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GENERAL THEOREMS ON EXPONENTIAL AND ROSENTHAL'S INEQUALITIES AND ON COMPLETE CONVERGENCE

ISTVÁN FAZEKAS, SÁNDOR PECSORA AND BETTINA PORVÁZSNYIK

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Abstract. Exponential inequalities are obtained under general conditions. Then it is shown that an exponential inequality implies both Rosenthal's inequality and complete convergence of sums of random variables. The general results are applied to weakly orthant dependent sequences.

1. Introduction

Analysing the proofs of asymptotic results for independent random variables (r.v.'s) one can see that certain exponential relations play fundamental role. Similar observation is true for some weakly dependent sequences of r.v.'s. It leads to the following definition. The r.v.'s $X_1, X_2, \ldots, X_n$ are called acceptable if

$$\mathbb{E}e^{\sum_{i=1}^{n} \lambda X_i} \leqslant \prod_{i=1}^{n} \mathbb{E}e^{\lambda X_i} \tag{1.1}$$

for any real number λ, see [1]. Then exponential inequalities and complete convergence theorems were obtained for acceptable sequences, furthermore some versions of the notion of acceptability were introduced, see e.g. [22] and [28].

In this paper we shall show that an appropriate version of inequality (1.1) implies an exponential inequality. Then the exponential inequality implies a Rosenthal's inequality. Moreover, the exponential inequality implies immediately complete convergence. We emphasize that to obtain the above results no additional dependence conditions are needed. Then our general theorems will be applied to weakly orthant dependent sequences.

Our main aim is to obtain general theorems and to point out the general features of methods of proofs. The same approach inspired the paper [8] where a general method was described to obtain Strong Laws of Large Numbers (SLLN). In that paper it was proved that a Hájek-Rényi type maximal inequality is always a consequence of an appropriate Kolmogorov type maximal inequality. Moreover, the Hájek-Rényi type maximal inequality immediately implies the SLLN. The benefit of that result was that no restriction was assumed on the dependence structure of the random variables. Then the

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results of [8] were widely applied for dependent r.v.'s (see the list in the review paper [7]).

In Section 2 of this paper exponential inequalities are studied. It is well-known that exponential inequalities played important role to obtain asymptotic results for sums independent r.v.'s. Classical exponential inequalities were obtained e.g. by Bernstein, Kolmogorov, Fuk and Nagaev (see the monographs [20], [23] and the paper [11]). Exponential bounds are used to prove the law of the iterated logarithm (see [14], [23]). Exponential inequalities were obtained also for dependent r.v.'s, e.g., in [4] for negatively associated, in [12] for negatively orthant dependent, in [29] for extended negatively orthant dependent r.v.'s. In [22] exponential inequalities were obtained for acceptable random variables and using them complete convergence results were proved. In our paper we shall assume the acceptability of the truncated r.v.'s, therefore our Theorem 2.1 and the results in [22] are different.

Section 3 is devoted to Rosenthal's inequality. Rosenthal's inequality plays an important role in the theory of independent r.v.'s. There are several methods to prove it (see, e.g., [14], [20], [19]). Rosenthal's inequality is true under weak dependence conditions like mixing (e.g. [9]) or negative dependence (e.g. [12], [30]). In this paper we shall prove that a general exponential inequality implies a Rosenthal inequality (Theorem 3.1).

In Section 4 complete convergence is studied. For independent identically distributed r.v.'s Kolmogorov's SLLN is one of the most important results of probability theory. The complete convergence in Kolmogorov's SLLN was proven by [16] and [5]. More general rate of convergence was obtained by Baum and Katz in [2]. The classical results of Baum and Katz were extended to arrays of independent dominated r.v.'s by [17]. The weakly mean dominated case was considered by Gut in [13]. Gut's results were extended to vector valued case in [6].

There is a vast literature of extensions of Baum-Katz type results to dependent random variables e.g. in [21] α-mixing, in [29] and in [30] extended negatively dependent, in [27] widely orthant dependent r.v.'s were considered.

In Section 4 of this paper, we want to give some general conditions for Baum-Katz type results. We show that an exponential inequality for the truncated and centered r.v.'s implies the Baum-Katz type convergence rate (Theorem 4.1). We prove our result for sequences of weakly mean dominated r.v.'s. We underline that we do not assume any restriction on the dependence structure of the r.v.'s.

In Section 5 widely orthant dependent (WOD) sequences are considered. This notion was introduced in [25]. It is known that extended negatively orthant dependent sequences, negatively orthant dependent sequences, negatively superadditive dependent sequences, negatively associated and independent sequences are WOD, see [27]. In Section 5, first we compare the notions of WOD sequence and widely acceptable sequence. It is known that a WOD sequence satisfies equation (2.1) so it is widely acceptable. On the other hand, our Example 5.1 shows that the class of widely acceptable r.v.'s is larger than the class of WOD r.v.'s. Therefore our results on widely acceptable sequences are more general than former results on WOD sequences. In the remaining part of Section 5 consequences of our general results are listed. Therefore we obtain exponential inequality (Theorem 5.1), Hoeffding's inequality (Theorem 5.2), Rosenthal's inequality (Theorem 5.3) and complete convergence (Theorem 5.4) for WOD sequences.

2. Exponential inequalities

Let $d>0$ be a real number, and let ξ be a random variable (r.v.). Throughout the paper

$$\xi^{(d)} = \min\{\xi, d\}$$

will denote the r.v. truncated from above.

For a sequence of r.v.'s $\eta_1, \eta_2, \dots, \eta_n$ we shall consider the condition

$$\mathbb{E}e^{\sum_{i=1}^n \lambda \eta_i} \leqslant g(n) \prod_{i=1}^n \mathbb{E}e^{\lambda \eta_i} \tag{2.1}$$

where $0<g(n)<\infty$. If condition (2.1) is satisfied for all $\lambda \in \mathbb{R}$ and for all n, then the sequence $\eta_1, \eta_2, \dots$ is called widely acceptable, see [28]. If $g(n)\equiv 1$, then we are at the notion of acceptable r.v.'s ([1], [22]). If (2.1) is true for $\eta_1, \eta_2, \dots, \eta_n$, then it remains true for $\eta_1 - a_1, \eta_2 - a_2, \dots, \eta_n - a_n$ for any real numbers $a_1, \dots, a_n$, in particular it remains true for $\eta_1 - \mathbb{E}\eta_1, \eta_2 - \mathbb{E}\eta_2, \dots, \eta_n - \mathbb{E}\eta_n$. If we assume condition (2.1) for positive values of λ and for the appropriately truncated r.v's, then we shall obtain a one-sided exponential inequality. If we assume condition (2.1) both for positive and negative values of λ, then shall we obtain a two-sided exponential inequality.

The following type exponential inequality was obtained for negatively orthant dependent r.v.'s in Lemma 3 of [12] and for extended negatively dependent r.v.'s in Lemma 1.2 of [29].

THEOREM 2.1. *Let $X_1, X_2, \dots, X_n$ be a sequence of zero mean r.v.'s, $d>0$. Let $S_n = \sum_{i=1}^n X_i$ be the sum and $B_n = \sum_{i=1}^n \mathbb{E}X_i^2$ be the sum of variances.*

Assume that (2.1) is satisfied for $\eta_i = X_i^{(d)}$, $i=1,2,\dots,n$ and $0<\lambda\leqslant\lambda_0$. Then for any x with $0<x\leqslant (B_n e^{d\lambda_0} - B_n)/d$, we have

$$\mathbb{P}(S_n > x) \leqslant \mathbb{P}\left(\max_{1\leqslant i\leqslant n} X_i > d\right) + g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1+\frac{xd}{B_n}\right)\right). \tag{2.2}$$

If (2.1) is satisfied both for $\eta_i = X_i^{(d)}$ $i=1,2,\dots,n$, and $\eta_i = (-X_i)^{(d)}$, $i = 1,2,\dots,n$ and $0<\lambda\leqslant\lambda_0$, then for any x with $0<x\leqslant (B_n e^{d\lambda_0} - B_n)/d$ we have

$$\mathbb{P}(|S_n| > x) \leqslant \mathbb{P}\left(\max_{1\leqslant i\leqslant n} |X_i| > d\right) + 2g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1+\frac{xd}{B_n}\right)\right). \tag{2.3}$$

Proof. The proof follows the classical ideas of [11] (see also [20]). The same method was applied for the proofs of Lemma 3 in [12] and Lemma 1.2 in [29]. First we prove (2.2). Let $\eta_i = X_i^{(d)}$, $i=1,2,\dots,n$ denote the truncated r.v.'s. As $\mathbb{E}X_i = 0$, so

$\mathbb{E}\eta_i \leqslant 0$. Let $F_i(x) = \mathbb{P}(X_i < x)$ be the distribution function of X_i. Algebraic calculation and $\mathbb{E}\eta_i \leqslant 0$ gives

$$\begin{aligned}
\mathbb{E}e^{\lambda\eta_i} &= \int_{-\infty}^{d} e^{\lambda x}\mathrm{d}F_i(x) + e^{\lambda d}\mathbb{P}(X_i \geqslant d) \\
&= \int_{-\infty}^{d} (e^{\lambda x} - 1 - \lambda x)\mathrm{d}F_i(x) + (e^{\lambda d} - 1 - \lambda d)\mathbb{P}(X_i \geqslant d) + 1 + \lambda\mathbb{E}\eta_i \\
&\leqslant \int_{-\infty}^{d} (e^{\lambda x} - 1 - \lambda x)\mathrm{d}F_i(x) + (e^{\lambda d} - 1 - \lambda d)\mathbb{P}(X_i \geqslant d) + 1 \\
&\leqslant 1 + \frac{e^{\lambda d} - 1 - \lambda d}{d^2}\left(\int_{-\infty}^{d} x^2\mathrm{d}F_i(x) + d^2\mathbb{P}(X_i \geqslant d)\right) \\
&\leqslant 1 + \frac{e^{\lambda d} - 1 - \lambda d}{d^2}\mathbb{E}X_i^2 \leqslant \exp\left(\frac{e^{\lambda d} - 1 - \lambda d}{d^2}\mathbb{E}X_i^2\right). \qquad (2.4)
\end{aligned}$$

Above we applied that the function $f(t) = (e^{\lambda t} - 1 - \lambda t)/t^2$ in monotone increasing, $\mathbb{E}\eta_i^2 \leqslant \mathbb{E}X_i^2$ and $1 + t \leqslant e^t$. Now, by (2.1) and (2.4),

$$e^{-\lambda x}\mathbb{E}e^{\sum_{i=1}^{n}\lambda\eta_i} \leqslant g(n)e^{-\lambda x}\prod_{i=1}^{n}\mathbb{E}e^{\lambda\eta_i} \leqslant g(n)\exp\left(-\lambda x + \frac{e^{\lambda d} - 1 - \lambda d}{d^2}B_n\right). \qquad (2.5)$$

The minimum of the function $f(\lambda) = -\lambda x + \frac{e^{\lambda d}-1-\lambda d}{d^2}B_n$ is at $\lambda = \left(\ln\left(1 + \frac{xd}{B_n}\right)\right)/d$. We can choose this value of λ because $0 < \lambda \leqslant \lambda_0$ is satisfied in view of condition $0 < x \leqslant (B_n e^{d\lambda_0} - B_n)/d$. Therefore the application of (2.1) is allowed for this λ. Moreover, we see that for this value of λ we have $\frac{e^{\lambda d}-1-\lambda d}{d^2}B_n \leqslant \frac{x}{d}$. Therefore (2.5) implies

$$e^{-\lambda x}\mathbb{E}e^{\sum_{i=1}^{n}\lambda\eta_i} \leqslant g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd}{B_n}\right)\right). \qquad (2.6)$$

By Markov's inequality and (2.6)

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} X_i^{(d)} > x\right) &= \mathbb{P}\left(e^{\lambda\sum_{i=1}^{n}\eta_i} > e^{\lambda x}\right) \leqslant e^{-\lambda x}\mathbb{E}e^{\sum_{i=1}^{n}\lambda\eta_i} \qquad (2.7) \\
&\leqslant g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd}{B_n}\right)\right).
\end{aligned}$$

Now

$$\mathbb{P}(S_n > x) \leqslant \mathbb{P}\left(\max_{1\leqslant i\leqslant n} X_i > d\right) + \mathbb{P}\left(\sum_{i=1}^{n} X_i^{(d)} > x\right),$$

therefore an application of (2.7) gives (2.2).

Now we turn to the proof of (2.3). If (2.1) is true for $\eta_i = (-X_i)^{(d)}$, $i = 1, 2, \ldots, n$ and $0 < \lambda \leqslant \lambda_0$, then (2.7) is true for the r.v.'s $-X_1, -X_2, \ldots, -X_n$, too. Applying (2.7) both for the r.v.'s $X_1, X_2, \dots, X_n$ and the r.v.'s $-X_1, -X_2, \dots, -X_n$, we get

$$\mathbb{P}(|S_n| > x) \leqslant \mathbb{P}\left(\max_{1\leqslant i\leqslant n} |X_i| > d\right) + \mathbb{P}\left(\sum_{i=1}^n X_i^{(d)} > x\right) + \mathbb{P}\left(\sum_{i=1}^n (-X_i)^{(d)} > x\right)$$
$$\leqslant \mathbb{P}\left(\max_{1\leqslant i\leqslant n} |X_i| > d\right) + 2g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd}{B_n}\right)\right),$$

so we obtain (2.3). $\square$

Now we turn to Hoeffding's inequality. It was obtained for independent random variables in [15]. Then it was extended to certain depended sequences. Our next theorem is a version of Theorem 2.3 in [22], where acceptable r.v.'s were considered.

THEOREM 2.2. *Let $X_1, X_2, \dots, X_n$ be a sequence of r.v.'s. Let $S_n = \sum_{i=1}^n X_i$ be the sum. Let the random variables be bounded, i.e. $a_i \leqslant X_i \leqslant b_i$ for $i = 1, 2, \dots, n$, where a_i and b_i are real numbers. Assume that (2.1) is satisfied with $\eta_i = X_i$, $i = 1, 2, \dots, n$ and $0 < \lambda \leqslant \lambda_0$. Then for any ε with $0 < \varepsilon \leqslant \frac{\lambda_0}{4}\sum_{i=1}^n (b_i - a_i)^2$, we have*

$$\mathbb{P}(S_n - \mathbb{E}S_n \geqslant \varepsilon) \leqslant g(n)\exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \tag{2.8}$$

Assume that (2.1) is satisfied for $\eta_i = X_i$, $i = 1, 2, \dots, n$ and $|\lambda| \leqslant \lambda_0$. Then for any ε with $0 < \varepsilon \leqslant \frac{\lambda_0}{4}\sum_{i=1}^n (b_i - a_i)^2$, we have

$$\mathbb{P}(|S_n - \mathbb{E}S_n| \geqslant \varepsilon) \leqslant 2g(n)\exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \tag{2.9}$$

Proof. We can follow the original proof given for independent r.v.'s (see [15], Theorem 2). For acceptable r.v.'s an appropriate version of the original proof is described in [22] (see the proof of Theorem 2.3 in [22]). $\square$

Now we turn to the maximal version of Hoeffding's inequality. We remark that in [22] the maximal Hoeffding's inequality for acceptable r.v.'s was not considered.

COROLLARY 2.1. *Let $X_1, X_2, \dots, X_n$ be a sequence of r.v.'s. Assume that the random variables are bounded, i.e. $a_i \leqslant X_i \leqslant b_i$ for $i = 1, 2, \dots, n$, where a_i and b_i are real numbers. For k, l with $1 \leqslant k \leqslant l \leqslant n$ denote by $M_{k,l}$ the following maximum*

$$M_{k,l} = \max_{k\leqslant j\leqslant l}\left|\sum_{t=k}^{j}(X_t - \mathbb{E}X_t)\right|. \tag{2.10}$$

Assume that

$$\mathbb{E}e^{\sum_{i=k}^l \lambda X_i} \leqslant C\prod_{i=k}^l \mathbb{E}e^{\lambda X_i} \tag{2.11}$$

for any $1 \leqslant k < l \leqslant n$ and any $\lambda \in \mathbb{R}$. Let $\varepsilon > 0$. Then for any $0 < \delta < 1$ there exists a $C_1 = C_1(\delta)$ such that

$$\mathbb{P}\left(M_{k,l} \geqslant \varepsilon\right) \leqslant 2CC_1\exp\left(-\frac{2\varepsilon^2(1-\delta)}{\sum_{i=k}^l (b_i - a_i)^2}\right) \tag{2.12}$$

for any $1 \leqslant k \leqslant l \leqslant n$.

Proof. By (2.9) we have

$$\mathbb{P}\left(\left|\sum_{t=k}^{l}(X_t-\mathbb{E}X_t)\right|\geqslant\varepsilon\right)\leqslant 2C\exp\left(-\frac{2\varepsilon^2}{\sum_{i=k}^{l}(b_i-a_i)^2}\right) \tag{2.13}$$

for any $1\leqslant k<l\leqslant n$. Here $g(k,l)=\sum_{i=k}^{l}(b_i-a_i)^2$ is a superadditive function. Therefore Theorem 1 of [18] implies the desired result. □

3. Rosenthal's inequality

We show that a general exponential inequality implies an appropriate Rosenthal inequality.

THEOREM 3.1. *Let $X_1,X_2,\dots,X_n$ be a sequence of zero mean r.v.'s, let $S_n=\sum_{i=1}^{n}X_i$ be their sum and B_n be a sequence of positive numbers. Assume that*

$$\mathbb{P}(|S_n|>x)\leqslant l(n)\mathbb{P}\left(\max_{1\leqslant i\leqslant n}|X_i|>d\right)+h(n)\exp\left(\frac{x}{d}-\frac{x}{d}\ln\left(1+\frac{xd}{B_n}\right)\right) \tag{3.1}$$

is satisfied for any $x>0$ and $d>0$ where $l(n)$ and $h(n)$ are some real numbers. Then, for $p>0$ we have

$$\mathbb{E}|S_n|^p\leqslant C_1 l(n)\mathbb{E}\max_{1\leqslant i\leqslant n}|X_i|^p+C_2h(n)B_n^{p/2}, \tag{3.2}$$

where $C_1=p^p$, $C_2=\frac{1}{2}p^{1+p/2}e^pB\left(\frac{p}{2},\frac{p}{2}\right)$ are absolute constants with $B(u,v)$ being the beta function.

Proof. We shall use the classical method, see [20]. Apply (3.1) with $d=x/p$. Then we obtain

$$\begin{aligned}\mathbb{E}|S_n|^p&=\int_0^\infty p\mathbb{P}(|S_n|\geqslant x)x^{p-1}\mathrm{d}x\\&\leqslant\int_0^\infty pl(n)\mathbb{P}\left(\max_{1\leqslant i\leqslant n}|X_i|>\frac{x}{p}\right)x^{p-1}\mathrm{d}x+\int_0^\infty ph(n)e^p\left(1+\frac{x^2}{pB_n}\right)^{-p}x^{p-1}\mathrm{d}x\\&=l(n)p^p\mathbb{E}\max_{1\leqslant i\leqslant n}|X_i|^p+h(n)\frac{1}{2}p^{1+p/2}e^pB\left(\frac{p}{2},\frac{p}{2}\right)B_n^{p/2},\end{aligned}$$

where in the last step we changed the variable in the second integral as $t=\frac{x^2}{pB_n}$ and $B(u,v)$ is the beta function

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t=\int_0^\infty t^{u-1}(1+t)^{-(u+v)}\mathrm{d}t,\qquad u>0,\ v>0.\quad\square$$

REMARK 3.1. Let $X_1,X_2,\dots,X_n$ be a sequence of zero mean r.v.'s, let $S_n=\sum_{i=1}^{n}X_i$ be their sum and $B_n=\sum_{i=1}^{n}\mathbb{E}X_i^2$ be the sum of variances. Assume that (2.1) is satisfied both for $\eta_i=X_i^{(d)}$, $i=1,2,\dots,n$ and for $\eta_i=(-X_i)^{(d)}$, $i=1,2,\dots,n$ for any $\lambda>0$ and $d>0$. Then Theorem 2.1 and inequality (3.2) imply

$$\mathbb{E}|S_n|^p\leqslant C_1\mathbb{E}\max_{1\leqslant i\leqslant n}|X_i|^p+2C_2g(n)B_n^{p/2}, \tag{3.3}$$

where $p>0$.

We remark that in [24] it was proved that Marcinkiewicz-Zygmund and Rosenthal inequalities imply complete moment convergence. Moreover, in [26] it was shown that a Rosenthal inequality implies complete convergence and complete moment convergence.

4. Complete convergence

In this section we shall show that a general exponential inequality implies a Baum-Katz type theorem.

Let Y_n, $i=1,2,\ldots$, be a sequence of random variables. We say that this sequence is weakly mean dominated (wmd) by the r.v. Y, if

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}\left(|Y_i|>t\right)\leqslant C\mathbb{P}\left(|Y|>t\right) \tag{4.1}$$

for all $t\geqslant 0$ and $n=1,2,\ldots$ (see [13]).

We shall often use the following technical lemma (see [6]).

LEMMA 4.1. *Let the sequence Y_n, $i=1,2,\ldots$ be weakly mean dominated by the r.v. Y. Let $t>0$ be fixed. Let $f:[0,\infty)\to[0,\infty)$ be a strictly increasing unbounded function with $f(0)=0$. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}|Y_i|\leqslant C\mathbb{E}|Y|; \tag{4.2}$$

the sequence $f(|Y_n|)$, $i=1,2,\ldots$ is weakly mean dominated by the r.v. $f(|Y|)$; the truncated sequence $|Y_n|^{(t)}$, $i=1,2,\ldots$ is weakly mean dominated by the truncated r.v. $|Y|^{(t)}$;

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}|Y_i|\mathbb{I}\left\{|Y_i|>t\right\}\leqslant C\mathbb{E}|Y|\mathbb{I}\left\{|Y|>t\right\}. \tag{4.3}$$

Given a r.v. X and a positive number t, we shall use the following truncated r.v.

$$\tilde{X}^{(t)}=-t\mathbb{I}\left\{X<-t\right\}+X\mathbb{I}\left\{|X|\leqslant t\right\}+t\mathbb{I}\left\{X>t\right\}. \tag{4.4}$$

Here $\mathbb{I}$ denotes the indicator function of a set.

THEOREM 4.1. *Let $X_1,X_2,\ldots$ be a sequence of r.v.'s, let $S_n=\sum_{i=1}^{n}X_i$ be their partial sum. Let $0<p<2$ and let α be a positive number. Assume that the exponential inequality is satisfied for the truncated and centered r.v.'s, that is*

$$\mathbb{P}\left(\left|\sum_{i=1}^{n}\left(\tilde{X}_i^{(t)}-\mathbb{E}\tilde{X}_i^{(t)}\right)\right|>x\right) \tag{4.5}$$
$$\leqslant\mathbb{P}\left(\max_{1\leqslant i\leqslant n}\left|\tilde{X}_i^{(t)}-\mathbb{E}\tilde{X}_i^{(t)}\right|>d\right)+g(n)\exp\left(\frac{x}{d}-\frac{x}{d}\ln\left(1+\frac{xd}{B_n}\right)\right)$$

for all $t>0$, $x>0$, $d>0$ and $n=1,2,\ldots$, where $B_n=\sum_{i=1}^{n}\mathbb{E}\left(\tilde{X}_i^{(t)}-\mathbb{E}\tilde{X}_i^{(t)}\right)^2$. Assume that $g(.)$ is regularly varying with exponent r, where $0<r<\alpha(2-p)$. Assume *that $X_1, X_2, \dots$ is weakly mean dominated by the r.v. X for which $\mathbb{E}|X|^p g(|X|^{1/\alpha}) < \infty$. If $0 < p < 1$, then assume $\alpha p > 1$. If $1 \leqslant p < 2$, then assume $\alpha p \geqslant 1$ and $\mathbb{E}X_i = 0$ for all i. Then we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\left(|S_n| > \varepsilon n^{\alpha}\right) < \infty.$$

Proof. For simplicity, denote $X'_{ni} = \tilde{X}_i^{(\varepsilon n^{\alpha}/4)}$. Now we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\left(|S_n| > \varepsilon n^{\alpha}\right) \leqslant & \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} \mathbb{P}\left(|X_i| > \frac{\varepsilon n^{\alpha}}{4}\right) \\
& + \sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\left(\left|\sum_{i=1}^{n} (X'_{ni} - \mathbb{E}X'_{ni})\right| > \frac{\varepsilon n^{\alpha}}{2}\right) \\
& + \sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{I}\left(\left|\sum_{i=1}^{n} \mathbb{E}X'_{ni}\right| > \frac{\varepsilon n^{\alpha}}{2}\right) = A_1 + A_2 + A_3,
\end{aligned}$$

say. Using the (wmd) assumption, we have

$$\begin{aligned}
A_1 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{P}\left(|X| > \frac{\varepsilon n^{\alpha}}{4}\right) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1} \sum_{i=n}^{\infty} \mathbb{P}\left(\frac{\varepsilon i^{\alpha}}{4} < |X| \leqslant \frac{\varepsilon (i+1)^{\alpha}}{4}\right) \\
&= C \sum_{i=1}^{\infty} \mathbb{P}\left(\frac{\varepsilon i^{\alpha}}{4} < |X| \leqslant \frac{\varepsilon (i+1)^{\alpha}}{4}\right) \sum_{n=1}^{i} n^{\alpha p-1} \\
&\leqslant C \sum_{i=1}^{\infty} i^{\alpha p} \mathbb{P}\left(\frac{\varepsilon i^{\alpha}}{4} < |X| \leqslant \frac{\varepsilon (i+1)^{\alpha}}{4}\right) \leqslant C \mathbb{E}|X|^p < \infty.
\end{aligned}$$

We have

$$\begin{aligned}
V &= n^{-\alpha} \left|\sum_{i=1}^{n} \mathbb{E}X'_{ni}\right| \\
&= n^{-\alpha} \left|\sum_{i=1}^{n} \left(-\frac{\varepsilon n^{\alpha}}{4} \mathbb{P}\left\{X_i < -\frac{\varepsilon n^{\alpha}}{4}\right\} + \mathbb{E}X_i \mathbb{I}\left\{|X_i| \leqslant \frac{\varepsilon n^{\alpha}}{4}\right\} + \frac{\varepsilon n^{\alpha}}{4} \mathbb{P}\left\{X_i > \frac{\varepsilon n^{\alpha}}{4}\right\}\right)\right|.
\end{aligned}$$

If $p \geqslant 1$, then using that $\mathbb{E}X_i = 0$, the (wmd) assumption and Lemma 4.1, we obtain

$$\begin{aligned}
V &\leqslant n^{-\alpha} \sum_{i=1}^{n} \left(\mathbb{E}|X_i| \mathbb{I}\left\{|X_i| > \frac{\varepsilon n^{\alpha}}{4}\right\} + \frac{\varepsilon n^{\alpha}}{4} \mathbb{P}\left\{|X_i| > \frac{\varepsilon n^{\alpha}}{4}\right\}\right) \\
&\leqslant 2 n^{-\alpha} n \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}|X_i| \mathbb{I}\left\{|X_i| > \frac{\varepsilon n^{\alpha}}{4}\right\} \leqslant C n^{1-\alpha} \mathbb{E}|X| \mathbb{I}\left\{|X| > \frac{\varepsilon n^{\alpha}}{4}\right\} \\
&\leqslant C n^{1-\alpha} \mathbb{E}|X| \frac{|X|^{p-1}}{n^{\alpha(p-1)}} \mathbb{I}\left\{|X| > \frac{\varepsilon n^{\alpha}}{4}\right\} = C n^{1-\alpha p} \mathbb{E}|X|^p \mathbb{I}\left\{|X| > \frac{\varepsilon n^{\alpha}}{4}\right\} \to 0,
\end{aligned}$$

as $n \to \infty$, because $\mathbb{E}|X|^p < \infty$ and $\alpha p \geqslant 1$.

If $p<1$, then using the (wmd) assumption and Lemma 4.1, we obtain

$$\begin{aligned} V &\leqslant n^{-\alpha}\sum_{i=1}^{n}\left(\mathbb{E}|X_i|\mathbb{I}\left\{|X_i|\leqslant \frac{\varepsilon n^\alpha}{4}\right\}+\frac{\varepsilon n^\alpha}{4}\mathbb{P}\left\{|X_i|>\frac{\varepsilon n^\alpha}{4}\right\}\right) \\ &\leqslant Cn^{1-\alpha}\mathbb{E}|X|\mathbb{I}\left\{|X|\leqslant \frac{\varepsilon n^\alpha}{4}\right\}+Cn^{1-\alpha}\frac{\varepsilon n^\alpha}{4}\mathbb{P}\left\{|X|>\frac{\varepsilon n^\alpha}{4}\right\} \\ &\leqslant Cn^{1-\alpha}\mathbb{E}|X|\frac{|X|^{p-1}}{(\varepsilon n^\alpha/4)^{p-1}}\mathbb{I}\left\{|X|\leqslant \frac{\varepsilon n^\alpha}{4}\right\}+Cn\frac{\mathbb{E}|X|^p}{(\varepsilon n^\alpha/4)^p} \\ &\leqslant Cn^{1-\alpha p}\mathbb{E}|X|^p\mathbb{I}\left\{|X|\leqslant \frac{\varepsilon n^\alpha}{4}\right\}+Cn^{1-\alpha p}\mathbb{E}|X|^p\to 0 \end{aligned}$$

as $n\to\infty$ because $\mathbb{E}|X|^p<\infty$ and $\alpha p>1$ in this case. Therefore

$$n^{-\alpha}\left|\sum_{i=1}^n \mathbb{E}X'_{ni}\right|<\frac{\varepsilon}{2}, \tag{4.6}$$

if $n>n_\varepsilon$. Therefore $A_3<\infty$. Applying equation (4.5) with $d=x=\varepsilon n^\alpha/2$ and using notation $B_n=\sum_{i=1}^n\mathbb{E}\left(X'_{ni}-\mathbb{E}X'_{ni}\right)^2$, we obtain

$$\begin{aligned} A_2 &= \sum_{n=1}^\infty n^{\alpha p-2}\mathbb{P}\left(\left|\sum_{i=1}^n (X'_{ni}-\mathbb{E}X'_{ni})\right|>\frac{\varepsilon n^\alpha}{2}\right) \\ &\leqslant \sum_{n=1}^\infty n^{\alpha p-2}\mathbb{P}\left(\max_{1\leqslant i\leqslant n}\left|X'_{ni}-\mathbb{E}X'_{ni}\right|>d\right)+\sum_{n=1}^\infty n^{\alpha p-2}g(n)\exp\left(\frac{x}{d}-\frac{x}{d}\ln\left(1+\frac{xd}{B_n}\right)\right). \end{aligned}$$

Here the first term is zero. So

$$\begin{aligned} A_2 &\leqslant \sum_{n=1}^\infty n^{\alpha p-2}g(n)e\frac{1}{1+\frac{xd}{B_n}}\leqslant C\sum_{n=1}^\infty n^{\alpha p-2-2\alpha}g(n)B_n \\ &\leqslant C\sum_{n=1}^\infty n^{\alpha p-2-2\alpha}g(n)\sum_{i=1}^n\mathbb{E}(X'_{ni})^2. \end{aligned}$$

Using (wmd) assumption, by Lemma 4.1, we obtain $\sum_{i=1}^n\mathbb{E}(X'_{ni})^2\leqslant n\mathbb{E}(X')^2$. So we have

$$\begin{aligned} A_2 &\leqslant C\sum_{n=1}^\infty n^{\alpha p-1-2\alpha}g(n)\left(\frac{\varepsilon n^\alpha}{4}\right)^2\mathbb{P}\left(|X|>\frac{\varepsilon n^\alpha}{4}\right) \\ &\quad +C\sum_{n=1}^\infty n^{\alpha p-1-2\alpha}g(n)\mathbb{E}\left(X^2\mathbb{I}\left\{|X|\leqslant\frac{\varepsilon n^\alpha}{4}\right\}\right)=B_1+B_2. \end{aligned}$$

We have

$$\begin{aligned} B_2 &\leqslant C\sum_{n=1}^\infty n^{\alpha p-1-2\alpha}g(n)\sum_{i=0}^{n-1}\mathbb{P}\left(\frac{\varepsilon i^\alpha}{4}<|X|\leqslant\frac{\varepsilon(i+1)^\alpha}{4}\right)\left(\frac{\varepsilon(i+1)^\alpha}{4}\right)^2 \\ &\leqslant C\sum_{i=0}^\infty\mathbb{P}\left(\frac{\varepsilon i^\alpha}{4}<|X|\leqslant\frac{\varepsilon(i+1)^\alpha}{4}\right)\left(\frac{\varepsilon(i+1)^\alpha}{4}\right)^2\sum_{n=i+1}^\infty n^{\alpha p-1-2\alpha}g(n). \end{aligned}$$

Now, using known properties of regularly varying functions (see [3], pp. 26–27) we obtain

$$B_2 \leqslant C\sum_{i=0}^{\infty} i^{\alpha p} g(i)\mathbb{P}\left(i < \left(\frac{4|X|}{\varepsilon}\right)^{1/\alpha} \leqslant i+1\right) \leqslant C\mathbb{E}|X|^p g\left(|X|^{1/\alpha}\right) < \infty.$$

Similarly, we have

$$\begin{aligned} B_1 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1} g(n)\mathbb{P}\left(|X| > \frac{\varepsilon n^{\alpha}}{4}\right) \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-1} g(n)\sum_{i=n}^{\infty}\mathbb{P}\left(\frac{\varepsilon i^{\alpha}}{4} < |X| \leqslant \frac{\varepsilon(i+1)^{\alpha}}{4}\right) \\ &\leqslant C\sum_{i=1}^{\infty}\mathbb{P}\left(\frac{\varepsilon i^{\alpha}}{4} < |X| \leqslant \frac{\varepsilon(i+1)^{\alpha}}{4}\right)\sum_{n=1}^{i} n^{\alpha p-1} g(n). \end{aligned}$$

Now, using again properties of regularly varying functions (see [3], pp. 26–27)

$$B_1 \leqslant C\sum_{i=0}^{\infty} i^{\alpha p} g(i)\mathbb{P}\left(i < \left(\frac{4|X|}{\varepsilon}\right)^{1/\alpha} \leqslant i+1\right) \leqslant C\mathbb{E}|X|^p g\left(|X|^{1/\alpha}\right) < \infty. \quad \square$$

5. Widely orthant dependent sequences of r.v.'s

The sequence of r.v.'s $X_1, X_2, \ldots$ is called widely orthant dependent (WOD) if for any positive integer n there exists a finite $g(n)$ so that for any real numbers $x_1, \ldots, x_n$ we have

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant g(n)\prod_{i=1}^{n}\mathbb{P}(X_i > x_i) \tag{5.1}$$

and

$$\mathbb{P}(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant g(n)\prod_{i=1}^{n}\mathbb{P}(X_i \leqslant x_i), \tag{5.2}$$

see [25]. It is known that extended negatively orthant dependent sequences, negatively orthant dependent sequences, negatively superadditive dependent sequences, negatively associated, and independent sequences are WOD, see [27]. In [27] exponential inequalities and complete convergence theorems were proved for WOD sequences.

In this section, first we compare the notions of WOD sequence and widely acceptable sequence. If $X_1, X_2, \ldots$ is a WOD sequence, then it is known that

$$\mathbb{E}e^{\sum_{i=1}^{n}\lambda X_i} \leqslant g(n)\prod_{i=1}^{n}\mathbb{E}e^{\lambda X_i} \tag{5.3}$$

for any real number λ. Recall that $X_1, X_2, \ldots$ are called widely acceptable it they satisfy equation (5.3). So if a sequence of r.v.'s is WOD, then it is widely acceptable with the same multiplier sequence $g(n)$. We shall give an example to show that the reverse statement is not true. Our Example 5.1 shows that the class of widely acceptable r.v.'s is larger than the class of WOD r.v.'s assuming the same multiplier sequence $g(n)$. In order to show it we shall use an example which can be considered as a discrete counterpart of the example given in [22]. That example is based on Feller's counterexample (see problem III/1 in [10]).

EXAMPLE 5.1. Let X and Y be discrete r.v.'s with the following joint distribution

$$\mathbb{P}(X=0,Y=0)=\tfrac{1}{16},\quad \mathbb{P}(X=0,Y=1)=\tfrac{3}{16},\quad \mathbb{P}(X=0,Y=2)=0,$$

$$\mathbb{P}(X=1,Y=0)=\tfrac{1}{16},\quad \mathbb{P}(X=1,Y=1)=\tfrac{4}{16},\quad \mathbb{P}(X=1,Y=2)=\tfrac{3}{16},$$

$$\mathbb{P}(X=2,Y=0)=\tfrac{2}{16},\quad \mathbb{P}(X=2,Y=1)=\tfrac{1}{16},\quad \mathbb{P}(X=2,Y=2)=\tfrac{1}{16}.$$

We see that X and Y are not independent, but the distribution of $X+Y$ is the same as the convolution of the distribution of X and the distribution of Y. Therefore

$$\mathbb{E}e^{\lambda(X+Y)}=\mathbb{E}e^{\lambda X}\mathbb{E}e^{\lambda Y}.$$

However, by direct calculation,

$$\mathbb{P}(X\leqslant u,Y\leqslant v)\leqslant g\mathbb{P}(X\leqslant u)\mathbb{P}(Y\leqslant v) \tag{5.4}$$

and

$$\mathbb{P}(X>u,Y>v)\leqslant g\mathbb{P}(X>u)\mathbb{P}(Y>v) \tag{5.5}$$

are not satisfied for $g=1$. For example,

$$\mathbb{P}(X>0,Y>1)=\frac{1}{4}=\frac{4}{3}\cdot\frac{3}{16}=\frac{4}{3}\mathbb{P}(X>0)\mathbb{P}(Y>1).$$

We can see that the smallest constant which satisfies the above two inequalities (5.4) - (5.5) is $g=4/3$. Now, let the two-dimensional r.v.'s $(X_1,X_2),(X_3,X_4),(X_5,X_6),\ldots$ be independent copies of (X,Y). Then inequality (5.3) is satisfied with $g(n)\equiv 1$ while (5.1) and (5.2) are satisfied with $g(n)=(4/3)^{n/2}$. So we can obtain sharper result if we use inequality (5.3) directly.

In the remaining part of this section we shall apply the results of the previous sections for WOD sequences. To this end first we list a few known facts on WOD sequences. If $X_1,X_2,\ldots$ is a WOD sequence and the real functions $f_1,f_2,\ldots$ are either all non-decreasing or all non-increasing, then the sequence $f_1(X_1),f_2(X_2),\ldots$ is also WOD. In particular, the truncated sequence $X_1^{(t)},X_2^{(t)},\ldots$ is WOD.

THEOREM 5.1. *Let $X_1,X_2,\ldots,X_n$ be a sequence of zero mean WOD r.v.'s. Let $S_n=\sum_{i=1}^n X_i$ be the sum and $B_n=\sum_{i=1}^n \mathbb{E}X_i^2$ be the sum of variances. Then for $d>0$ and $x>0$ we have*

$$\mathbb{P}(S_n>x)\leqslant\mathbb{P}\left(\max_{1\leqslant i\leqslant n}X_i>d\right)+g(n)\exp\left(\frac{x}{d}-\frac{x}{d}\ln\left(1+\frac{xd}{B_n}\right)\right) \tag{5.6}$$

and

$$\mathbb{P}(|S_n|>x)\leqslant\mathbb{P}\left(\max_{1\leqslant i\leqslant n}|X_i|>d\right)+2g(n)\exp\left(\frac{x}{d}-\frac{x}{d}\ln\left(1+\frac{xd}{B_n}\right)\right). \tag{5.7}$$

Proof. It is a simple consequence of Theorem 2.1 and inequality (5.3). We remark that (5.7) is a special case of Lemma 2.2 in [27], where no detailed proof was presented. □

Now we turn to Hoeffding's inequality for WOD sequences.

THEOREM 5.2. *Let $X_1, X_2, \ldots, X_n$ be a WOD sequence of r.v.'s. Let $S_n = \sum_{i=1}^n X_i$ be the sum. Let the random variables be bounded, i.e. $a_i \leqslant X_i \leqslant b_i$ for $i = 1, 2, \ldots, n$, where a_i and b_i are real numbers. Let $\varepsilon > 0$. Then we have*

$$\mathbb{P}(S_n - \mathbb{E}S_n \geqslant \varepsilon) \leqslant g(n) \exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right), \tag{5.8}$$

$$\mathbb{P}(|S_n - \mathbb{E}S_n| \geqslant \varepsilon) \leqslant 2g(n) \exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \tag{5.9}$$

Let $M_{k,l}$ the maximum defined by (2.10). Assume that (5.1) and (5.2) are satisfied with $g(n) = C$. Then for any $0 < \delta < 1$ there exists a $C_1 = C_1(\delta)$ such that

$$\mathbb{P}\left(M_{k,l} \geqslant \varepsilon\right) \leqslant 2CC_1 \exp\left(-\frac{2\varepsilon^2(1-\delta)}{\sum_{i=k}^l (b_i - a_i)^2}\right) \tag{5.10}$$

for any $1 \leqslant k \leqslant l \leqslant n$.

Proof. It is a simple consequence of Theorem 2.2, Corollary 2.1 and inequality (5.3). □

Now we turn to the Rosenthal inequality for WOD sequences.

THEOREM 5.3. *Let $X_1, X_2, \ldots, X_n$ be a WOD sequence of zero mean r.v.'s, let $S_n = \sum_{i=1}^n X_i$ be their sum and $B_n = \sum_{i=1}^n \mathbb{E}X_i^2$. Then, for $p > 0$ we have*

$$\mathbb{E}|S_n|^p \leqslant C_1 \mathbb{E} \max_{1 \leqslant i \leqslant n} |X_i|^p + C_2 g(n) B_n^{p/2}, \tag{5.11}$$

where C_1 and C_2 are absolute constants.

Proof. It is a simple consequence of Remark 3.1 and inequality (5.3). □

The following complete convergence theorem is a version of Corollary 3.2 of [27].

THEOREM 5.4. *Let $X_1, X_2, \ldots$ be a WOD sequence of r.v.'s, let $S_n = \sum_{i=1}^n X_i$ be their partial sum. Let $0 < p < 2$ and let α be a positive number. Assume that $g(.)$ is regularly varying with exponent r, where $0 < r < \alpha(2-p)$. Assume that $X_1, X_2, \ldots$ is weakly mean dominated by the r.v. X for which $\mathbb{E}|X|^p g(|X|^{1/\alpha}) < \infty$. If $0 < p < 1$, then assume $\alpha p > 1$. If $1 \leqslant p < 2$, then assume $\alpha p \geqslant 1$ and $\mathbb{E}X_i = 0$ for all i. Then we have*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} \mathbb{P}(|S_n| > \varepsilon n^\alpha) < \infty.$$

Proof. It is a simple consequence of Theorems 4.1 and 5.1. □

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REFERENCES

[1] R. G. ANTONINI, Y. KOZACHENKO, A. VOLODIN, *Convergence of series of dependent φ-sub-Gaussian random variables*, J. Math. Anal. Appl. **338**, no. 2 (2008), 1188–1203.

[2] L. E. BAUM, M. KATZ, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. **120** (1965), 108–123.

[3] N. H. BINGHAM, C. M. GOLDIE, J. L. TEUGELS, *Regular variation*, Cambridge University Press, Cambridge, 1987.

[4] T. C. CHRISTOFIDES, M. HADJIKYRIAKOU, *Exponential inequalities for N-demimartingales and negatively associated random variables*, Statist. Probab. Lett. **79**, no. 19 (2009), 2060–2065.

[5] P. ERDŐS, *On a theorem of Hsu and Robbins*, Ann. Math. Statistics **20** (1949), 286–291.

[6] I. FAZEKAS, *Convergence rates in the law of large numbers for arrays*, Publ. Math. Debrecen **41**, no. 1–2 (1992), 53–71.

[7] I. FAZEKAS, *On a general approach to the strong laws of large numbers*, J. Math. Sci. (N. Y.) **200**, no. 4 (2014), 411–423.

[8] I. FAZEKAS, O. I. KLESOV, *A general approach to the strong laws of large numbers*, Teor. Veroyatnost. i Primenen. **45** (2000), no. 3, 568–583, Theory Probab. Appl. **45** (2002), no. 3, 436–449.

[9] I. FAZEKAS, A. G. KUKUSH, T. TÓMÁCS, *On the Rosenthal inequality for mixing fields*, Ukrainian Math. J. **52**, no. 2 (2000), 305–318.

[10] W. FELLER, *An introduction to probability theory and its applications*, vol. II, Wiley, New York, 1971.

[11] D. H. FUK, S. V. NAGAEV, *Probabilistic inequalities for sums of independent random variables*, Teor. Verojatnost. i Primenen. **16** (1971), 660–675.

[12] SHIXIN GAN, PINGYAN CHEN, DEHUA QIU, *Rosenthal inequality for NOD sequences and its applications*, Wuhan Univ. J. Nat. Sci. **16**, no. 3 (2011), 185–189.

[13] A. GUT, *Complete convergence for arrays*, Period. Math. Hungar. **25**, no. 1 (1992), 51–75.

[14] A. GUT, *Probability: a graduate course*, Springer, New York, 2013.

[15] W. HOEFFDING, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58** (1963), 13–30.

[16] P. L. HSU, H. ROBBINS, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U. S. A. **33** (1947), 25–31.

[17] T. C. HU, F. MÓRICZ, R. L. TAYLOR, *Strong laws of large numbers for arrays of rowwise independent random variables*, Acta Math. Hungar. **54**, no. 1–2 (1989), 153–162.

[18] F. MÓRICZ, *Exponential estimates for the maximum of partial sums, I, Sequences of rv's*, Acta Math. Acad. Sci. Hungar. **33**, no. 1–2 (1979), 159–167.

[19] S. V. NAGAEV, I. F. PINELIS, *Some inequalities for the distribution of sums of independent random variables*, Teor. Veroyatn. Primen. **22** (1977), 254–263.

[20] V. V. PETROV, *Limit theorems of probability theory. Sequences of independent random variables*, The Clarendon Press, Oxford University Press, New York, 1995.

[21] QI MAN SHAO, *Complete convergence for α-mixing sequences*, Statist. Probab. Lett. **16**, no. 4 (1993), 279–287.

[22] AITING SHEN, SHUHE HU, ANDREI VOLODIN, XUEJUN WANG, *Some exponential inequalities for acceptable random variables and complete convergence*, J. Inequal. Appl. **2011:142** (2011), 10 pp.

[23] W. F. STOUT, *Almost sure convergence*, Probab. Math. Statistics, vol. **24**, Academic Press, New York-London, 1974.

[24] SOO HAK SUNG, *Moment inequalities and complete moment convergence*, J. Inequal. Appl., Art. ID 271265 (2009), 14 pp.

[25] KAIYONG WANG, YUEBAO WANG, QINGWU GAO, *Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate*, Methodol. Comput. Appl. Probab. **15**, no. 1 (2013), 109–124.

[26] XINGHUI WANG, SHUHE HU, *Complete convergence and complete moment convergence for a class of random variables*, J. Inequal. Appl. **2012**: 229 (2012), 12 pp.

[27] XUEJUN WANG, CHEN XU, TIEN-CHUNG HU, ANDREI VOLODIN, SHUHE HU, *On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models*, TEST **23**, no. 3 (2014), 607–629.

[28] YUEBAO WANG, YAWEI LI, QINGWU GAO, *On the exponential inequality for acceptable random variables*, J. Inequal. Appl. **2011**: 40 (2011), 10 pp.

[29] YONGFENG WU, MEI GUAN, *Convergence properties of the partial sums for sequences of end random variables*, J. Korean Math. Soc. **49**, no. 6 (2012), 1097–1110.

[30] YONGFENG WU, MINGZHU SONG, CHUNHUA WANG, *Complete Moment Convergence and Mean Convergence for Arrays of Rowwise Extended Negatively Dependent Random Variables*, Scientific World Journal, Article ID 2014:478612 (2014), doi: 10.1155/2014/478612.

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István Fazekas
Faculty of Informatics
University of Debrecen
P. O. Box 400, 4002 Debrecen, Hungary
e-mail: fazekas.istvan@inf.unideb.hu

Sándor Pecsora
Faculty of Informatics
University of Debrecen
P. O. Box 400, 4002 Debrecen, Hungary
e-mail: pecsora.sandor@inf.unideb.hu

Bettina Porvázsnyik
Faculty of Informatics
University of Debrecen
P. O. Box 400, 4002 Debrecen, Hungary
e-mail: porvazsnyik.bettina@inf.unideb.hu