

SOME GENERALIZATIONS OF NUMERICAL RADIUS ON OFF-DIAGONAL PART OF 2×2 OPERATOR MATRICES

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Abstract. We generalize several inequalities involving powers of the numerical radius for off-diagonal part of 2×2 operator matrices of the form $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, where B, C are two operators.

In particular, if $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, then we get

$$\frac{1}{2^{\frac{3}{2}(r-1)}} \max\{\|\mu\|, \|\eta\|\} \leq w^r(T) \leq \frac{1}{2^{r+1}} \max\{\|\mu\|, \|\eta\|\},$$

where $r \geq 2$, $\mu = |(C - B^*) + i(C + B^*)|^r + |(B^* - C) + i(C + B^*)|^r$ and $\eta = |(B - C^*) + i(B + C^*)|^r + |(C^* - B) + i(B + C^*)|^r$.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ denotes the C^* -algebra of all bounded linear operators on \mathcal{H} . In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra \mathbb{M}_n of all $n \times n$ matrices with entries in the complex field. The numerical radius of $T \in \mathbb{B}(\mathcal{H})$ is defined by

$$w(T) := \sup\{|\langle Tx, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $T \in \mathbb{B}(\mathcal{H})$, $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$; see [11]. An important inequality for $w(A)$ is the power inequality stating that $w(A^n) \leq w(A)^n$ ($n = 1, 2, \dots$). It has been shown in [8], that if $T \in \mathbb{B}(\mathcal{H})$, then

$$w(T) \leq \frac{1}{2} (\|T\| + |T^*|), \tag{1.1}$$

where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T . Recently in [12] the authors showed

$$w^{2r}(T) \leq \frac{1}{2} \left(\|A\|^{2r} + \left\| \frac{1}{p} f^{pr}(|A^2|) + \frac{1}{q} g^{qr}(|(A^*)^2|) \right\| \right), \tag{1.2}$$

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in which f, g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$), $r \geq 1, p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $pr \geq 2$.

Let $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ be Hilbert spaces, and consider the direct sum $\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j$. With respect to this decomposition, every operator $T \in \mathbb{B}(\mathcal{H})$ has an $n \times n$ operator matrix representation $T = [T_{ij}]$ with entries $T_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$, the space of all bounded linear operators from \mathcal{H}_j to \mathcal{H}_i . Operator matrices provide a usual tool for studying Hilbert space operators, which have been extensively studied in the literatures. The classical Young inequality says that if $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for positive real numbers a, b . A refinement of the scalar Young inequality is presented in [3] as following $(a^{\frac{1}{p}} b^{\frac{1}{q}})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\frac{a}{p} + \frac{b}{q})^m$, where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$ and $m = 1, 2, \dots$. In particular, if $p = q = 2$, then

$$(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + \left(\frac{1}{2}\right)^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq 2^{-m} (a + b)^m. \tag{1.3}$$

Let $T_1, T_2, \dots, T_n \in \mathbb{B}(\mathcal{H})$. The functional w_p of operators T_1, \dots, T_n for $p \geq 1$ is defined in [13] as following

$$w_p(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{i=1}^n |\langle T_i x, x \rangle|^p \right)^{\frac{1}{p}}.$$

In [14] the authors showed the following inequality

$$w_p^p(A_1^* T_1 B_1, \dots, A_n^* T_n B_n) \leq \frac{1}{2} \left\| \sum_{i=1}^n ([B_i^* f^2(|T_i|) B_i]^p + [A_i^* g^2(|T_i^*|) A_i]^p) \right\| - \inf_{\|x\|=1} \zeta(X),$$

where $A_i, B_i, T_i \in \mathbb{B}(\mathcal{H})$ ($i = 1, 2, \dots, n$), f, g are nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ ($t \in [0, \infty)$), $p, r \geq m, m = 1, 2, \dots$, and

$$\zeta(X) = 2^{-m} \sum_{i=1}^n \left(\langle [B_i^* f^2(|T_i|) B_i]^{\frac{p}{m}} x, x \rangle^{\frac{m}{2}} - \langle [A_i^* g^2(|T_i^*|) A_i]^{\frac{p}{m}} x, x \rangle^{\frac{m}{2}} \right)^2.$$

For further information about numerical radius inequalities we refer the reader to [1, 4, 14] and references therein.

In this paper, we establish some generalizations of inequalities that is based on the off-diagonal parts of 2×2 operator matrices. We also show some inequalities involving powers of the numerical radius for the off-diagonal parts of 2×2 operator matrices.

2. Main results

To prove our first result, we need several well known lemmas.

LEMMA 2.1. [6, 15] Let $A \in \mathbb{B}(\mathcal{H}_1), B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1), C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathbb{B}(\mathcal{H}_2)$. Then the following statements hold:

(a) $w \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) = \max\{w(A), w(D)\};$

- (b) $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right);$
- (c) $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \frac{1}{2} \sup_{\theta \in \mathcal{R}} \| e^{i\theta} B + e^{-i\theta} C^* \|;$
- (d) $w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \max\{w(A+B), w(A-B)\}.$

In particular,

$$w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B).$$

The second lemma is a simple consequence of the classical Jensen and Young inequalities; see [5].

LEMMA 2.2. *Let $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$$

for $r \geq 1$.

The next lemma follows from the spectral theorem for positive operators and Jensen inequality; see [7].

LEMMA 2.3. (McCarty inequality) *Let $T \in \mathbb{B}(\mathcal{H})$, $T \geq 0$ and $x \in \mathcal{H}$ be a unit vector. Then*

- (a) $\langle Tx, x \rangle^r \leq \langle T^r x, x \rangle$ for $r \geq 1$;
- (b) $\langle T^r x, x \rangle \leq \langle Tx, x \rangle^r$ for $0 < r \leq 1$.

The following lemma is a consequence of convexity of the absolute value function.

LEMMA 2.4. *Let $T \in \mathbb{B}(\mathcal{H})$ be self-adjoint and $x \in \mathcal{H}$ be a unit vector. Then*

$$|\langle Tx, x \rangle| \leq \langle |T| x, x \rangle.$$

LEMMA 2.5. [7, Theorem 1] *Let $T \in \mathbb{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*

(a) *If f, g are nonnegative continuous functions on $[0, \infty)$ which are satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$), then*

$$|\langle Tx, y \rangle| \leq \| f(|T|)x \| \| g(|T^*|)y \|;$$

(b) *If $0 \leq \alpha \leq 1$, then*

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle.$$

Now we are in a position to state the main results of this section.

THEOREM 2.6. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ and f, g be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then

$$w^r(T) \leq \max \left\{ \left\| \frac{1}{p} f^{pr}(|C|) + \frac{1}{q} g^{qr}(|B^*|) \right\|, \left\| \frac{1}{p} f^{pr}(|B|) + \frac{1}{q} g^{qr}(|C^*|) \right\| \right\}, \quad (2.1)$$

in which $r \geq 1, p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $pr \geq 2$.

Proof. For any unit vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ we have

$$\begin{aligned} |\langle TX, X \rangle|^r &\leq \|f(|T|)X\|^r \|g(|T^*|)X\|^r && \text{(by Lemma 2.5)} \\ &= \langle f^2(|T|)X, X \rangle^{\frac{r}{2}} \langle g^2(|T^*|)X, X \rangle^{\frac{r}{2}} \\ &\leq \frac{1}{p} \left\langle f^2 \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) X, X \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle g^2 \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) X, X \right\rangle^{\frac{qr}{2}} \\ &&& \text{(by Lemma 2.2)} \\ &\leq \frac{1}{p} \left\langle \begin{bmatrix} f^{pr} |C| & 0 \\ 0 & f^{pr} |B| \end{bmatrix} X, X \right\rangle + \frac{1}{q} \left\langle \begin{bmatrix} g^{qr} |B^*| & 0 \\ 0 & g^{qr} |C^*| \end{bmatrix} X, X \right\rangle \\ &&& \text{(by Lemma 2.3(a))} \\ &= \left\langle \begin{bmatrix} \frac{1}{p} f^{pr}(|C|) + \frac{1}{q} g^{qr}(|B^*|) & 0 \\ 0 & \frac{1}{p} f^{pr}(|B|) + \frac{1}{q} g^{qr}(|C^*|) \end{bmatrix} X, X \right\rangle. \end{aligned}$$

Then

$$|\langle TX, X \rangle|^r \leq \left\langle \begin{bmatrix} \frac{1}{p} f^{pr}(|C|) + \frac{1}{q} g^{qr}(|B^*|) & 0 \\ 0 & \frac{1}{p} f^{pr}(|B|) + \frac{1}{q} g^{qr}(|C^*|) \end{bmatrix} X, X \right\rangle.$$

Now, applying the definition of numerical radius and Lemma 2.1(a), we have

$$w^r(T) \leq \max \left\{ \left\| \frac{1}{p} f^{pr}(|C|) + \frac{1}{q} g^{qr}(|B^*|) \right\|, \left\| \frac{1}{p} f^{pr}(|B|) + \frac{1}{q} g^{qr}(|C^*|) \right\| \right\}. \quad \square$$

COROLLARY 2.7. [2, Corollary 3] Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ be a positive operator matrix and $r \geq 1$. Then

$$w(T) = \frac{1}{2} \|B + C\|.$$

Proof. Putting $f(t) = g(t) = t^{\frac{1}{2}}, r = 1$ and $p = q = 2$ in inequality (2.1) and applying Lemma 2.1(c), we get the equality. \square

THEOREM 2.8. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ and f, g be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then

$$w^{2r}(T) \leq \max \left\{ \left\| \frac{1}{p} f^{2pr}(|C|) + \frac{1}{q} g^{2qr}(|B^*|) \right\|, \left\| \frac{1}{p} f^{2pr}(|B|) + \frac{1}{q} g^{2qr}(|C^*|) \right\| \right\}, \tag{2.2}$$

where $r \geq 1$ and $p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $pr \geq 1$.

Proof. Assume that $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ is a unit vector. Then

$$\begin{aligned} |\langle TX, X \rangle|^{2r} &\leq \|f(|T|)X\|^{2r} \|g(|T^*|)X\|^{2r} \quad (\text{by Lemma 2.5}) \\ &= \langle f^2(|T|)X, X \rangle^r \langle g^2(|T^*|)X, X \rangle^r \\ &\leq \frac{1}{p} \left\langle f^2 \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) X, X \right\rangle^{rp} + \frac{1}{q} \left\langle g^2 \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) X, X \right\rangle^{rq} \\ &\quad (\text{by Lemma 2.2}) \\ &\leq \frac{1}{p} \left\langle \begin{bmatrix} f^{2pr} |C| & 0 \\ 0 & f^{2pr} |B| \end{bmatrix} X, X \right\rangle + \frac{1}{q} \left\langle \begin{bmatrix} g^{2qr} |B^*| & 0 \\ 0 & g^{2qr} |C^*| \end{bmatrix} X, X \right\rangle \\ &\quad (\text{by Lemma 2.3(a)}) \\ &= \left\langle \begin{bmatrix} \frac{1}{p} f^{2pr}(|C|) + \frac{1}{q} g^{2qr}(|B^*|) & 0 \\ 0 & \frac{1}{p} f^{2pr}(|B|) + \frac{1}{q} g^{2qr}(|C^*|) \end{bmatrix} X, X \right\rangle. \end{aligned}$$

Thus

$$|\langle TX, X \rangle|^{2r} \leq \left\langle \begin{bmatrix} \frac{1}{p} f^{2pr}(|C|) + \frac{1}{q} g^{2qr}(|B^*|) & 0 \\ 0 & \frac{1}{p} f^{2pr}(|B|) + \frac{1}{q} g^{2qr}(|C^*|) \end{bmatrix} X, X \right\rangle.$$

Now by the definition of numerical radius and Lemma 2.1(a), we have

$$w^{2r}(T) \leq \max \left\{ \left\| \frac{1}{p} f^{2pr}(|C|) + \frac{1}{q} g^{2qr}(|B^*|) \right\|, \left\| \frac{1}{p} f^{2pr}(|B|) + \frac{1}{q} g^{2qr}(|C^*|) \right\| \right\}. \quad \square$$

Inequality (2.2) induces several numerical radius inequalities as follows.

COROLLARY 2.9. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$. Then

$$w^{2r}(T) \leq \frac{1}{2} \max \{ \| |C|^{4r\alpha} + |B^*|^{4r(1-\alpha)} \|, \| |B|^{4r\alpha} + |C^*|^{4r(1-\alpha)} \| \}$$

for any $r \geq 1$ and $0 \leq \alpha \leq 1$.

Proof. Letting $f(t) = t^\alpha$, $g(t) = t^{1-\alpha}$ and $p = q = 2$ in inequality (2.2), we get the desired inequality. \square

COROLLARY 2.10. *Let $B \in \mathbb{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 1$. Then*

$$w^{2r}(B) \leq \frac{1}{2} \left\| \|B\|^{4r\alpha} + \|B^*\|^{4r(1-\alpha)} \right\|. \tag{2.3}$$

Proof. We put $f(t) = t^\alpha$, $g(t) = t^{1-\alpha}$, $p = q = 2$ and $T = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$ and apply Lemma 2.1(d), we get the desired result. \square

THEOREM 2.11. *Let $T_i = \begin{bmatrix} 0 & B_i \\ C_i & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_2 \oplus \mathcal{H}_1)$ for any $i = 1, 2, \dots, n$. Then*

$$\begin{aligned} & w_p^p(T_1, T_2, \dots, T_n) \\ & \leq \max \left\{ \left\| \sum_{i=1}^n \alpha |C_i|^p + (1-\alpha) |B_i^*|^p \right\|, \left\| \sum_{i=1}^n \alpha |B_i|^p + (1-\alpha) |C_i^*|^p \right\| \right\} \end{aligned} \tag{2.4}$$

for $0 \leq \alpha \leq 1$ and $p \geq 2$.

Proof. For any unit vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have

$$\begin{aligned} & \sum_{i=1}^n |\langle T_i X, X \rangle|^p \\ & = \sum_{i=1}^n (|\langle T_i X, X \rangle|^2)^{\frac{p}{2}} \\ & \leq \sum_{i=1}^n (\langle |T_i|^{2\alpha} X, X \rangle \langle |T_i^*|^{2(1-\alpha)} X, X \rangle)^{\frac{p}{2}} \quad (\text{by Lemma 2.5 (b)}) \\ & \leq \sum_{i=1}^n \langle |T_i|^{p\alpha} X, X \rangle \langle |T_i^*|^{p(1-\alpha)} X, X \rangle \quad (\text{by Lemma 2.3 (b)}) \\ & \leq \sum_{i=1}^n \langle |T_i|^p X, X \rangle^\alpha \langle |T_i^*|^p X, X \rangle^{1-\alpha} \\ & \leq \sum_{i=1}^n (\alpha \langle |T_i|^p X, X \rangle + (1-\alpha) \langle |T_i^*|^p X, X \rangle) \quad (\text{by Lemma 2.2}) \\ & = \sum_{i=1}^n \left(\alpha \left\langle \begin{bmatrix} |C_i|^p & 0 \\ 0 & |B_i|^p \end{bmatrix} X, X \right\rangle + (1-\alpha) \left\langle \begin{bmatrix} |B_i^*|^p & 0 \\ 0 & |C_i^*|^p \end{bmatrix} X, X \right\rangle \right) \\ & = \sum_{i=1}^n \left\langle \begin{bmatrix} \alpha |C_i|^p + (1-\alpha) |B_i^*|^p & 0 \\ 0 & \alpha |B_i|^p + (1-\alpha) |C_i^*|^p \end{bmatrix} X, X \right\rangle \\ & = \left\langle \begin{bmatrix} \sum_{i=1}^n \alpha |C_i|^p + (1-\alpha) |B_i^*|^p & 0 \\ 0 & \sum_{i=1}^n \alpha |B_i|^p + (1-\alpha) |C_i^*|^p \end{bmatrix} X, X \right\rangle. \end{aligned}$$

By the definition of numerical radius and Lemma 2.1, we have

$$w_p^p(T_1, T_2, \dots, T_n) \leq \max \left\{ \left\| \sum_{i=1}^n \alpha |C_i|^p + (1-\alpha) |B_i^*|^p \right\|, \left\| \sum_{i=1}^n \alpha |B_i|^p + (1-\alpha) |C_i^*|^p \right\| \right\}. \quad \square$$

REMARK 2.12. As a special case for $\alpha = \frac{1}{2}$ and $B_i = C_i$ for any $i = 1, 2, \dots, n$, we have the following inequality

$$w_p^p(B_1, B_2, \dots, B_n) \leq \frac{1}{2} \left\| \sum_{i=1}^n |B_i|^p + |B_i^*|^p \right\|,$$

which already shown in [13, Proposition 3.9].

Now using a refinement of the classical Young inequality, we have the following theorem.

THEOREM 2.13. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ and f, g be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then for $m = 1, 2, \dots$ and $p, r \geq m$

$$w^r(T) \leq \left(\frac{1}{2}\right)^m \max \{ \|f^{\frac{2r}{m}}(|C|) + g^{\frac{2r}{m}}(|B^*|)\|^m, \|f^{\frac{2r}{m}}(|B|) + g^{\frac{2r}{m}}(|C^*|)\|^m \} - \inf_{\|X\|=1} \zeta(X), \quad (2.5)$$

where

$$\zeta(X) = 2^{-m} \left(\left\langle f^{\frac{2r}{m}} \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) X, X \right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}} \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) X, X \right\rangle^{\frac{m}{2}} \right)^2.$$

Proof. Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ be a unit vector. Applying Lemmas 2.5, 2.3 and inequality (1.3), respectively, we have

$$\begin{aligned} |\langle TX, X \rangle|^r &\leq \|f(|T|)X\|^r \|g(|T^*|)X\|^r \\ &= \left(\langle f^2(|T|)X, X \rangle^{\frac{r}{2m}} \langle g^2(|T^*|)X, X \rangle^{\frac{r}{2m}} \right)^m \\ &\leq \left(\langle f^{\frac{2r}{m}}(|T|)X, X \rangle^{\frac{1}{2}} \langle g^{\frac{2r}{m}}(|T^*|)X, X \rangle^{\frac{1}{2}} \right)^m \\ &\quad - 2^{-m} \left(\langle f^{\frac{2r}{m}}(|T|)X, X \rangle^{\frac{m}{2}} - \langle g^{\frac{2r}{m}}(|T^*|)X, X \rangle^{\frac{m}{2}} \right)^2 \\ &\leq \left(\frac{1}{2} \left\langle f^{\frac{2r}{m}} \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) X, X \right\rangle + \frac{1}{2} \left\langle g^{\frac{2r}{m}} \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) X, X \right\rangle \right)^m \end{aligned}$$

$$\begin{aligned}
 & -2^{-m} \left(\left\langle f^{\frac{2r}{m}} \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) X, X \right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}} \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) X, X \right\rangle^{\frac{m}{2}} \right)^2 \\
 & = \left(\frac{1}{2} \left\langle \begin{bmatrix} f^{\frac{2r}{m}}(|C|) + g^{\frac{2r}{m}}(|B^*|) & 0 \\ 0 & f^{\frac{2r}{m}}(|B|) + g^{\frac{2r}{m}}(|C^*|) \end{bmatrix} X, X \right\rangle \right)^m \\
 & -2^{-m} \left(\left\langle f^{\frac{2r}{m}} \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) X, X \right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}} \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) X, X \right\rangle^{\frac{m}{2}} \right)^2.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 w^r(T) \leq & \left(\frac{1}{2}\right)^m \max\{\|f^{\frac{2r}{m}}(|C|) + g^{\frac{2r}{m}}(|B^*|)\|^m, \|f^{\frac{2r}{m}}(|B|) + g^{\frac{2r}{m}}(|C^*|)\|^m\} \\
 & - \inf_{\|X\|=1} \zeta(X).
 \end{aligned}$$

Hence we get the desired inequality. \square

REMARK 2.14. In inequality (2.5) if $m = 1$, then we get a refinement of inequality (2.1).

3. Numerical radius of the operator matrix 2×2

In this section, we estimate numerical radius of matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

LEMMA 3.1. Let $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Then

$$w^r(T) \leq \frac{1}{2} \max\{\| |A|^r + |A^*|^r \|, \| |D|^r + |D^*|^r \| \} \tag{3.1}$$

for $r \geq 1$.

Proof. Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ be any unit vector. Then

$$\begin{aligned}
 |\langle TX, X \rangle| & \leq \langle |T|X, X \rangle^{\frac{1}{2}} \langle |T^*|X, X \rangle^{\frac{1}{2}} \\
 & \leq \frac{1}{2} \left\langle \begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix} X, X \right\rangle + \frac{1}{2} \left\langle \begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix} X, X \right\rangle \\
 & \leq \left(\frac{1}{2} \left\langle \begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix} X, X \right\rangle^r + \frac{1}{2} \left\langle \begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix} X, X \right\rangle^r \right)^{\frac{1}{r}} \\
 & \leq \left(\frac{1}{2} \left\langle \begin{bmatrix} |A|^r & 0 \\ 0 & |D|^r \end{bmatrix} X, X \right\rangle + \frac{1}{2} \left\langle \begin{bmatrix} |A^*|^r & 0 \\ 0 & |D^*|^r \end{bmatrix} X, X \right\rangle \right)^{\frac{1}{r}}
 \end{aligned}$$

$$= \left(\left\langle \left[\begin{array}{cc} \frac{1}{2}(|A|^r + |A^*|^r) & 0 \\ 0 & \frac{1}{2}(|D|^r + |D^*|^r) \end{array} \right] X, X \right\rangle \right)^{\frac{1}{r}},$$

and so

$$|\langle TX, X \rangle|^r \leq \left\langle \left[\begin{array}{cc} \frac{1}{2}(|A|^r + |A^*|^r) & 0 \\ 0 & \frac{1}{2}(|D|^r + |D^*|^r) \end{array} \right] X, X \right\rangle.$$

Therefore

$$w^r(T) \leq \frac{1}{2} \max\{\| |A|^r + |A^*|^r \|, \| |D|^r + |D^*|^r \| \}. \quad \square$$

REMARK 3.2. By letting $r = 1$ and $A = D$ in inequality (3.1), we obtain inequality (1.1), that is

$$w(A) \leq \frac{1}{2} \| |A| + |A^*| \|.$$

The following proposition follows from inequalities (2.1) and (3.1).

PROPOSITION 3.3. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then

$$w(T) \leq \frac{1}{2} \max\{\| |C| + |B^*| \|, \| |B| + |C^*| \| \} + \frac{1}{2} \max\{\| |A| + |A^*| \|, \| |D| + |D^*| \| \}.$$

In particular,

$$w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leq \frac{1}{2} (\| |A| + |A^*| \| + \| |B| + |B^*| \|).$$

THEOREM 3.4. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_2 \oplus \mathcal{H}_1)$ and $r \geq 2$. Then

$$\frac{1}{2^{\frac{3}{2}(r-1)}} \max\{\| \mu \|, \| \eta \| \} \leq w^r(T) \leq \left(\frac{1}{2}\right)^{r+1} \max\{\| \mu \|, \| \eta \| \}, \tag{3.2}$$

where

$$\mu = |(C - B^*) + i(C + B^*)|^r + |(B^* - C) + i(C + B^*)|^r,$$

and

$$\eta = |(B - C^*) + i(B + C^*)|^r + |(C^* - B) + i(B + C^*)|^r.$$

Proof. Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ be a unit vector. Let $T = S + iW$ be the Cartesian decomposition of T . Then applying [9, Theorem 1], we have

$$w^2(T) \geq \frac{1}{2} \|(S \pm W)^2\|.$$

Therefore

$$w^r(T) \geq 2^{-\frac{r}{2}} \|(S \pm W)^2\|^{\frac{r}{2}} = 2^{-\frac{r}{2}} \| |S \pm W|^r \|,$$

and so

$$\begin{aligned} 2w^r(T) &\geq 2^{-\frac{r}{2}} (\| |S + W|^r \| + \| |S - W|^r \|) \\ &\geq 2^{-\frac{r}{2}} \| |S + W|^r + |S - W|^r \| \\ &\geq 2^{-\frac{r}{2}-1} | \langle (|S + W|^r + |S - W|^r)X, X \rangle | \\ &= 2^{-\frac{r}{2}-1} \left| \left\langle \begin{bmatrix} (\frac{1}{2})^r \mu & 0 \\ 0 & (\frac{1}{2})^r \eta \end{bmatrix} X, X \right\rangle \right|, \end{aligned}$$

where

$$\mu = |(C - B^*) + i(C + B^*)|^r + |(B^* - C) + i(C + B^*)|^r,$$

and

$$\eta = |(B - C^*) + i(B + C^*)|^r + |(C^* - B) + i(B + C^*)|^r.$$

Taking the supremum over $X \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with $\|X\| = 1$ in the above inequality and applying the numerical radius of diagonal matrices, we deduce the first inequality.

For the second inequality, we have

$$\begin{aligned} |\langle TX, X \rangle|^r &= (\langle SX, X \rangle^2 + \langle WX, X \rangle^2)^{\frac{r}{2}} \\ &= 2^{-\frac{r}{2}} (\langle (S + W)X, X \rangle^2 + \langle (S - W)X, X \rangle^2)^{\frac{r}{2}} \\ &\leq 2^{-\frac{r}{2}} 2^{\frac{r}{2}-1} (|\langle (S + W)X, X \rangle|^r + |\langle (S - W)X, X \rangle|^r) \\ &\quad \text{(since } f(t) = t^{\frac{r}{2}} \text{ is convex)} \\ &\leq \frac{1}{2} (\langle |S + W|X, X \rangle^r + \langle |S - W|X, X \rangle^r) \\ &\leq \frac{1}{2} (\langle |S + W|^r X, X \rangle + \langle |S - W|^r X, X \rangle) \\ &= \frac{1}{2} \langle (|S + W|^r + |S - W|^r)X, X \rangle \\ &= \frac{1}{2} \left\langle \begin{bmatrix} (\frac{1}{2})^r \mu & 0 \\ 0 & (\frac{1}{2})^r \eta \end{bmatrix} X, X \right\rangle. \end{aligned}$$

Now, applying the definition of numerical radius and Lemma 2.1, we get the desired inequality. \square

REMARK 3.5. If $T^2 = 0$, then $w(T) = \frac{1}{2} \|T\|$, $\|T^*T + TT^*\| = \|T\|^2$, and

$$\| |S + W|^r + |S - W|^r \| = 2^{-\frac{r}{2}+1} \|T^*T + TT^*\|^{\frac{r}{2}} = 2^{-\frac{r}{2}+1} \|T\|^r.$$

On the other hand, from $\| |S + W|^r + |S - W|^r \| = \sup_{\|X\|=1} | \langle |S + W|^r + |S - W|^r X, X \rangle |$, we conclude that $(\frac{1}{2})^r \max\{\|\mu\|, \|\eta\|\} = 2^{-\frac{r}{2}+1} \|T\|^r$. Therefore $2^{\frac{3}{2}r-1} \max\{\|\mu\|, \|\eta\|\} = 2^{-r} \|T\|^r = w^r(T)$, where μ and η are defined above.

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