + \sum_{i=1}^j p_{z_i} / P_{z_1}^n = 1$, by using the Jensen's inequality we have

$$\begin{aligned} & \sum_{i=1}^j \frac{P_{z_i+1}^{z_i+1}}{P_{z_1}^n} f\left(\frac{1}{P_{z_i+1}^{z_i+1}} \sum_{k=z_i+1}^{z_i+1} p_i x_i\right) + \frac{\sum_{i=1}^j p_{z_i}}{P_{z_1}^n} f\left(\frac{\sum_{i=1}^j p_{z_i} x_{z_i}}{\sum_{i=1}^j p_{z_i}}\right) \\ & \geq f\left(\sum_{i=1}^j \frac{P_{z_i+1}^{z_i+1}}{P_{z_1}^n} \left(\frac{1}{P_{z_i+1}^{z_i+1}} \sum_{k=z_i+1}^{z_i+1} p_i x_i\right) + \frac{\sum_{i=1}^j p_{z_i}}{P_{z_1}^n} \left(\frac{\sum_{i=1}^j p_{z_i} x_{z_i}}{\sum_{i=1}^j p_{z_i}}\right)\right) = f\left(\frac{1}{P_{z_1}^n} \sum_{i=z_1}^n p_i x_i\right). \end{aligned}$$

This derives

$$F_2(f, \mathbf{x}, \mathbf{p}; z'_1, \dots, z'_l) - \left[\sum_{i=1}^j p_{z_i} f(x_{z_i}) - \left(\sum_{i=1}^j p_{z_i} \right) f\left(\frac{\sum_{i=1}^j p_{z_i} x_{z_i}}{\sum_{i=1}^j p_{z_i}}\right) \right] \geq 0,$$

or

$$F_2(f, \mathbf{x}, \mathbf{p}; z'_1, \dots, z'_l) \geq \sum_{i=1}^j p_{z_i} f(x_{z_i}) - \left(\sum_{i=1}^j p_{z_i} \right) f\left(\frac{\sum_{i=1}^j p_{z_i} x_{z_i}}{\sum_{i=1}^j p_{z_i}}\right).$$

From the definition, we have $S_{2,l} \geq F_1(f, \mathbf{x}, \mathbf{p}; z'_1, \dots, z'_l)$. Therefore, the inequality $S_{2,l} \geq S_j$ holds. \square

So compared with the previous inequalities in Lemma 1 and Lemma 4, we can see the new inequalities from Theorem 4 to Theorem 7 are better improved by the conclusions from Theorem 8 to Theorem 11.

4. Entropy upper bounds

As the consistent work we present new upper bounds for the Shannon's entropy.

THEOREM 12.

$$H(X) \leq - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \log \left[\left(1 - \sum_{i=1}^{n-1} p_{\mu_i} \right)^{1 - \sum_{i=1}^{n-1} p_{\mu_i}} \left(\prod_{i=1}^{n-1} p_{\mu_i}^{p_{\mu_i}} \right) \right]. \tag{25}$$

Proof. Applying Theorem 2 with $f(x) = -\log x$ and $x_i = 1/p_i$, $i = 1, 2, \dots, n$, after some calculations in the last part of the inequalities we obtain the assertion. \square

COROLLARY 1. *If $j \in \{2, 3, \dots, n-1\}$ is determined, then*

$$\begin{aligned} H(X) & \leq - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_j \leq n} \log \left[\left(\frac{1 - \sum_{i=1}^j p_{\mu_i}}{n - j} \right)^{1 - \sum_{i=1}^j p_{\mu_i}} \left(\prod_{i=1}^j p_{\mu_i}^{p_{\mu_i}} \right) \right] \\ & \leq \log n - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_j \leq n} \log \left[\left(\frac{j}{\sum_{i=1}^j p_{\mu_i}} \right)^{\sum_{i=1}^j p_{\mu_i}} \left(\prod_{i=1}^j p_{\mu_i}^{p_{\mu_i}} \right) \right]. \end{aligned}$$

Proof. Applying Theorem 3 with $f(x) = -\log x$, $x_i = 1/p_i$, $i = 1, 2, \dots, n$, we obtain the assertion. \square

When $j = n - 1$, compared with the previous upper bound in Lemma 5, the new upper bound in Theorem 4 is stronger.

COROLLARY 2. *If ϕ, ψ are defined as above, then*

$$\begin{aligned} H(X) &\leq -\log \left[\phi^\phi \psi^\psi \left(\frac{1 - \phi - \psi}{n - 2} \right)^{1 - \phi - \psi} \right] \\ &\leq \log n - \left[\phi \log \left(\frac{2\phi}{\phi + \psi} \right) + \psi \log \left(\frac{2\psi}{\phi + \psi} \right) \right]. \end{aligned}$$

Proof. Applying Theorem 3 with $f(x) = -\log x$, $x_i = 1/p_i$, $i = 1, 2, \dots, n$; $j = 2$, $p_{\mu_1} = \phi$, $p_{\mu_2} = \psi$, we obtain the assertion. \square

Compared with the previous upper bound in Lemma 2, the new upper bound in Corollary 2 is stronger.

Analogously, applying expressions (19), (20), (21), (22) with $f(x) = -\log x$, $x_i = 1/p_i$, $i = 1, 2, \dots, n$, we obtain the new upper bounds for the Shannon’s entropy as follows:

THEOREM 13.

$$H(X) \leq \log n - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \log \left[\left(\frac{\mu_{n-1}}{P_1^{\mu_{n-1}}} \right)^{P_1^{\mu_{n-1}}} \prod_{i=1}^{n-1} \left(\frac{P^{\mu_i}}{\mu_i - \mu_{i-1}} \right)^{P^{\mu_i}} \right], \tag{26}$$

where $\mu_0 = 0$.

REMARK 2. If $\mu_i = i$, $1 \leq i \leq n - 1$, then the expression (26) is exactly (6).

THEOREM 14.

$$H(X) \leq \log n - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \log \left[\left(\frac{n + 1 - \mu_1}{P_{\mu_1}^n} \right)^{P_{\mu_1}^n} \prod_{i=1}^{n-1} \left(\frac{P^{\mu_{i+1}-1}}{\mu_{i+1} - \mu_i} \right)^{P^{\mu_{i+1}-1}} \right], \tag{27}$$

where $\mu_n = n + 1$.

REMARK 3. If $\mu_i = i + 1$, $1 \leq i \leq n - 1$, then the expression (27) is exactly (6).

THEOREM 15.

$$H(X) \leq - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \log \left[\left(\frac{P^n}{n - \mu_{n-1}} \right)^{P^n} \prod_{i=1}^{n-1} \left(\frac{P^{\mu_i}}{\mu_i - \mu_{i-1}} \right)^{P^{\mu_i}} \right], \tag{28}$$

where $\mu_0 = 0$.

REMARK 4. If $\mu_i = i$, $1 \leq i \leq n-1$, then the expression (28) is exactly (25).

THEOREM 16.

$$H(X) \leq - \max_{1 \leq \mu_1 < \mu_2 < \dots < \mu_{n-1} \leq n} \log \left[\left(\frac{P_1^{\mu_1-1}}{\mu_1-1} \right)^{P_1^{\mu_1-1}} \prod_{i=1}^{n-1} \left(\frac{P_{\mu_i}^{\mu_{i+1}-1}}{\mu_{i+1}-\mu_i} \right)^{P_{\mu_i}^{\mu_{i+1}-1}} \right], \quad (29)$$

where $\mu_n = n+1$.

REMARK 5. If $\mu_i = i+1$, $1 \leq i \leq n-1$, then the expression (29) is exactly (25).

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Guoxiang Lu
School of Statistics and Mathematics
Zhongnan University of Economics and Law
No. 182 Nanhu Avenue, Wuhan 430073, China
e-mail: lgxmath@zuel.edu.cn