

PARTIAL SUMS OF MITTAG–LEFFLER FUNCTION

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Abstract. In the present investigation, Mittag-Leffler function with their normalization are considered. In this paper, we will study the ratio of a function of the form (1.4) to its sequence of partial sums $(\mathbb{E}_{\lambda,\mu})_n(z) = z + \sum_{k=1}^n \frac{\Gamma(\mu)}{\Gamma(\lambda k + \mu)} z^{k+1}$. We will determine lower bounds for $\Re \left\{ \frac{\mathbb{E}_{\lambda,\mu}(z)}{(\mathbb{E}_{\lambda,\mu})_n(z)} \right\}$, $\Re \left\{ \frac{(\mathbb{E}'_{\lambda,\mu})_n(z)}{\mathbb{E}'_{\lambda,\mu}(z)} \right\}$, $\Re \left\{ \frac{\mathbb{E}'_{\lambda,\mu}(z)}{(\mathbb{E}'_{\lambda,\mu})_n(z)} \right\}$ and $\Re \left\{ \frac{(\mathbb{E}'_{\lambda,\mu})_n(z)}{\mathbb{E}'_{\lambda,\mu}(z)} \right\}$. Results obtained are new and their usefulness are depicted by deducing several interesting examples.

1. Introduction

Let \mathcal{A} denote the family of all functions f that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \quad (1.1)$$

Recently, several researchers studied families of analytic functions involving special functions $\mathcal{F} \subset \mathcal{A}$, to find different conditions such that the members of \mathcal{F} have certain geometric properties like univalence, starlikeness or convexity in \mathbb{D} . In this context many results are available in the literature regarding the hypergeometric functions [11, 19, 16, 15], Bessel functions [2, 3, 4, 5, 17], Wright function [18] and Mittag-Leffler functions [1].

The function $E_\lambda(z)$ defined by

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)} \quad (z \in \mathbb{C}, \Re(\lambda) > 0), \quad (1.2)$$

was introduced by Mittag-Leffler [12] and is, therefore, known as the Mittag-Leffler function. A more general function $E_{\lambda,\mu}(z)$, generalizing $E_\lambda(z)$, is defined by

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)} \quad (z, \lambda, \mu \in \mathbb{C}; \Re(\lambda) > 0). \quad (1.3)$$

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This function, sometime called, a Mittag-Leffler type function, first appeared in a paper by Wiman [24, 25]. The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, superdiffusive transport and in the study of complex systems. These functions interpolate between a purely exponential law and power-law like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts [8, 9, 7, 20]. The most essential properties of these entire functions, investigated by many mathematicians, can be found in [6].

Observe that, Mittag-Leffler function $E_{\lambda,\mu}$ does not belong to the family \mathcal{A} . Thus, it is natural to consider the following normalization of Mittag-Leffler function:

$$\begin{aligned} \mathbb{E}_{\lambda,\mu}(z) &= \Gamma(\mu)zE_{\lambda,\mu}(z) \\ &:= \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} z^{n+1} \quad (z, \lambda, \mu \in \mathbb{C}; \Re(\lambda) > 0, \mu \neq 0, -1, \dots). \end{aligned} \tag{1.4}$$

Whilst formula (1.4) holds for complex-valued λ, μ and $z \in \mathbb{C}$, however in this paper we shall restrict our attention to the case of real-valued λ, μ and $z \in \mathbb{D}$. Observe that, the function $\mathbb{E}_{\lambda,\mu}$ contains many well known functions as its special case, for example

$$\left\{ \begin{aligned} \mathbb{E}_{0,1}(z) &= \frac{z}{1-z}, & \mathbb{E}_{1,1}(z) &= ze^z, & \mathbb{E}_{2,1}(z) &= z \cosh(\sqrt{z}), \\ \mathbb{E}_{1,2}(z) &= e^z - 1, & \mathbb{E}_{1,3}(z) &= \frac{2(e^z - z - 1)}{z}, \\ \mathbb{E}_{1,4}(z) &= \frac{6(e^z - 1 - z) - 3z^2}{z^2}, & \mathbb{E}_{2,2}(z) &= \sqrt{z} \sinh(\sqrt{z}), \\ \mathbb{E}_{3,1}(z) &= \frac{z}{2} \left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right) \right]. \end{aligned} \right. \tag{1.5}$$

If f, g are analytic functions in \mathbb{D} , then f is said to be subordinate to g , written as $f(z) \prec g(z)$ ($z \in \mathbb{D}$), if there exists an analytic function w with $w(0) = 0$ and $|w(z)| \leq 1$ ($z \in \mathbb{D}$) such that $f(z) = g(w(z))$. In particular, if g is univalent in \mathbb{D} , then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

For more details one can refer [10].

In this paper, we will study the ratio of a function of the form (1.4) to its sequence of partial sums

$$(\mathbb{E}_{\lambda,\mu})_n(z) = z + \sum_{k=1}^n \frac{\Gamma(\mu)}{\Gamma(\lambda k + \mu)} z^{k+1} \tag{1.6}$$

$$(\mathbb{E}_{\lambda,\mu})_0(z) = z. \tag{1.7}$$

We will determine lower bounds for $\Re \left\{ \frac{\mathbb{E}_{\lambda,\mu}(z)}{(\mathbb{E}_{\lambda,\mu})_n(z)} \right\}$, $\Re \left\{ \frac{(\mathbb{E}_{\lambda,\mu})_n(z)}{\mathbb{E}_{\lambda,\mu}(z)} \right\}$, $\Re \left\{ \frac{\mathbb{E}'_{\lambda,\mu}(z)}{(\mathbb{E}'_{\lambda,\mu})_n(z)} \right\}$ and $\Re \left\{ \frac{(\mathbb{E}'_{\lambda,\mu})_n(z)}{\mathbb{E}'_{\lambda,\mu}(z)} \right\}$.

For various known results concerning with partial sums of analytic univalent functions one can refer the works of Owa et. al [14], Sheil-Small [21], Silverman [22], Silvia [23] and Orhan et. al [13].

2. Main results

To prove main results we need following Lemma:

LEMMA 2.1. *If $\lambda \geq 1$ and $\mu \geq 1$ then the function $\mathbb{E}_{\lambda,\mu} : \mathbb{D} \rightarrow \mathbb{C}$ given by (1.4), satisfies the following inequalities:*

$$|\mathbb{E}_{\lambda,\mu}(z)| \leq \frac{\mu^2 + \mu + 1}{\mu^2} \tag{2.1}$$

and

$$|\mathbb{E}'_{\lambda,\mu}(z)| \leq \frac{\mu^3 + 2\mu^2 + 3\mu + 1}{\mu^3}. \tag{2.2}$$

Proof. Under the hypothesis, it is easy to check that $\Gamma(n+\mu) \leq \Gamma(\lambda n+\mu)$, $n \in \mathbb{N}$, which is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \leq \frac{\Gamma(\mu)}{\Gamma(n+\mu)} = \frac{1}{\mu(\mu+1)(\mu+2)\cdots(\mu+n-1)}, \quad n \in \mathbb{N}. \tag{2.3}$$

Using this inequality, we get

$$\begin{aligned} |\mathbb{E}_{\lambda,\mu}(z)| &\leq |z| + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} |z|^{n+1} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(n+\mu)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{\mu(\mu+1)\cdots(\mu+n-1)} \\ &\leq 1 + \frac{1}{\mu} \sum_{n=0}^{\infty} \frac{1}{(\mu+1)^n} \\ &= \frac{\mu^2 + \mu + 1}{\mu^2}. \end{aligned}$$

Similarly

$$\begin{aligned}
 |\mathbb{E}'_{\lambda, \mu}(z)| &\leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)\Gamma(\mu)}{\Gamma(\lambda n + \mu)} |z|^n \leq 1 + \sum_{n=1}^{\infty} \frac{n\Gamma(\mu)}{\Gamma(\lambda n + \mu)} + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \\
 &\leq 1 + \sum_{n=1}^{\infty} \frac{n}{\mu(\mu+1)\cdots(\mu+n-1)} + \sum_{n=1}^{\infty} \frac{1}{\mu(\mu+1)\cdots(\mu+n-1)} \\
 &\leq 1 + \frac{1}{\mu} \left[1 + \frac{2}{(\mu+1)} + \frac{3}{(\mu+1)^2} + \dots \right] + \frac{1}{\mu} \sum_{n=0}^{\infty} \frac{1}{(\mu+1)^n} \\
 &= 1 + \frac{1}{\mu} \left[\frac{(\mu+1)}{\mu} \right]^2 + \frac{\mu+1}{\mu^2} = \frac{\mu^3 + 2\mu^2 + 3\mu + 1}{\mu^3}. \quad \square
 \end{aligned}$$

THEOREM 2.1. *If $\lambda \geq 1$, $\mu \geq 1$ and $\mu^2 - \mu - 1 \geq 0$, then*

$$\Re \left\{ \frac{\mathbb{E}_{\lambda, \mu}(z)}{(\mathbb{E}_{\lambda, \mu})_n(z)} \right\} \geq \frac{\mu^2 - \mu - 1}{\mu^2} \quad (z \in \mathbb{D}), \tag{2.4}$$

and

$$\Re \left\{ \frac{(\mathbb{E}_{\lambda, \mu})_n(z)}{\mathbb{E}_{\lambda, \mu}(z)} \right\} \geq \frac{\mu^2}{\mu^2 + \mu + 1} \quad (z \in \mathbb{D}). \tag{2.5}$$

Proof. It is easy to see from (2.1) of Lemma 2.1 that

$$1 + \sum_{k=1}^{\infty} b_k \leq \frac{\mu^2 + \mu + 1}{\mu^2}$$

which is equivalent to

$$\frac{\mu^2}{\mu+1} \sum_{k=1}^{\infty} b_k \leq 1 \quad \left(\text{where } b_k = \frac{\Gamma(\mu)}{\Gamma(\lambda k + \mu)} \right). \tag{2.6}$$

To prove 2.4, we have to show that

$$\frac{\mu^2}{\mu+1} \left[\frac{\mathbb{E}_{\lambda, \mu}(z)}{(\mathbb{E}_{\lambda, \mu})_n(z)} - \frac{\mu^2 - \mu - 1}{\mu^2} \right] \prec \frac{1+z}{1-z}. \tag{2.7}$$

Using definition of subordination, and putting the values of $\mathbb{E}_{\lambda, \mu}$ and $(\mathbb{E}_{\lambda, \mu})_n$, we have

$$\frac{1 + \sum_{k=1}^n b_k z^k + \frac{\mu^2}{\mu+1} \sum_{k=n+1}^{\infty} b_k z^k}{1 + \sum_{k=1}^n b_k z^k} = \frac{1 + w(z)}{1 - w(z)}.$$

Our assertion 2.4 is true if we show that $w(0) = 0$ and $|w(z)| < 1$ provided $z \in \mathbb{D}$. Simplifying for $w(z)$, we get

$$w(z) = \frac{\frac{\mu^2}{\mu+1} \sum_{k=n+1}^{\infty} b_k z^k}{2 + 2 \sum_{k=1}^n b_k z^k + \frac{\mu^2}{\mu+1} \sum_{k=n+1}^{\infty} b_k z^k}.$$

Obviously $w(0) = 0$ and

$$|w(z)| \leq \frac{\frac{\mu^2}{\mu+1} \sum_{k=n+1}^{\infty} b_k}{2 - 2 \sum_{k=1}^n b_k - \frac{\mu^2}{\mu+1} \sum_{k=n+1}^{\infty} b_k} \leq 1$$

provided

$$\sum_{k=1}^n b_k + \frac{\mu^2}{\mu+1} \sum_{k=n+1}^{\infty} b_k \leq 1. \tag{2.8}$$

It suffices to show that the left hand side of (2.8) is bounded above by left hand side of (2.6), which is equivalent to

$$\left(\frac{\mu^2}{\mu+1} - 1 \right) \sum_{k=1}^n b_k \geq 0.$$

This is true as $\mu^2 - \mu - 1 \geq 0$.

To prove the result (2.5), we write

$$\frac{\mu^2 + \mu + 1}{\mu + 1} \left[\frac{(\mathbb{E}_{\lambda, \mu}(z))_n}{\mathbb{E}_{\lambda, \mu}(z)} - \frac{\mu^2}{\mu^2 + \mu + 1} \right] = \frac{1 + w(z)}{1 - w(z)}.$$

Substituting the values of $\mathbb{E}_{\lambda, \mu}$ and $(\mathbb{E}_{\lambda, \mu}(z))_n$ and simplifying for $w(z)$, we have

$$w(z) = \frac{-\frac{\mu^2 + \mu + 1}{\mu + 1} \sum_{k=n+1}^{\infty} b_k z^k}{2 + 2 \sum_{k=1}^n b_k z^k - \frac{\mu^2 - \mu - 1}{\mu + 1} \sum_{k=n+1}^{\infty} b_k z^k}.$$

Obviously $w(0) = 0$ and

$$|w(z)| \leq \frac{\frac{\mu^2 + \mu + 1}{\mu + 1} \sum_{k=n+1}^{\infty} b_k}{2 - 2 \sum_{k=1}^n b_k - \frac{\mu^2 - \mu - 1}{\mu + 1} \sum_{k=n+1}^{\infty} b_k} \leq 1 \tag{2.9}$$

as (2.8) is true for $\mu^2 - \mu - 1 \geq 0$. \square

THEOREM 2.2. *If $\lambda \geq 1$ and $\mu \geq 1$ with $\mu^3 - 2\mu^2 - 3\mu - 1 \geq 0$, then*

$$\Re \left\{ \frac{\mathbb{E}'_{\lambda, \mu}(z)}{(\mathbb{E}'_{\lambda, \mu})_n(z)} \right\} \geq \frac{\mu^3 - 2\mu^2 - 3\mu - 1}{\mu^3} \quad (z \in \mathbb{D}), \tag{2.10}$$

and

$$\Re \left\{ \frac{(\mathbb{E}'_{\lambda, \mu})_n(z)}{\mathbb{E}'_{\lambda, \mu}(z)} \right\} \geq \frac{\mu^3}{\mu^3 + 2\mu^2 + 3\mu + 1} \quad (z \in \mathbb{D}). \tag{2.11}$$

Proof. It is easy to see from (2.2) of Lemma 2.1 that

$$1 + \sum_{k=1}^{\infty} b_k(k+1) \leq \frac{\mu^3 + 2\mu^2 + 3\mu + 1}{\mu^3}$$

which is equivalent to

$$\frac{\mu^3}{2\mu^2 + 3\mu + 1} \sum_{k=1}^{\infty} b_k(k+1) \leq 1 \left(\text{where } b_k = \frac{\Gamma(\mu)}{\Gamma(\lambda k + \mu)} \right). \tag{2.12}$$

To prove (2.10), we have to show that

$$\frac{\mu^3}{2\mu^2 + 3\mu + 1} \left[\frac{\mathbb{E}'_{\lambda, \mu}(z)}{(\mathbb{E}'_{\lambda, \mu}(z))_n} - \frac{\mu^3 - 2\mu^2 - 3\mu - 1}{\mu^3} \right] < \frac{1+z}{1-z}. \tag{2.13}$$

Using definition of subordination, and putting the values of $\mathbb{E}_{\lambda, \mu}$ and $(\mathbb{E}_{\lambda, \mu}(z))_n$, we have

$$\frac{1 + \sum_{k=1}^n b_k(k+1)z^k + \frac{\mu^3}{2\mu^2+3\mu+1} \sum_{k=n+1}^{\infty} (k+1)b_k z^k}{1 + \sum_{k=1}^n (k+1)b_k z^k} = \frac{1+w(z)}{1-w(z)}.$$

Our assertion (2.4) is true if we show that $w(0) = 0$ and $|w(z)| < 1$ provided $z \in \mathbb{D}$. Simplifying for $w(z)$, we get

$$w(z) = \frac{\frac{\mu^3}{2\mu^2+3\mu+1} \sum_{k=n+1}^{\infty} (k+1)b_k z^k}{2 + 2 \sum_{k=1}^n (k+1)b_k z^k + \frac{\mu^3}{2\mu^2+3\mu+1} \sum_{k=n+1}^{\infty} (k+1)b_k z^k}$$

Obviously $w(0) = 0$ and

$$|w(z)| \leq \frac{\frac{\mu^3}{2\mu^2+3\mu+1} \sum_{k=n+1}^{\infty} (k+1)b_k}{2 - 2 \sum_{k=1}^n (k+1)b_k - \frac{\mu^3}{2\mu^2+3\mu+1} \sum_{k=n+1}^{\infty} (k+1)b_k} \leq 1$$

provided

$$\sum_{k=1}^n (k+1)b_k + \frac{\mu^3}{2\mu^2 + 3\mu + 1} \sum_{k=n+1}^{\infty} (k+1)b_k \leq 1. \tag{2.14}$$

It suffices to show that the left hand side of (2.14) is bounded above by left hand side of (2.12), which is equivalent to

$$\left(\frac{\mu^3}{2\mu^2 + 3\mu + 1} \right) \sum_{k=1}^n b_k \geq 0.$$

This is true in view of hypothesis.

To prove the result (2.11), we write

$$\frac{\mu^3 + 2\mu^2 + 3\mu + 1}{2\mu^2 + 3\mu + 1} \left[\frac{(\mathbb{E}'_{\lambda, \mu}(z))_n}{\mathbb{E}'_{\lambda, \mu}(z)} - \frac{\mu^3}{\mu^3 + 2\mu^2 + 3\mu + 1} \right] = \frac{1 + w(z)}{1 - w(z)}.$$

Substituting the values of $\mathbb{E}'_{\lambda, \mu}$ and $(\mathbb{E}'_{\lambda, \mu}(z))_n$ and simplifying for $w(z)$, we have

$$w(z) = \frac{-\frac{\mu^3 + 2\mu^2 + 3\mu + 1}{2\mu^2 + 3\mu + 1} \sum_{k=n+1}^{\infty} (k+1)b_k z^k}{2 + 2 \sum_{k=1}^n (k+1)b_k z^k + \left(1 - \frac{\mu^3}{2\mu^2 + 3\mu + 1}\right) \sum_{k=n+1}^{\infty} (k+1)b_k z^k}.$$

Obviously $w(0) = 0$ and

$$|w(z)| \leq \frac{\frac{\mu^3 + 2\mu^2 + 3\mu + 1}{2\mu^2 + 3\mu + 1} \sum_{k=n+1}^{\infty} (k+1)b_k}{2 - 2 \sum_{k=1}^n (k+1)b_k - \frac{\mu^3 - 2\mu^2 - 3\mu - 1}{2\mu^2 + 3\mu + 1} \sum_{k=n+1}^{\infty} (k+1)b_k} \leq 1 \tag{2.15}$$

as (2.14) is true under the hypothesis. \square

3. Application

In view of (1.1) and Theorem 2.1 and Theorem 2.2, we get the following inequalities:

- (1) Choosing $\lambda = 1, \mu = 2$ and $n = 0$, the hypothesis of Theorem 2.1 is satisfied but hypothesis of Theorem 2.2 is not satisfied, thus we get the following results

$$\Re \left\{ \frac{e^z - 1}{z} \right\} \geq \frac{1}{4} \quad (z \in \mathbb{D}), \tag{3.1}$$

$$\Re \left\{ \frac{z}{e^z - 1} \right\} \geq \frac{4}{7} \quad (z \in \mathbb{D}). \tag{3.2}$$

- (2) Choosing $\lambda = 2, \mu = 2$ and $n = 0$, the hypothesis of Theorem 2.1 is satisfied but hypothesis of Theorem 2.2 is not satisfied, thus we get the following results

$$\Re \left\{ \frac{\sinh \sqrt{z}}{\sqrt{z}} \right\} \geq \frac{1}{4} \quad (z \in \mathbb{D}), \tag{3.3}$$

$$\Re \left\{ \frac{\sqrt{z}}{\sinh \sqrt{z}} \right\} \geq \frac{4}{7} \tag{3.4}$$

One can find out more examples in view of (1.1). The following are graphs of some of the functions discusses above. These figures depicts validity of our results.

i We have the image domains of $f_1(z) = \frac{e^z-1}{z}$, $f_2(z) = \frac{z}{e^z-1}$, in Figure 1.

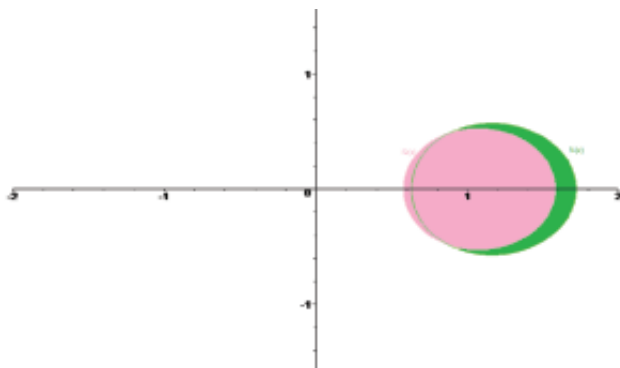


Figure 1.

ii We have the image domains of $f_3(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}$, $f_4(z) = \frac{\sqrt{z}}{\sinh(\sqrt{z})}$, in Figure 2.

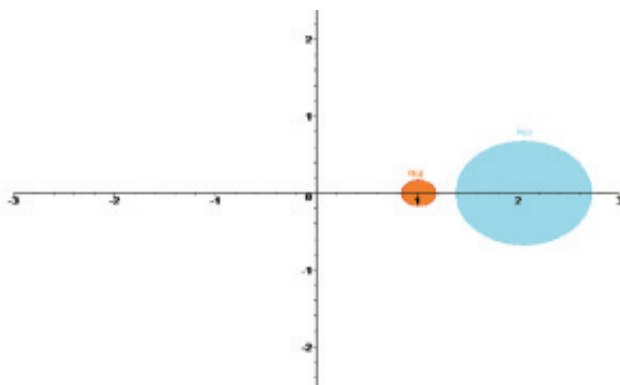


Figure 2.

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