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IMPROVEMENTS OF GENERALIZED HÖLDER'S INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper, we present some new improvements of generalized Hölder's inequalities, and then we obtain a new refinement of Minkowski inequality. Moreover, the obtained results are used to improve Chung's inequality and Beckenbach-type inequality proposed by Wang.

1. Introduction

The classical Hölder inequality states that if $a_k \geqslant 0,\ b_k \geqslant 0\ \ (k = 1,2,\cdots,n)$, $p \geqslant q > 1,\ \frac{1}{p}+\frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \tag{1}$$

The sign of inequality (1) is reversed for $p < 0$. (For $p < 0$, we assume that $a_k, b_k > 0$.) Hölder's inequality is of cardinal importance in mathematical analysis and in the field of applied mathematics. Due to the importance of Hölder's inequality (1), it has received considerable attention by many authors, and has motivated a large number of research papers giving it various generalizations, improvements and applications (e.g. [1, 2, 3, 7, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23]). Among various generalizations of (1), Vasić and Pečarić in [16] presented the following interesting theorem.

THEOREM A. *Let* $A_{ij} > 0\ \ (i = 1,2,\cdots,n,\ j = 1,2,\cdots,m)$.
(a) If $\beta_j \geqslant 0$, *and if* $\sum_{j=1}^{m} \beta_j \geqslant 1$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij}^{\beta_j} \leqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}\right)^{\beta_j}. \tag{2}$$

(b) If $\beta_j \leqslant 0 (j = 1,2,\cdots,m)$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij}^{\beta_j} \geqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}\right)^{\beta_j}. \tag{3}$$

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(c) If $\beta_1 > 0,\ \beta_j \leqslant 0\ \ (j = 2,3,\cdots,m)$*, and if* $\sum_{j=1}^m \beta_j \leqslant 1$*, then*

$$\sum_{i=1}^n \prod_{j=1}^m A_{ij}^{\beta_j} \geqslant \prod_{j=1}^m \left(\sum_{i=1}^n A_{ij}\right)^{\beta_j}. \tag{4}$$

The above inequalities are called as generalized Hölder's inequalities.

The main objective of this paper is to establish some new improvements of inequalities (1)–(4), and their applications. The rest of this paper is organized as follows. In Section 2, we present a series of sharpened versions of inequalities (1)–(4). And then we obtain a new refinement of Minkowski inequality. In Section 3, we apply the results to improve Chung's inequality and Beckenbach-type inequality proposed by Wang.

2. Improvements of generalized Hölder's inequalities

We begin this section with four lemmas, which will be used in the sequel.

LEMMA 2.1. *Let* $0 < X_{ij} < 1\ \ (i = 1,2,\cdots,n,\ \ j = 1,2,\cdots,m),\ m \geqslant 2$*, let* $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0,\ \sum_{j=1}^m \frac{1}{\lambda_j} \geqslant 1$ *such that* $1 - \sum_{i=1}^n X_{ij}^{\lambda_j} > 0$*, and let* $X_{i(m+1)} = X_{i1}$*,* $\lambda_{m+1} = \lambda_1$*. Then*

$$\prod_{j=1}^m \left(1 - \sum_{i=1}^n X_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \sum_{i=1}^n \prod_{j=1}^m X_{ij} \leqslant \prod_{j=1}^m \left[1 - \left(\sum_{i=1}^n X_{ij}^{\lambda_j} - \sum_{i=1}^n X_{i(j+1)}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}}. \tag{5}$$

Proof. From the hypotheses of Lemma 2.1, it is easy to see that

$$\frac{1}{\lambda_m} \geqslant \frac{1}{\lambda_{m-1}} \geqslant \ldots \geqslant \frac{1}{\lambda_2} \geqslant \frac{1}{\lambda_1} > 0,$$

and

$$\frac{1}{2\lambda_{j+1}} - \frac{1}{2\lambda_j} \geqslant 0 \quad (j = 1,2,\ldots,m-1).$$

Noting that $\left(\frac{1}{2\lambda_2} - \frac{1}{2\lambda_1}\right) + \frac{1}{2\lambda_1} + \frac{1}{2\lambda_1} + \left(\frac{1}{2\lambda_3} - \frac{1}{2\lambda_2}\right) + \frac{1}{2\lambda_2} + \frac{1}{2\lambda_2} + \cdots + \left(\frac{1}{2\lambda_m} - \frac{1}{2\lambda_{m-1}}\right) + \frac{1}{2\lambda_{m-1}} + \frac{1}{2\lambda_{m-1}} + \left(\frac{1}{2\lambda_m} - \frac{1}{2\lambda_1}\right) + \frac{1}{2\lambda_1} + \frac{1}{2\lambda_1} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_m} \geqslant 1$, from the inequality (2), we get

$$\begin{aligned}
&\prod_{j=1}^m \left[1 - \left(\sum_{i=1}^n X_{ij}^{\lambda_j} - \sum_{i=1}^n X_{i(j+1)}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}} \\
&= \prod_{j=1}^m \Bigg\{ \left[\left(1 - \sum_{i=1}^n X_{i(j+1)}^{\lambda_{j+1}}\right) + \sum_{i=1}^n X_{ij}^{\lambda_j}\right]^{\frac{1}{2\lambda_j}} \left[\left(1 - \sum_{i=1}^n X_{ij}^{\lambda_j}\right) + \sum_{i=1}^n X_{i(j+1)}^{\lambda_{j+1}}\right]^{\frac{1}{2\lambda_j}} \\
&\quad \times \left[\left(1 - \sum_{i=1}^n X_{i(j+1)}^{\lambda_{j+1}}\right) + \sum_{i=1}^n X_{i(j+1)}^{\lambda_{j+1}}\right]^{\frac{1}{2\lambda_{j+1}} - \frac{1}{2\lambda_j}} \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \prod_{j=1}^{m}\left\{\left(1-\sum_{i=1}^{n} X_{i(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(1-\sum_{i=1}^{n} X_{ij}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(1-\sum_{i=1}^{n} X_{i(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right\}\\
&\quad+\prod_{j=1}^{m}\left[\left(X_{1j}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{1(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{1(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\\
&\quad+\prod_{j=1}^{m}\left[\left(X_{2j}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{2(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{2(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\\
&\quad\cdots\\
&\quad+\prod_{j=1}^{m}\left[\left(X_{nj}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{n(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{n(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\\
&=\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n} X_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{i=1}^{n}\prod_{j=1}^{m} X_{ij}, \qquad (6)
\end{aligned}
$$

and then (5) follows. The proof of Lemma 2.1 is completed. □

LEMMA 2.2. *Let* $X_{ij} > 1$ $(i = 1,2,\cdots,n,\ j = 1,2,\cdots,m)$, $m \geqslant 2$, *let* $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m < 0$ *such that* $1-\sum_{i=1}^{n} X_{ij}^{\lambda_j} > 0$, *and let* $X_{i(m+1)} = X_{i1}$, $\lambda_{m+1} = \lambda_1$. *Then*

$$
\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n} X_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{i=1}^{n}\prod_{j=1}^{m} X_{ij} \geqslant \prod_{j=1}^{m}\left[1-\left(\sum_{i=1}^{n} X_{ij}^{\lambda_j}-\sum_{i=1}^{n} X_{i(j+1)}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}}. \qquad (7)
$$

Proof. By the same method as in Lemma 2.1, and using inequality (3), we can deduce the desired inequality (7). □

LEMMA 2.3. *Let* $0 < X_{im} < 1$, $X_{ij} > 1$ $(i = 1,2,\cdots,n,\ j = 1,2,\cdots,m-1)$, $m \geqslant 2$, *let* $\lambda_m > 0$, $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{m-1} < 0$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$ *such that* $1-\sum_{i=1}^{n} X_{ij}^{\lambda_j} > 0$, *and let* $X_{i(m+1)} = X_{i1}$, $\lambda_{m+1} = \lambda_1$. *Then*

$$
\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n} X_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{i=1}^{n}\prod_{j=1}^{m} X_{ij} \geqslant \prod_{j=1}^{m-1}\left[1-\left(\sum_{i=1}^{n} X_{ij}^{\lambda_j}-\sum_{i=1}^{n} X_{i(j+1)}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}}. \qquad (8)
$$

Proof. In view of the assumptions in Lemma 2.3, we have

$$
\frac{1}{\lambda_m} > 0, \frac{1}{\lambda_{m-1}} \leqslant \frac{1}{\lambda_{m-2}} \leqslant \ldots \leqslant \frac{1}{\lambda_2} \leqslant \frac{1}{\lambda_1} < 0
$$

and

$$
\frac{1}{2\lambda_j}-\frac{1}{2\lambda_{j-1}} \leqslant 0 \quad (j = 2,3,\ldots,m-1), \qquad \frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}} > 0.
$$

Consequently, according to $\frac{1}{2\lambda_1}+\frac{1}{2\lambda_1}+\left(\frac{1}{2\lambda_2}-\frac{1}{2\lambda_1}\right)+\frac{1}{2\lambda_2}+\frac{1}{2\lambda_2}+\left(\frac{1}{2\lambda_3}-\frac{1}{2\lambda_2}\right)+\ldots+\frac{1}{2\lambda_{m-2}}+\frac{1}{2\lambda_{m-2}}+\left(\frac{1}{2\lambda_{m-1}}-\frac{1}{2\lambda_{m-2}}\right)+\frac{1}{2\lambda_{m-1}}+\frac{1}{2\lambda_{m-1}}+\left(\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}\right)+\frac{1}{2\lambda_1} = \frac{1}{\lambda_1}+$

$\frac{1}{\lambda_2}+\ldots+\frac{1}{\lambda_m}\leqslant 1$, by using inequality (4), we obtain

$$\begin{aligned}
&\prod_{j=1}^{m-1}\left[1-\left(\sum_{i=1}^{n}X_{ij}^{\lambda_j}-\sum_{i=1}^{n}X_{i(j+1)}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}}\\
&=\left\{\prod_{j=1}^{m-1}\left[\sum_{i=1}^{n}X_{ij}^{\lambda_j}+\left(1-\sum_{i=1}^{n}X_{i(j+1)}^{\lambda_{j+1}}\right)\right]^{\frac{1}{2\lambda_j}}\right\}\\
&\quad\times\left\{\prod_{j=1}^{m-1}\left[\sum_{i=1}^{n}X_{i(j+1)}^{\lambda_{j+1}}+\left(1-\sum_{i=1}^{n}X_{ij}^{\lambda_j}\right)\right]^{\frac{1}{2\lambda_j}}\right\}\\
&\quad\times\left\{\prod_{j=1}^{m-2}\left[\sum_{i=1}^{n}X_{i(j+1)}^{\lambda_{j+1}}+\left(1-\sum_{i=1}^{n}X_{i(j+1)}^{\lambda_{j+1}}\right)\right]^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right\}\\
&\quad\times\left[\sum_{i=1}^{n}X_{im}^{\lambda_m}+\left(1-\sum_{i=1}^{n}X_{im}^{\lambda_m}\right)\right]^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left[\sum_{i=1}^{n}X_{i1}^{\lambda_1}+\left(1-\sum_{i=1}^{n}X_{i1}^{\lambda_1}\right)\right]^{\frac{1}{2\lambda_1}}\\
&\leqslant\left\{\prod_{j=1}^{m-2}\left[\left(X_{1j}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{1(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{1(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(X_{1(m-1)}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_{1m}^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_{1m}^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(X_{11}^{\lambda_1}\right)^{\frac{1}{2\lambda_1}}\\
&\quad+\left\{\prod_{j=1}^{m-2}\left[\left(X_{2j}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{2(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{2(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(X_{2(m-1)}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_{2m}^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_{2m}^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(X_{21}^{\lambda_1}\right)^{\frac{1}{2\lambda_1}}\\
&\quad+\cdots\\
&\quad+\left\{\prod_{j=1}^{m-2}\left[\left(X_{nj}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{n(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{n(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(X_{n(m-1)}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_{nm}^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_{nm}^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(X_{n1}^{\lambda_1}\right)^{\frac{1}{2\lambda_1}}\\
&\quad+\left\{\prod_{j=1}^{m-2}\left[\left(1-\sum_{i=1}^{n}X_{i(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(1-\sum_{i=1}^{n}X_{ij}^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(1-\sum_{i=1}^{n}X_{i(j+1)}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(1-\sum_{i=1}^{n}X_{im}^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(1-\sum_{i=1}^{n}X_{i(m-1)}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\\
&\quad\times\left(1-\sum_{i=1}^{n}X_{im}^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(1-\sum_{i=1}^{n}X_{i1}^{\lambda_1}\right)^{\frac{1}{2\lambda_1}}\\
&=\prod_{j=1}^{m}X_j+\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}.
\end{aligned}\tag{9}$$

The proof of Lemma 2.3 is completed. □

LEMMA 2.4. [5] *If* $x > -1$, $\alpha > 1$ *or* $\alpha < 0$, *then*

$$(1+x)^{\alpha} \geqslant 1 + \alpha x. \tag{10}$$

The inequality is reversed for $0 \leqslant \alpha \leqslant 1$.

Next, we present new improvements of inequalities (2), (3) and (4).

THEOREM 2.5. *Let* $A_{ij} \geqslant 0$ $(i = 1,2,\cdots,n,\ j = 1,2,\cdots,m)$, *let* s *be any given natural number* $(1 \leqslant s \leqslant n)$, *and let* $m \geqslant 2$, $\lambda_{m+1} = \lambda_1$, $A_{i(m+1)} = A_{i1}$.

(a) If $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, *then*

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} &\leqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \prod_{j=1}^{m}\left[1-\left(\frac{A_{sj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}-\frac{A_{s(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n} A_{k(j+1)}^{\lambda_{j+1}}}\right)^2\right]^{\frac{1}{2\lambda_j}}\\ &\leqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}.\end{aligned} \tag{11}$$

(b) If $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m < 0$, *then*

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} &\geqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \prod_{j=1}^{m}\left[1-\left(\frac{A_{sj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}-\frac{A_{s(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n} A_{k(j+1)}^{\lambda_{j+1}}}\right)^2\right]^{\frac{1}{2\lambda_j}}\\ &\geqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}.\end{aligned} \tag{12}$$

(c) If $\lambda_m > 0$, $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{m-1} < 0$, *and if* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$, *then*

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} &\geqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \prod_{j=1}^{m-1}\left[1-\left(\frac{A_{sj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}-\frac{A_{s(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n} A_{k(j+1)}^{\lambda_{j+1}}}\right)^2\right]^{\frac{1}{2\lambda_j}}\\ &\geqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}.\end{aligned} \tag{13}$$

Proof. (a). For any given natural number s $(1 \leqslant s \leqslant n)$, from the following substitution:

$$X_{ij} = \left(\frac{A_{ij}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}} \quad (i = 1,2,\cdots,n,\ j = 1,2,\cdots,m), \tag{14}$$

we have

$$0 < X_{ij} < 1, \quad 1 - \sum_{1 \leqslant i \leqslant n,\quad i \neq s} X_{ij}^{\lambda_j} > 0.$$

Consequently, by using the substitution (14) and inequality (5), we find

$$\begin{aligned}
&\prod_{j=1}^{m}\left[1-\sum_{1\leqslant i\leqslant n, i\neq s}\left(\frac{A_{ij}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}\right)\right]^{\frac{1}{\lambda_j}}+\sum_{1\leqslant i\leqslant n, i\neq s}\left(\prod_{j=1}^{m}\frac{A_{ij}}{\left(\sum_{k=1}^{n}A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}\right)\\
&\leqslant\prod_{j=1}^{m}\left\{1-\left[\sum_{1\leqslant i\leqslant n, i\neq s}\left(\frac{A_{ij}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}\right)-\sum_{1\leqslant i\leqslant n, i\neq s}\left(\frac{A_{i(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n}A_{k(j+1)}^{\lambda_{j+1}}}\right)\right]^{2}\right\}^{\frac{1}{2\lambda_j}}, \qquad (15)
\end{aligned}$$

and then we obtain

$$\begin{aligned}
&\frac{\prod_{j=1}^{m}A_{sj}}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n}A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}+\frac{\sum_{1\leqslant i\leqslant n, i\neq s}\prod_{j=1}^{m}A_{ij}}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n}A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}\\
&\leqslant\prod_{j=1}^{m}\left[1-\left(\frac{A_{sj}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}-\frac{A_{s(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n}A_{k(j+1)}^{\lambda_{j+1}}}\right)^{2}\right]^{\frac{1}{2\lambda_j}}, \qquad (16)
\end{aligned}$$

which implies

$$\frac{\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n}A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}\leqslant\prod_{j=1}^{m}\left[1-\left(\frac{A_{sj}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}-\frac{A_{s(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n}A_{k(j+1)}^{\lambda_{j+1}}}\right)^{2}\right]^{\frac{1}{2\lambda_j}}. \qquad (17)$$

So, the desired inequality (11) is valid.

(b). In view of

$$X_{ij}=\left(\frac{A_{ij}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}}>1$$

for all $\lambda_j<0\ \ (j=1,2,\cdots,m)$, and

$$1-\sum_{1\leqslant i\leqslant n,\ \ i\neq s}X_{ij}^{\lambda_j}>0,$$

using similar reasoning as in Case (a), but applying Lemma 2.2 in place of Lemma 2.1, we immediately obtain inequality (12).

(c). Noting that

$$0<X_{im}=\left(\frac{A_{im}^{\lambda_m}}{\sum_{k=1}^{n}A_{km}^{\lambda_m}}\right)^{\frac{1}{\lambda_m}}<1,\quad X_{ij}=\left(\frac{A_{ij}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}}>1$$

for $\lambda_m>0,\ \lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_{m-1}<0$, and

$$1-\sum_{1\leqslant i\leqslant n, i\neq s}X_{ij}^{\lambda_j}>0,$$

making the similar methods as in Case (a), but using Lemma 2.3 in place of Lemma 2.1, we get inequality (13).

The proof of Theorem 2.5 is completed. □

Letting $s=1$ in (11), (12) and (13), respectively, we obtain the following corollary.

COROLLARY 2.6. *Let* $A_{ij} \geqslant 0$ $(i=1,2,\cdots,n,\ j=1,2,\cdots,m)$, *let* $m \geqslant 2$, *and let* $\lambda_{m+1}=\lambda_1$, $A_{i(m+1)}=A_{i1}$.

(a) If $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$, *and if* $\sum_{j=1}^{m}\frac{1}{\lambda_j} \geqslant 1$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \leqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \prod_{j=1}^{m}\left[1-\left(\frac{A_{1j}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}-\frac{A_{1(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n}A_{k(j+1)}^{\lambda_{j+1}}}\right)^2\right]^{\frac{1}{2\lambda_j}}$$
$$\leqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \tag{18}$$

(b) If $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m < 0$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \geqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \prod_{j=1}^{m}\left[1-\left(\frac{A_{1j}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}-\frac{A_{1(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n}A_{k(j+1)}^{\lambda_{j+1}}}\right)^2\right]^{\frac{1}{2\lambda_j}}$$
$$\geqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \tag{19}$$

(c) If $\lambda_m > 0,\ \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{m-1} < 0,\ \sum_{j=1}^{m}\frac{1}{\lambda_j} \leqslant 1$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \geqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \prod_{j=1}^{m-1}\left[1-\left(\frac{A_{1j}^{\lambda_j}}{\sum_{k=1}^{n}A_{kj}^{\lambda_j}}-\frac{A_{1(j+1)}^{\lambda_{j+1}}}{\sum_{k=1}^{n}A_{k(j+1)}^{\lambda_{j+1}}}\right)^2\right]^{\frac{1}{2\lambda_j}}$$
$$\geqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \tag{20}$$

In particular, putting $m=2$, $s=1$, $\lambda_1=p$, $\lambda_2=q$, $A_{i1}=a_i$, $A_{i2}=b_i$ in Corollary 2.6, we have the new improvements of Hölder's inequality as follows.

COROLLARY 2.7. *Let* $a_i \geqslant 0, b_i \geqslant 0 (i=1,2,\cdots,n)$. *If* $p \geqslant q > 0$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\sum_{i=1}^{n}a_ib_i \leqslant \left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}$$
$$\times\left[1-\left(\frac{a_1^p}{\sum_{k=1}^{n}a_k^p}-\frac{b_1^q}{\sum_{k=1}^{n}b_k^q}\right)^2\right]^{\frac{1}{2p}}\left[1-\left(\frac{a_1^p}{\sum_{k=1}^{n}a_k^p}-\frac{b_1^q}{\sum_{k=1}^{n}b_k^q}\right)^2\right]^{\frac{1}{2q}}$$

$$= \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \left[1 - \left(\frac{a_1^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_1^q}{\sum_{k=1}^{n} b_k^q}\right)^2\right]^{\frac{1}{2}}$$
$$\leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}. \tag{21}$$

If $p<0,\ q>0,\ \frac{1}{p}+\frac{1}{q}=1$, *then*

$$\sum_{i=1}^{n} a_i b_i \geqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \left[1 - \left(\frac{a_1^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_1^q}{\sum_{k=1}^{n} b_k^q}\right)^2\right]^{\frac{1}{2p}}. \tag{22}$$

Moreover, from Corollary 2.7, we get the following refinement of Minkowski inequality.

COROLLARY 2.8. *Let* $a_k>0,\ b_k>0\ \ (k=1,2,\cdots,n)$.
(I) If $p>1$, *then*

$$\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{\frac{1}{p}} \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}$$
$$-\frac{1}{2}\left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\frac{a_1^p}{\sum_{k=1}^{n} a_k^p} - \frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2$$
$$-\frac{1}{2}\left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \left(\frac{b_1^p}{\sum_{k=1}^{n} b_k^p} - \frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2. \tag{23}$$

(II) If $0<p<1$, *then*

$$\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{1/p} \geqslant \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}$$
$$+\frac{1-p}{2p}\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\frac{a_1^p}{\sum_{k=1}^{n} a_k^p} - \frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2$$
$$+\frac{1-p}{2p}\left(\sum_{k=1}^{n} b_k^p\right)^{1/p} \left(\frac{b_1^p}{\sum_{k=1}^{n} b_k^p} - \frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2. \tag{24}$$

Proof. (I). For $p>1$, we write

$$\sum_{k=1}^{n}(a_k+b_k)^p = \sum_{k=1}^{n} a_k(a_k+b_k)^{p-1} + \sum_{k=1}^{n} b_k(a_k+b_k)^{p-1},$$

and using inequality 21, we have

$$\begin{aligned}\sum_{k=1}^{n}(a_k+b_k)^p \leqslant & \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{\frac{p-1}{p}}\\ &\times\left[1-\left(\frac{a_1^p}{\sum_{k=1}^{n}a_k^p}-\frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right]^{\frac{1}{2}}\\ &+\left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{\frac{p-1}{p}}\\ &\times\left[1-\left(\frac{b_1^p}{\sum_{k=1}^{n}b_k^p}-\frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right]^{\frac{1}{2}}.\end{aligned} \tag{25}$$

Then, by using Lemma 2.4, we obtain

$$\begin{aligned}\sum_{k=1}^{n}(a_k+b_k)^p \leqslant & \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{\frac{p-1}{p}}\\ &\times\left[1-\frac{1}{2}\left(\frac{a_1^p}{\sum_{k=1}^{n}a_k^p}-\frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right]\\ &+\left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{\frac{p-1}{p}}\\ &\times\left[1-\frac{1}{2}\left(\frac{b_1^p}{\sum_{k=1}^{n}b_k^p}-\frac{(a_1+b_1)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right].\end{aligned}$$

Dividing both sides by $\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{\frac{p-1}{p}}$, we obtain the desired inequality (23).

(II). For $0<p<1$, by inequality (22) and in the same way, we obtained the desired inequality (24). The proof of Corollary 2.8 is completed. □

3. Applications

In this section, we provide two applications of our new inequalities. Firstly, we give here a refined and generalized version of Chung's inequality.

The famous Chung inequality [6] states that if $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$ and $\sum_{j=1}^{k} a_j \leqslant \sum_{j=1}^{k} b_j$ for $k=1,2,\ldots,n$, then

$$\sum_{j=1}^{n} a_j^2 \leqslant \sum_{j=1}^{n} b_j^2. \tag{26}$$

THEOREM 3.1. *Let* $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$, *and let* $\sum_{j=1}^k a_j \leqslant \sum_{j=1}^k b_j$ *for* $k = 1,2,\ldots,n$. *Then, for* $p > 1$ *we have*

$$\sum_{j=1}^n a_j^p \leqslant \left(\sum_{j=1}^n b_j^p\right)\left[1-\left(\frac{a_1^p}{\sum_{j=1}^n a_j^p}-\frac{b_1^p}{\sum_{j=1}^n b_j^p}\right)^2\right]^{\frac{p}{2}} \leqslant \sum_{j=1}^n b_j^2. \tag{27}$$

Proof. From the hypotheses in Theorem 3.1 we find that

$$\begin{aligned}\sum_{j=1}^n a_j^p &= \sum_{k=1}^{n-1}\left[\left(a_k^{p-1}-a_{k+1}^{p-1}\right)\sum_{j=1}^k a_j\right]+a_n^{p-1}\sum_{j=1}^n a_j \\ &\leqslant \sum_{k=1}^{n-1}\left[\left(a_k^{p-1}-a_{k+1}^{p-1}\right)\sum_{j=1}^k b_j\right]+a_n^{p-1}\sum_{j=1}^n b_j \\ &= \sum_{j=1}^n a_j^{p-1}b_j. \end{aligned}\tag{28}$$

By using inequality (21), we have

$$\sum_{j=1}^n a_j^{p-1}b_j \leqslant \left(\sum_{j=1}^n a_j^p\right)^{\frac{p-1}{p}}\left(\sum_{j=1}^n b_j^p\right)^{\frac{1}{p}}\left[1-\left(\frac{a_1^p}{\sum_{j=1}^n a_j^p}-\frac{b_1^p}{\sum_{j=1}^n b_j^p}\right)^2\right]^{\frac{1}{2}}. \tag{29}$$

Hence, combining inequalities (28) and (29) yields inequality (27). The proof of Theorem 3.1 is completed. □

Nextly, we present a new refinement of Beckenbach-type inequality proposed by Wang. The famous Beckenbach inequality [4] has been generalized and refined in several directions (e.g. [8]). In [17], Wang obtained an interesting Beckenbach-type inequality as follows.

THEOREM B. *Let* $f(x)$, $g(x)$ *be positive integrable functions defined on* $[0,T]$, *and let* a, b, c *be positive numbers. If* $p \geqslant q > 0$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\frac{\left(a+c\int_0^T \varphi^p(x)\mathrm{dx}\right)^{\frac{1}{p}}}{b+c\int_0^T \varphi(x)g(x)\mathrm{dx}} \leqslant \frac{\left(a+c\int_0^T f^p(x)\mathrm{dx}\right)^{\frac{1}{p}}}{b+c\int_0^T f(x)g(x)\mathrm{dx}}, \tag{30}$$

where $\varphi(x) = (\frac{ag(x)}{b})^{\frac{q}{p}}$.

THEOREM 3.2. *Let* $f(x)$, $g(x)$ *be positive integrable functions defined on* $[0,T]$, *and let* a, b, c *be positive numbers. If* $p \geqslant q > 0$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}\frac{\left(a+c\int_0^T \varphi^p(x)\mathrm{dx}\right)^{\frac{1}{p}}}{b+c\int_0^T \varphi(x)g(x)\mathrm{dx}} \leqslant{}& \frac{\left(a+c\int_0^T f^p(x)\mathrm{dx}\right)^{\frac{1}{p}}}{b+c\int_0^T f(x)g(x)\mathrm{dx}} \\ &\times\left[1-\frac{1}{2}\left(\frac{a}{a+c\int_0^T f^p(x)\mathrm{dx}}-\frac{a^{-\frac{q}{p}}b^q}{a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{dx}}\right)^2\right],\end{aligned}\tag{31}$$

where $\varphi(x) = \left(\frac{ag(x)}{b}\right)^{\frac{q}{p}}$.

Proof. A simple calculation gives

$$\frac{\left(a+c\int_0^T \varphi^p(x)\mathrm{dx}\right)^{\frac{1}{p}}}{b+c\int_0^T \varphi(x)g(x)\mathrm{dx}} = \left(a^{-\frac{q}{p}}b^q + c\int_0^T g^q(x)\mathrm{dx}\right)^{-\frac{1}{q}}. \tag{32}$$

By using Hölder's inequality 1 and inequality 21 we get

$$\begin{aligned} b+c\int_0^T f(x)g(x)\mathrm{dx} &\leqslant b+c\Big(\int_0^T f^p(x)\mathrm{dx}\Big)^{\frac{1}{p}}\Big(\int_0^T g^q(x)\mathrm{dx}\Big)^{\frac{1}{q}} \\ &= a^{\frac{1}{p}}(ba^{-\frac{1}{p}})+c\Big(\int_0^T f^p(x)\mathrm{dx}\Big)^{\frac{1}{p}}\Big(\int_0^T g^q(x)\mathrm{dx}\Big)^{\frac{1}{q}} \\ &\leqslant \left(a+c\int_0^T f^p(x)\mathrm{dx}\right)^{\frac{1}{p}}\left(a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{dx}\right)^{\frac{1}{q}} \\ &\quad \times\left[1-\left(\frac{a}{a+c\int_0^T f^p(x)\mathrm{dx}}-\frac{a^{-\frac{q}{p}}b^q}{a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{dx}}\right)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{33}$$

which implies

$$\begin{aligned} &\left(a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{dx}\right)^{-\frac{1}{q}} \\ &\leqslant \frac{\left(a+c\int_0^T f^p(x)\mathrm{dx}\right)^{\frac{1}{p}}}{b+c\int_0^T f(x)g(x)\mathrm{dx}} \\ &\quad \times\left[1-\left(\frac{a}{a+c\int_0^T f^p(x)\mathrm{dx}}-\frac{a^{-\frac{q}{p}}b^q}{a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{dx}}\right)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{34}$$

Moreover, from Lemma 2.4 we have

$$\begin{aligned} &\left(a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{dx}\right)^{-\frac{1}{q}} \\ &\leqslant \frac{\left(a+c\int_0^T f^p(x)\mathrm{dx}\right)^{\frac{1}{p}}}{b+c\int_0^T f(x)g(x)\mathrm{dx}} \\ &\quad \times\left[1-\frac{1}{2}\left(\frac{a}{a+c\int_0^T f^p(x)\mathrm{dx}}-\frac{a^{-\frac{q}{p}}b^q}{a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{dx}}\right)^2\right]. \end{aligned} \tag{35}$$

Hence, combining inequalities (32) and (35) yields inequality (31). The proof of Theorem 3.2 is completed. □

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