Journal of Mathematical Inequalities
Volume 12, Number 2 (2018), 473–485 doi:10.7153/jmi-2018-12-35

A PROOF OF AN OPEN PROBLEM OF YUSUKE NISHIZAWA FOR A POWER–EXPONENTIAL FUNCTION

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(*Communicated by T. Burić*)

Abstract. This paper presents a proof of the following conjecture, stated by Yusuke Nishizawa in [Appl. Math. Comput. 269, (2015), 146–154.]: for $0 < x < \pi/2$ the inequality $\dfrac{\sin x}{x} > \left(\dfrac{2}{\pi} + \dfrac{\pi-2}{\pi^3}(\pi^2-4x^2)\right)^{\theta(x)}$ holds, where $\theta(x) = -\dfrac{(\pi^3-24\pi+48)x^3}{3(\pi-2)\pi^3} + \dfrac{\pi^3}{24(\pi-2)}$.

1. Introduction

In [1], Nishizawa proved the following power-exponential inequalities:

THEOREM 1. *For* $0 < x < \pi/2$, *we have*

$$\left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2-4x^2)\right)^{\theta_1} < \frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2-4x^2)\right)^{\vartheta_1}$$

with the best possible constants $\theta_1 = 1$ *and* $\vartheta_1 = 0$.

THEOREM 2. *For* $0 < x < \pi/2$, *we have*

$$\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2-4x^2)\right)^{\theta_2} < \frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2-4x^2)\right)^{\vartheta_2}$$

with the best possible constants $\theta_2 = \pi^3/(24(\pi-2)) = 1.13169\ldots$ *and* $\vartheta_2 = 1$.

THEOREM 3. *For* $0 < x < \pi/2$, *we have*

$$\frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2-4x^2)\right)^{\vartheta(x)}$$

where $\vartheta(x) = \dfrac{4x^2}{\pi^2}$.

Mathematics subject classification (2010): 26D05, 41A10.
Keywords and phrases: Mixed logarithmic-trigonometric polynomial functions and inequalities.
Authors were supported in part by Serbian Ministry of Education, Science and Technological Development, Projects ON 174032, III 44006 and TR 32023.

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Paper JMI-12-35

THE OPEN PROBLEM OF YUSUKE NISHIZAWA. Considering the previous theorems, Nishizawa stated the following open problem (Problem 3.1 of [1]):
For $0 < x < \pi/2$*, we have*

$$\frac{\sin x}{x} > \left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)} \tag{1}$$

where $\theta(x)$ *is the function of* x *and* $\theta(x) = -\dfrac{(\pi^3 - 24\pi + 48)x^3}{3(\pi-2)\pi^3} + \dfrac{\pi^3}{24(\pi-2)}$.

This paper provides a proof of Nishizawa's open problem, using approximations and methods from [4] and [14]. Let us notice that method from paper [14] relates to proving mixed trigonometric polynomial inequalities

$$f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x > 0, \tag{2}$$

where $\alpha_i \in \mathbb{R}$, $p_i, q_i, r_i \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $x \in (\delta_1, \delta_2)$, $\delta_1 \leqslant 0 \leqslant \delta_2$, with $\delta_1 < \delta_2$. The function $f(x)$ is called mixed trigonometric polynomial function.

In this paper the power-exponential inequality (1) can be rewritten as one example of the inequality of the following form:

$$F(x) = f(x) + \sum_{j=1}^{m} P_j(x) \ln(f_j(x)) > 0, \tag{3}$$

where $f(x)$ and $f_j(x)$ are mixed trigonometric polynomial functions, with $f_j(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$; $P_j(x)$ is a real polynomial of degree k_j and $m \in \mathbb{N}$. The function $F(x)$ is called mixed logarithmic-trigonometric polynomial function and the inequality $F(x) > 0$ is called mixed logarithmic-trigonometric polynomial inequality.

Assuming that in mixed logarithmic-trigonometric polynomial function appear only monomials and logarithmic functions, and no trigonometric sine and cosine functions, we can call that function mixed logarithmic polynomial function. One application of mixed logarithmic polynomial functions, in purpose of proving some power-exponential inequalities, was given in the papers [1], [9], [10].

Let us assume that the degree of the zero polynomial is -1. Then, for a mixed logarithmic-trigonometric polynomial function $F(x)$ it is not difficult to show the following result:

> *The derivative* $F^{(K+1)}(x)$ *is quotient of two mixed trigonometric polynomial functions, where* $K = \max\{k_j \,|\, j = 1, \ldots, m\}$. (4)

In the proof of Nishazawa's open problem, we will use the following statement.

PROPOSITION 1. *Let* $\Phi : (0, \delta) \longrightarrow \mathbb{R}$ $(\delta > 0)$*, be a* $(K+1)$*-times differentiable function* $(K \in \mathbb{N} \cup \{0\})$ *such that*

(i) $\lim\limits_{x \to 0+} \Phi(x) \geqslant 0$,

(ii) $\lim_{x\to 0+} \Phi^{(j)}(x) \geqslant 0$ *holds true for every* $j \in \{1,2,\ldots,K\}$,
(iii) $\Phi^{(\kappa+1)}(x) > 0$ *for* $0 < x < c$ *where* $c \in (0,\delta]$.
Then, for $0 < x < c$, *inequality*

$$\Phi(x) > 0 \tag{5}$$

holds true.

In the next section the proof of Nishizawa's open problem also makes use of the fact that for the constant π and a given rational function $R(x)$, it is possible to determine either $R(\pi) > 0$ or $R(\pi) < 0$. Stated is a consequence of the fact that for an arbitrarily small $\varepsilon > 0$, there exist fractions p/q and r/s such that $p/q > \pi > r/s$ and $p/q - r/s < \varepsilon$. Fractions p/q and r/s can be chosen as two consequential convergents in the development of continued fractions of π.

2. The proof of Nishizawa's open problem

As $\frac{\sin x}{x} > 0$ and $\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2) > 0$ for $x \in (0,\pi/2)$, the power-exponential inequality (1) is equivalent to the following inequality:

$$\ln\frac{\sin x}{x} > \ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)} \tag{6}$$

or

$$\ln\frac{\sin x}{x} - \theta(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right) > 0, \tag{7}$$

where $\theta(x) = -\dfrac{(\pi^3 - 24\pi + 48)x^3}{3(\pi-2)\pi^3} + \dfrac{\pi^3}{24(\pi-2)}$.

Let us notice that the previous inequality is a mixed logarithmic-trigonometric polynomial inequality

$$F(x) = \ln\sin x - \ln x - \theta(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right) > 0 \tag{8}$$

for $x \in (0,\pi/2)$.

Let us further consider the following mixed logarithmic-trigonometric polynomial function

$$F_1(x) = \ln\sin x - \ln x - \theta_1(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right) \tag{9}$$

where $\theta_1(x) = \dfrac{(\pi^3 - 60\pi + 120)}{720(\pi-2)}x^2 + \dfrac{\pi^3}{24(\pi-2)}$.

As

$$\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2) < \frac{2}{\pi} + \frac{\pi-2}{\pi^3}\pi^2 = 1 \tag{10}$$

we can conclude that for $x \in (0, \pi/2)$:

$$\begin{aligned} & F(x) - F_1(x) \geqslant 0 \\ \Longleftrightarrow\ & \theta(x) - \theta_1(x) \geqslant 0 \\ \Longleftrightarrow\ & -\frac{1}{3(\pi-2)}\left(\frac{\pi^3-60\pi+120}{240} + \frac{\pi^3-24\pi+48}{\pi^3}x\right)x^2 \geqslant 0. \end{aligned} \tag{11}$$

As the last inequality holds for $x \in (0, c]$, where $c = -\dfrac{\left(\pi^3-60(\pi-2)\right)\pi^3}{240(\pi^3-24(\pi-2))} = 1.342\,37\ldots$ we distinguish two cases $x \in (0, c]$ or $x \in (c, \pi/2)$.

2.1. The case 1: $x \in (0, c]$

It is enough to prove that $F_1(x) > 0$ for $x \in (0, c]$. We have:

$$F_1'''(x) = \frac{2A(x)\sin^3 x + B(x)\cos x}{45x^3 C(x)\sin^3 x} \tag{12}$$

where

$$\begin{aligned} C(x) &= -64(\pi-2)^3x^6 + 48\pi^3(\pi-2)^2x^4 - 12\pi^6(\pi-2)x^2 + \pi^9 \\ &= \left(\pi^3 - 4(\pi-2)x^2\right)^3, \end{aligned} \tag{13}$$

$$B(x) = 90x^3\left(\pi^3 - 4(\pi-2)x^2\right)^3 = 90x^3C(x) \tag{14}$$

and

$$\begin{aligned} A(x) = \ & 8(\pi-2)^2(\pi^3-60\pi+120)x^8 \\ & -6(\pi-2)(\pi^6-100\pi^4+200\pi^3-480(\pi-2)^2)x^6 \\ & +3\pi^3(\pi^6-720(\pi-2)^2)x^4 \\ & +540\pi^6(\pi-2)x^2 \\ & -45\pi^9. \end{aligned} \tag{15}$$

Let us determine the sign of the polynomials $C(x), B(x)$ and $A(x)$ for $x \in (0, c]$. By substituting $t = 4(\pi-2)x^2$ for $x \in (0, c]$, the polynomial $C(x)$ can be transformed into the polynomial $C_1(t) = \left(\pi^3 - t\right)^3$ for $t \in (0, 4(\pi-2)c^2]$. Obviously, the sign of the polynomial $C_1(t)$ coincides with the sign of the polynomial $\pi^3 - t$ for $t \in (0, 4(\pi-2)c^2]$.

Since $\left(\frac{\pi^3-60(\pi-2)}{\pi^3-24(\pi-2)}\right)^2 < 1$, we have

$$\begin{aligned}
\pi^3-4(\pi-2)c^2 &= \pi^3-4(\pi-2)\left(\frac{(\pi^3-60(\pi-2))}{240(\pi^3-24(\pi-2))}\,\pi^3\right)^2 \\
&= \pi^3\left(1-(\pi-2)\frac{1}{14400}\left(\frac{(\pi^3-60(\pi-2))}{(\pi^3-24(\pi-2))}\right)^2\pi^3\right) \\
&> \pi^3\left(1-(\pi-2)\frac{1}{14400}\pi^3\right) \\
&= \pi^3\left(\frac{14400-(\pi-2)\pi^3}{14400}\right) \\
&> 0.
\end{aligned} \tag{16}$$

Therefore, we can conclude that $C_1(t)>0$ for $t\in(0,4(\pi-2)c^2]\subset(0,\pi^3)$, i.e. $C(x)>0$ for $x\in(0,c]$ and $B(x)>0$ for $x\in(0,c]$.

Let us prove that

$$A(x)<0, \tag{17}$$

for $x\in(0,c]$. We note that $A(x)$ can be written as

$$A(x)=2(\pi-2)x^6\varphi_1(x)+3\pi^3x^2\varphi_2(x)-45\pi^9, \tag{18}$$

where

$$\varphi_1(x)=4(\pi-2)(\pi^3-60\pi+120)x^2-3\left(\pi^6-100\pi^4+200\pi^3-480(\pi-2)^2\right) \tag{19}$$

and

$$\varphi_2(x)=(\pi^6-720(\pi-2)^2)x^2+180\pi^3(\pi-2). \tag{20}$$

As $\pi^3-60\pi-120<0$ and $\pi^6-720(\pi-2)^2>0$, we have the following estimation, for $x\in(0,c]$:

$$A(x)\leqslant 2(\pi-2)c^6\varphi_1(0)+3\pi^3c^2\varphi_2(c)-45\pi^9=-138097.86896\ldots<0. \tag{21}$$

In view of all the above, we can conclude that for $x\in(0,c]$:

$$C(x)>0,\ \ B(x)>0\ \text{ and }\ A(x)<0. \tag{22}$$

Now we prove that

$$g_1(x)=2A(x)\sin^3x+B(x)\cos x>0 \tag{23}$$

for $x\in(0,c]$. Let us note that $g_1(x)$ is a mixed trigonometric polynomial function, and that the proof of previous inequality will be proved applying the methods from [4] and [14]. In particular, we use the following inequalities from [14]:

$$\cos x>1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}, \qquad \left(x\in(0,c)\subseteq\left(0,\sqrt{90}\right)\right); \tag{24}$$

and

$$\sin x > x - \frac{x^3}{2!} + \frac{x^5}{5!}, \qquad \left(x \in (0,c) \subseteq \left(0,\sqrt{72}\right)\right). \tag{25}$$

Therefore, for $x \in (0,\pi/2)$, we have:

$$\begin{aligned} g_1(x) &= 2A(x)\sin^3 x + B(x)\cos x \\ &> 2A(x)\left(x - \tfrac{x^3}{2!} + \tfrac{x^5}{5!}\right)^3 + B(x)\left(1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!}\right) \\ &= \frac{x^9}{864000} P_{14}(x), \end{aligned} \tag{26}$$

where $P_{14}(x)$ is the polynomial of the 14^{th} degree, as follows:

$$\begin{aligned} &P_{14}(x) = \sum_{i=0}^{14} a_i x^i \\ &= 8\left(\pi^3 - 60\pi + 120\right)(\pi-2)^2 x^{14} \\ &\quad -6(\pi-2)\left(\pi^6 - 20\pi^4 + 40\pi^3 - 5280\pi^2 + 21120\pi - 21120\right)x^{12} \\ &\quad +3\left(\pi^9 + 40(\pi-2)\left(3\pi^6 - 214\pi^4 + 428\pi^3 - 7680\pi^2 + 30720\pi - 30720\right)\right)x^{10} \\ &\quad -20\left(9\pi^9 + (\pi-2)\left(441\pi^6 - 44320\pi^4 + 88640\pi^3 - 762240\pi^2 + 3048960\pi - 3048960\right)\right)x^8 \\ &\quad +15\left(309\pi^9 + 400(\pi-2)\left(17\pi^6 - 2552\pi^4 + 5104\pi^3 - 24576\pi^2 + 98304\pi - 98304\right)\right)x^6 \\ &\quad -300\left(215\pi^9 + 72(\pi-2)\left(13\pi^6 - 6880\pi^4 + 13760\pi^3 - 34560\pi^2 + 138240\pi - 138240\right)\right)x^4 \\ &\quad +5400\left(91\pi^9 - 80(\pi-2)\left(13\pi^6 + 1744\pi^4 - 3488\pi^3 + 1920\pi^2 - 7680\pi + 7680\right)\right)x^2 \\ &\quad -36000\pi^3\left(47\pi^6 - 1440(\pi-2)\left(\pi^3 + 20\pi - 40\right)\right). \end{aligned} \tag{27}$$

Therefore, for inequality (23) it is sufficient to prove that

$$P_{14}(x) > 0, \tag{28}$$

for $x \in (0,c]$. It is easy to check that non-zero coefficients $a_i, i \in \{14,12,10,8,6,4,2,0\}$ of the polynomial $P_{14}(x)$ satisfy the following conditions: $a_{14} < 0$, $a_{12} > 0$, $a_{10} < 0$, $a_8 > 0$, $a_6 < 0$, $a_4 > 0$, $a_2 < 0$ and $a_0 > 0$. Thus, for the proof of $P_{14}(x) = x^{12}\left(a_{14}x^2 + a_{12}\right) + x^8\left(a_{10}x^2 + a_8\right) + x^4\left(a_6x^2 + a_4\right) + \left(a_2x^2 + a_0\right) > 0$, for $x \in (0,c]$, it is sufficient and easy to check that the following inequalities hold: $a_{14}c^2 + a_{12} > 0$, $a_{10}c^2 + a_8 > 0$, $a_6c^2 + a_4 > 0$ and $a_2c^2 + a_0 > 0$. Thus we may conclude that $P_{14}(x) > 0$ and based on $F_1'''(x) = g_1(x)/\left(45x^3C(x)\sin^3 x\right) > P_{14}(x)x^9/\left(38880000x^3C(x)\sin^3 x\right)$ the inequality

$$F_1'''(x) > 0 \tag{29}$$

was proved for $x \in (0,c]$. Let us notice that

$$\lim_{x \to +0} F_1''(x) = 0, \tag{30}$$

$$\lim_{x \to +0} F_1'(x) = 0 \tag{31}$$

and

$$\lim_{x\to+0} F_1(x) = 0. \tag{32}$$

Then based on Proposition 1 we can conclude that

$$F_1(x) > 0 \text{ for } x \in (0,c] \tag{33}$$

which also proves that $F(x) > 0$ for $x \in (0,c]$.

2.2. The case 2: $x \in (c,\pi/2)$

In this subsection we prove that $F(x) > 0$ for $x \in \left(c, \frac{\pi}{2}\right)$. Let us consider the mixed logarithmic-trigonometric polynomial function

$$G(x) = F\left(\frac{\pi}{2} - x\right) = \ln\cos x - \ln\left(\frac{\pi}{2} - x\right) - \omega(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}\left(\pi^2 - 4\left(\frac{\pi}{2} - x\right)^2\right)\right), \tag{34}$$

where $\omega(x) = \theta\left(\frac{\pi}{2} - x\right) = -\frac{(\pi^3 - 24\pi + 48)\left(\frac{\pi}{2} - x\right)^3}{3(\pi-2)\pi^3} + \frac{\pi^3}{24\pi - 48}$ and $x \in \left(0, \frac{\pi}{2}\right)$. We have to prove the following mixed logarithmic-trigonometric polynomial inequality

$$G(x) > 0 \tag{35}$$

for $x \in (0,c_1)$, where $c_1 = \frac{\pi}{2} - c = 0.22842\ldots$. Let us further consider the new mixed logarithmic-trigonometric polynomial function

$$G_1(x) = \ln\cos x - \ln\left(\frac{\pi}{2} - x\right) - \omega_1(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}\left(\pi^2 - 4\left(\frac{\pi}{2} - x\right)^2\right)\right), \tag{36}$$

where $\omega_1(x) = \frac{x}{5} + 1$ and $x \in (0,c_1)$. Based on the inequality (10) we can conclude that for $x \in (0,c_1)$:

$$\begin{aligned} & G(x) > G_1(x) \\ \iff & \omega(x) > \omega_1(x) \\ \iff & \frac{x}{(\pi-2)\pi^3}\Big(\left(20\pi^3 - 480\pi + 960\right)x^2 \\ & \qquad + \left(-30\pi^4 + 720\pi^2 - 1440\pi\right)x \\ & \qquad + 15\pi^5 - 12\pi^4 - 336\pi^3 + 720\pi^2\Big) > 0. \end{aligned} \tag{37}$$

As the last inequality holds for $x \in (0,c_1)$, in order to prove (35), it is enough to prove

$$G_1(x) > 0, \tag{38}$$

for $x \in (0,c_1)$. We have:

$$G_1''(x) = \frac{P(x)\cos^2 x - \sin^2 x Q(x)}{Q(x)\cos^2 x} \tag{39}$$

where

$$\begin{aligned} P(x) = & \left(-80\pi^2+320\pi-320\right)x^6 \\ & +\left(240\pi^3-992\pi^2+1088\pi-128\right)x^5 \\ & +\left(-260\pi^4+1216\pi^3-1344\pi^2-576\pi+960\right)x^4 \\ & +\left(120\pi^5-728\pi^4+720\pi^3+1472\pi^2-1920\pi\right)x^3 \\ & +\left(-20\pi^6+212\pi^5-132\pi^4-1184\pi^3+1440\pi^2\right)x^2 \\ & +\left(-24\pi^6-16\pi^5+408\pi^4-480\pi^3\right)x \\ & +11\pi^6-52\pi^5+60\pi^4 \end{aligned} \tag{40}$$

and

$$Q(x)=5\,(\pi-2x)^2\left((-2\pi+4)x^2+\left(2\pi^2-4\pi\right)x+\pi^2\right)^2. \tag{41}$$

Obviously, $Q(x)>0$ for $x\in(0,c_1)$. Let us prove that $P(x)>0$ for $x\in(0,c_1)$. Note that $P(x)$ can be written as

$$P(x)=\phi_1(x)+4x^3(\pi-2)\left((-20\pi+40)x^3+\phi_2(x)\right), \tag{42}$$

where

$$\begin{aligned} \phi_1(x) = & \left(-20\pi^6+212\pi^5-132\pi^4-1184\pi^3+1440\pi^2\right)x^2 \\ & +\left(-24\pi^6-16\pi^5+408\pi^4-480\pi^3\right)x+\left(11\pi^6-52\pi^5+60\pi^4\right) \end{aligned} \tag{43}$$

and

$$\begin{aligned} \phi_2(x) = & \left(60\pi^2-128\pi+16\right)x^2+\left(-65\pi^3+174\pi^2+12\pi-120\right)x \\ & +\left(30\pi^4-122\pi^3-64\pi^2+240\pi\right) \end{aligned} \tag{44}$$

are quadratic trinomials. Let us denote by y_1 the minimum of the trinomial $\phi_1(x)$ and by y_2 the minimum of the trinomial $\phi_2(x)$ over $[0,c_1]$ respectively. It then becomes possible to verify that $y_1>271$ and $y_2=\phi_2(c_1)>-815$. Thus

$$\begin{aligned} P(x) & \geqslant y_1+4x^3(\pi-2)\Big((-20\pi+40)x^3+y_2\Big) \\ & > 271+4x^3(\pi-2)\Big((40-20\pi)x^3-815\Big) \\ & > 271-4\left(\frac{23}{100}\right)^3(\pi-2)\left((20\pi-40)\left(\frac{23}{100}\right)^3+815\right) \\ & > 225>0, \end{aligned} \tag{45}$$

for $x\in\left(0,\frac{23}{100}\right)$. Therefore $P(x)>0$ for $x\in(0,c_1)\subset\left(0,\frac{23}{100}\right)$.

Now we prove that:

$$g_2(x)=P(x)\cos^2 x-Q(x)\sin^2 x>0 \tag{46}$$

for $x \in (0, c_1)$. Let us note that $g_2(x)$ is a mixed trigonometric polynomial function, and that the proof of previous inequality will be proved applying the methods from [4] and [14]. In particular, we use the following inequalities from [14]:

$$\cos x > 1 - \frac{x^2}{2}, \qquad \big(x \in (0, c_1) \subseteq \big(0, \sqrt{30}\big)\big), \tag{47}$$

and

$$\sin x < x, \qquad \big(x \in (0, c_1) \subseteq \big(0, \sqrt{20}\big)\big). \tag{48}$$

Therefore

$$g_2(x) > T_{10}(x) = P(x)\left(1 - \frac{x^2}{2}\right)^2 - Q(x)x^2 \tag{49}$$

for $x \in (0, c_1)$ and it is enough to prove

$$T_{10}(x) > 0, \tag{50}$$

for $x \in (0, c_1)$. For the polynomial

$$\begin{aligned} T_{10}(x) = {} & \big(-20\pi^2 + 80\pi - 80\big)x^{10} \\ & + \big(60\pi^3 - 248\pi^2 + 272\pi - 32\big)x^9 \\ & + \big(-65\pi^4 + 304\pi^3 - 336\pi^2 - 144\pi + 240\big)x^8 \\ & + \big(30\pi^5 - 182\pi^4 + 180\pi^3 + 400\pi^2 - 608\pi + 128\big)x^7 \\ & + \big(-5\pi^6 + 53\pi^5 - 33\pi^4 - 392\pi^3 + 424\pi^2 + 896\pi - 1280\big)x^6 \\ & + \big(-6\pi^6 - 4\pi^5 + 190\pi^4 + 200\pi^3 - 2464\pi^2 + 3008\pi - 128\big)x^5 \\ & + \big(960 + 2400\pi^3 - 2784\pi^2 - 576\pi - 413\pi^4 + 11/4\pi^6 - 45\pi^5\big)x^4 \\ & + \big(4\pi^6 + 196\pi^5 - 1136\pi^4 + 1200\pi^3 + 1472\pi^2 - 1920\pi\big)x^3 \\ & + \big(-36\pi^6 + 264\pi^5 - 192\pi^4 - 1184\pi^3 + 1440\pi^2\big)x^2 \\ & + \big(-24\pi^6 - 16\pi^5 + 408\pi^4 - 480\pi^3\big)x \\ & + 11\pi^6 - 52\pi^5 + 60\pi^4 \end{aligned} \tag{51}$$

we form the polynomials:

$$\begin{aligned} \psi_1(x) &= \Big((-20\pi^2 + 80\pi - 80)x + (60\pi^3 - 248\pi^2 + 272\pi - 32)\Big)x^9, \\ \psi_2(x) &= \Big((-65\pi^4 + 304\pi^3 - 336\pi^2 - 144\pi + 240)x^2 \\ &\quad +(30\pi^5 - 182\pi^4 + 180\pi^3 + 400\pi^2 - 608\pi + 128)x \\ &\quad +(-5\pi^6 + 53\pi^5 - 33\pi^4 - 392\pi^3 + 424\pi^2 + 896\pi - 1280)\Big)x^6, \end{aligned}$$

$$
\begin{aligned}
\psi_3(x) = &\Big((-6\pi^6-4\pi^5+190\pi^4+200\pi^3-2464\pi^2+3008\pi-128)x^2\\
&+(960+2400\pi^3-2784\pi^2-576\pi-413\pi^4+\tfrac{11}{4}\pi^6-45\pi^5)x\\
&+(4\pi^6+196\pi^5-1136\pi^4+1200\pi^3+1472\pi^2-1920\pi)\Big)x^3,\\
\psi_4(x) = &\Big((-36\pi^6+264\pi^5-192\pi^4-1184\pi^3+1440\pi^2)x^2\\
&+(-24\pi^6-16\pi^5+408\pi^4-480\pi^3)x\\
&+(11\pi^6-52\pi^5+60\pi^4)\Big).
\end{aligned}
\tag{52}
$$

It is easy to check that

$$
\begin{aligned}
&\psi_1(x) > 0,\\
&\psi_2(x) > 0,\\
&\psi_3(x) > \Big((-6\pi^6-4\pi^5+190\pi^4+200\pi^3-2464\pi^2+3008\pi-128)\left(\tfrac{23}{100}\right)^2\\
&\quad+(960+2400\pi^3-2784\pi^2-576\pi-413\pi^4+\tfrac{11}{4}\pi^6-45\pi^5)\left(\tfrac{23}{100}\right)\\
&\quad+(4\pi^6+196\pi^5-1136\pi^4+1200\pi^3+1472\pi^2-1920\pi)\Big)\left(\tfrac{23}{100}\right)^3\\
&\quad> -27,\\
&\psi_4(x) > \Big((-36\pi^6+264\pi^5-192\pi^4-1184\pi^3+1440\pi^2)\left(\tfrac{23}{100}\right)^2\\
&\quad+(-24\pi^6-16\pi^5+408\pi^4-480\pi^3)\left(\tfrac{23}{100}\right)\\
&\quad+(11\pi^6-52\pi^5+60\pi^4)\Big)\left(\tfrac{23}{100}\right)^3\\
&\quad> 54,
\end{aligned}
\tag{53}
$$

for $x \in (0, c_1) \subset \left(0, \frac{23}{100}\right)$. Thus we may conclude that

$$
T_{10}(x) = \psi_1(x)+\psi_2(x)+\psi_3(x)+\psi_4(x) > 27 > 0, \tag{54}
$$

for $x \in (0, c_1)$. Based on $G_1''(x) = g_2(x)/\big(Q(x)\cos^2 x\big) > T_{10}(x)/\big(Q(x)\cos^2 x\big)$, the inequality

$$
G_1''(x) > 0, \tag{55}
$$

was proved for $x \in (0, c_1)$. Let us notice that

$$
\begin{aligned}
G_1'(0) &= \frac{1}{5}\ln\left(\frac{\pi}{2}\right) - \frac{2\pi-6}{\pi} = \frac{1}{5}\ln\left(1+\left(\frac{\pi}{2}-1\right)\right) - \frac{2\pi-6}{\pi}\\
&> \frac{1}{5}\sum_{k=1}^{4}\left(\frac{\left(\frac{\pi}{2}-1\right)^{2k-1}}{2k-1} - \frac{\left(\frac{\pi}{2}-1\right)^{2k}}{2k}\right) - \frac{2\pi-6}{\pi}\\
&= -\frac{\pi^8}{10240} + \frac{\pi^7}{560} - \frac{7\pi^6}{480} + \frac{7\pi^5}{100} - \frac{7\pi^4}{32} + \frac{7\pi^3}{15} - \frac{7\pi^2}{10} + \frac{4\pi}{5} - \frac{3561}{1400} + \frac{6}{\pi} > 0
\end{aligned}
\tag{56}
$$

and

$$G_1(0) = 0. \tag{57}$$

Based on Proposition 1 it follows that $G_1(x) > 0$ for $x \in (0, c_1)$. This also proves that $G(x) > 0$ for $x \in (0, c_1)$, which in turn proves that $F(x) > 0$ for $x \in \left(c, \frac{\pi}{2}\right)$. Therefore, we can conclude that $F(x) > 0$ for any $x \in \left(0, \frac{\pi}{2}\right)$. The proof of Nishizava's open problem is now completed. □

Based on previous proof of the Nishizawa's conjecture we obtained the following statement.

THEOREM 4. *For $0 < x < \pi/2$, we have*

$$\left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)} < \frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta(x)}$$

where $\theta(x) = -\dfrac{(\pi^3 - 24\pi + 48)}{3(\pi - 2)\pi^3}x^3 + \dfrac{\pi^3}{24(\pi - 2)}$ *and* $\vartheta(x) = \dfrac{4}{\pi^2}x^2$.

Finally, through analyzing Theorems 1–4 and using the fact (4), it is not difficult to show the following result.

PROPOSITION 2. *For $0 < x < \pi/2$, we have*

$$\begin{aligned} &\left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta_2} \\ &< \left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)} \\ &< \frac{\sin x}{x} \\ &< \left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta_2} \\ &< \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta(x)} \\ &< \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta_1}, \end{aligned} \tag{58}$$

and

$$\left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta_2} - \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta_1} = \begin{cases} > 0 : x \in (0, x_1) \\ < 0 : x \in (x_1, \pi/2) \\ = 0 : x = x_1 = 1.12698\ldots, \end{cases}$$

and

$$\left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)} - \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta_1} = \begin{cases} > 0 : x \in (0, x_2) \\ < 0 : x \in (x_2, \pi/2) \\ = 0 : x = x_2 = 1.40089\ldots; \end{cases}$$

where $\theta(x) = -\dfrac{(\pi^3-24\pi+48)}{3(\pi-2)\pi^3}x^3 + \dfrac{\pi^3}{24(\pi-2)}$ *and* $\vartheta(x) = \dfrac{4}{\pi^2}x^2$ *are the functions;* $\theta_1 = 1,\ \theta_2 = \dfrac{\pi^3}{24(\pi-2)} = 1.13169\ldots,\ \vartheta_1 = 0$ *and* $\vartheta_2 = 1$ *are the best possible constants;* $x_1 = 1.12698\ldots,\ x_2 = 1.40089\ldots$ *are unique determined roots of considered functions.*

3. Conclusions

This paper proved an open problem stated by Nishizawa in [1], applying computation method from [4] and [14]. We note that proofs of polynomial inequalities (17), (28), (45) and (50) can be based on reducing (by differentiation) of the corresponding polynomials to polynomials of a degree up to four (as illustrated in papers [12]–[15]), which allows symbolic radical representation of roots.

Our approach, based on the fact (4), allows new proofs of some power-exponential inequalities from papers [1], [5]–[12] and monographs [2], [3].

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(Received August 17, 2016)

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