A NOTE ON THE STRONG CONVERGENCE FOR
WEIGHTED SUMS OF ρ^*–MIXING RANDOM VARIABLES

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Abstract. In this work, the authors investigate the strong convergence for weighted sums of ρ^*-mixing random variables and obtain an improved convergence theorem so called the complete moment convergence in some sense. The result archived not only generalizes the corresponding ones of Sung (*Stat. Papers* 52: 447–454 (2011)), Zhou et al. (*J. Inequal. Appl.* 2011, Article ID 157816 (2011)), Sung (*Stat. Papers* 54: 773–781 (2013)) and Wu et al. (*Lith. Math. J.* 54 (2): 220–228 (2014)), but also improves them, respectively.

1. Introduction

Sung [1] proved the following strong law of large numbers for weighted sums of identically distributed negatively associated random variables.

THEOREM A. *Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed negatively associated random variables, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of real constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n), \tag{1.1}$$

for some $0 < \alpha \leqslant 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ for $1 < \alpha \leqslant 2$. If

$$
\begin{aligned}
E|X|^{\alpha} &< \infty, & \text{for} \quad \alpha > \gamma, \\
E|X|^{\alpha}\log(1+|X|) &< \infty, & \text{for} \quad \alpha = \gamma, \\
E|X|^{\gamma} &< \infty, & \text{for} \quad \alpha < \gamma.
\end{aligned}
\tag{1.2}
$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leqslant j \leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.3}$$

Recently, Zhou et al. [2] partially extended Theorem A for negatively associated random variable to ρ^*-mixing random variables and obtained the following theorem.

THEOREM B. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of identically distributed* ρ^*-*mixing random variables, and let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of real constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha, \gamma\}} = O(n), \qquad (1.4)$$

for some $0 < \alpha \leqslant 2$ *and* $\gamma > 0$ *with* $\alpha \neq \gamma$*. Set* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$*. Assume further that* $EX = 0$ *for* $1 < \alpha \leqslant 2$*. If* (1.2) *is satisfied for the cases* $\alpha > \gamma$ *and* $\alpha < \gamma$*, then* (1.3) *holds.*

Zhou et al. [2] left an open problem whether the case $\alpha = \gamma$ of Theorem A holds for ρ^*-mixing random variables. Sung [3] solved the open problem and obtained the following result.

THEOREM C. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of identically distributed* ρ^*-*mixing random variables, and let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of real constants satisfying* (1.4) *for* $\alpha = \gamma$ *with some* $0 < \alpha \leqslant 2$*. Set* $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$*. Assume further that* $EX = 0$ *for* $1 < \alpha \leqslant 2$*. If* (1.2) *is satisfied for the case* $\alpha = \gamma$*, then* (1.3) *holds.*

Sung [3] also presented an open problem whether the case $\alpha < \gamma$ of Theorem A holds for ρ^*-mixing random variables. Wu et al. [4] settled out this open problem.

It is interesting and meaningful to find the optimal moment conditions for (1.3). For the case $\alpha < \gamma$, the moment condition $E|X|^\gamma < \infty$ is optimal. Namely, if (1.3) satisfies for all array $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ satisfying (1.1), then the moment condition $E|X|^\gamma < \infty$ can be obtained easily. It is not known whether the moment conditions for the cases $\alpha \geqslant \gamma$ are optimal. But for the case $\alpha > \gamma$, Li et al. [5] improved the moment condition from $E|X|^\alpha < \infty$ to $E|X|^\alpha/(\log(1 + |X|))^{\alpha/\gamma - 1} < \infty$.

Inspired by the results of Wu et al. [6], in this paper, we shall further study the strong convergence properties for weighted sums of ρ^*-mixing random variables without assumption of identical distribution, and obtain an improved convergence theorem so called the complete moment convergence in some sense. The theorem archived not only generalizes the corresponding ones of the above listed, but also improves them under the same conditions, respectively.

In the following, some definitions will be restated for easy reference.

DEFINITION 1.1. Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. For any $S \subset \mathbb{N} = \{1, 2, \cdots\}$, define $\mathscr{F}_S = \sigma(X_i, i \in S)$. Given two σ-algebra A and B in $\mathscr{F}$, put

$$\rho(A, B) = \sup \left\{ \frac{|EXY - EXEY|}{\sqrt{\mathrm{Var}X} \cdot \sqrt{\mathrm{Var}Y}}; X \in L_2(A), Y \in L_2(B) \right\}. \qquad (1.5)$$

Define the ρ^*-mixing coefficients by

$$\rho^*(n) = \sup \{\rho(F_S, F_T): \text{ finite subsets } S, T \subset \mathbb{N} \text{ with dist}(S, T) \geqslant n\}, \qquad (1.6)$$

where $\mathrm{dist}(S, T) = \inf\{|s - t|; s \in S, t \in T\}$. Then the sequence $\{X_n, n \geqslant 1\}$ is called ρ^*-mixing if there exists $n \in \mathbb{N}$ such that $\rho^*(n) < 1$. Obviously, $0 \leqslant \rho^*(n+1) \leqslant \rho^*(n) \leqslant \rho^*(0) = 1$.

Since the concept of ρ^*-mixing random variables was firstly introduced by Bradley [7], many applications have been established. For more details, one can refer to [2–5], [7–16] among others.

DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to converge completely to a constant λ if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - \lambda| > \varepsilon) < \infty.$$

This notion was firstly introduced by Hsu and Robbins [17].

DEFINITION 1.3. Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If for all $\varepsilon \geqslant 0$,

$$\sum_{n=1}^{\infty} a_n E \left(b_n^{-1} |X_n| - \varepsilon \right)_+^q < \infty,$$

then $\{X_n, n \geqslant 1\}$ is called to converge in the sense of complete moment convergence. The concept of the complete moment convergence was introduced by Chow [18]. It is well known that the complete moment convergence can imply the complete convergence. Thus, the complete moment convergence is stronger than the complete convergence.

The definition of stochastic domination below will play an important role throughout this article.

DEFINITION 1.4. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| \geqslant x) \leqslant CP(|X| \geqslant x),$$

for all $x \geqslant 0$ and $n \geqslant 1$.

Throughout this paper, let $I(A)$ be the indicator function of the set A. The symbols C, C_1, $C_2 \cdots$ denote positive constants, which may be different in various places, $a_n = O(b_n)$ stands for $a_n \leqslant Cb_n$ for all $n \geqslant 1$. Set $\log x = \ln \max(x, e)$ and $x^+ = xI(x \geqslant 0)$.

2. Main results and proofs

To prove the main results of this paper, we need the following lemmas.

LEMMA 2.1. (Utev and Peligrad [8]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of ρ^*-mixing random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p \geqslant 2$ and all $n \geqslant 1$. Then there exists a positive constant $C = C(p, n, \rho^*(n))$ depending only on p, n and $\rho^*(n)$ such that*

$$E\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} X_i \right|^p \right) \leqslant C \left(\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right). \qquad (2.1)$$

In particular, if $p = 2$,

$$E\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} X_i \right|^2 \right) \leqslant C \sum_{i=1}^{n} EX_i^2. \qquad (2.2)$$

LEMMA 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For all $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leqslant b) \leqslant C_1 \left(E|X|^\alpha I(|X| \leqslant b) + b^\alpha P(|X| > b) \right), \quad (2.3)$$

$$E|X_n|^\alpha I(|X_n| > b) \leqslant C_2 E|X|^\alpha I(|X| > b), \quad (2.4)$$

where C_1 and C_2 are different positive constants. Consequently, $E|X_n|^\alpha \leqslant CE|X|^\alpha$.

LEMMA 2.3. (Wu et al. [4]) *Let $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ be an array of real constants satisfying (1.1) for some $\alpha > 0$, and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Then*

$$\sum_{n=1}^\infty \frac{1}{n b_n^\alpha} \sum_{i=1}^n E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n) \leqslant \begin{cases} CE|X|^\alpha, & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1+|X|), & \text{for } \alpha = \gamma, \\ CE|X|^\gamma, & \text{for } \alpha < \gamma. \end{cases} \quad (2.5)$$

LEMMA 2.4. (Wu et al. [4]) *Let $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ be an array of real constants satisfying (1.1) for some $\alpha > 0$, and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. If $q > \max\{\alpha, \gamma\}$, then*

$$\sum_{n=1}^\infty \frac{1}{n b_n^q} \sum_{i=1}^n E|a_{ni}X|^q I(|a_{ni}X| \leqslant b_n) \leqslant \begin{cases} CE|X|^\alpha, & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1+|X|), & \text{for } \alpha = \gamma, \\ CE|X|^\gamma, & \text{for } \alpha < \gamma. \end{cases} \quad (2.6)$$

Now, we state and prove the main results of this paper.

THEOREM 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of ρ^*-mixing random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of real constants satisfying (1.1) for some $0 < \alpha \leqslant 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, assume that $EX_n = 0$ for $1 < \alpha \leqslant 2$. If (1.2) holds, then*

$$\sum_{n=1}^\infty \frac{1}{n} E \left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^j a_{ni}X_i \right| - \varepsilon \right)_+^\alpha < \infty \quad \text{for all } \varepsilon > 0. \quad (2.7)$$

Proof of Theorem 2.1. For all $\varepsilon > 0$, noting that

$$\sum_{n=1}^\infty \frac{1}{n} E \left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^j a_{ni}X_i \right| - \varepsilon \right)_+^\alpha$$

$$= \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty P \left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^j a_{ni}X_i \right| - \varepsilon > t^{1/\alpha} \right) dt$$

$$= \sum_{n=1}^\infty \frac{1}{n} \int_0^1 P \left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^j a_{ni}X_i \right| > \varepsilon + t^{1/\alpha} \right) dt$$

$$+ \sum_{n=1}^\infty \frac{1}{n} \int_1^\infty P \left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^j a_{ni}X_i \right| > \varepsilon + t^{1/\alpha} \right) dt$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{n} P \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} \int_{1}^{\infty} P \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > b_n t^{1/\alpha} \right) dt$$

$$\triangleq I + J. \tag{2.8}$$

To prove (2.7), it suffices to prove $I < \infty$ and $J < \infty$. By the above corresponding results of Zhou et al. [2], Sung [3] and Wu et al. [4], we can obtain that $I < \infty$. Here omit the details.

Without loss of generality, assume that $a_{ni} \geqslant 0$ (otherwise, we will use a_{ni}^+ and a_{ni}^- instead of a_{ni}, and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). For any $t \geqslant 1$ and all $n \geqslant 1$, define

$$Y_{ni} = a_{ni} X_i I \left(|a_{ni} X_i| \leqslant b_n t^{1/\alpha} \right), i \geqslant 1;$$

$$A = \bigcap_{i=1}^{n} (Y_{ni} = a_{ni} X_i),$$

$$B = \bar{A} = \bigcup_{i=1}^{n} (Y_{ni} \neq a_{ni} X_i) = \bigcup_{i=1}^{n} \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right).$$

It is easy to check that for all $\varepsilon > 0$,

$$\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > b_n t^{1/\alpha} \right) \subset \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} Y_{ni} \right| > b_n t^{1/\alpha} \right) \bigcup \left(\bigcup_{i=1}^{n} \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right) \right),$$

which implies that

$$P \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > b_n t^{1/\alpha} \right)$$

$$\leqslant P \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} Y_{ni} \right| > b_n t^{1/\alpha} \right) + P \left(\bigcup_{i=1}^{n} \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right) \right)$$

$$\leqslant P \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right| > b_n t^{1/\alpha} - \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} EY_{ni} \right| \right)$$

$$+ \sum_{i=1}^{n} P \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right). \tag{2.9}$$

Firstly, we will show that

$$\sup_{t \geqslant 1} \frac{1}{b_n t^{1/\alpha}} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} EY_{ni} \right| \to 0 \quad \text{as } n \to \infty. \tag{2.10}$$

For $0 < \alpha \leqslant 1$, it follows from (2.3) of Lemma 2.2, the Markov inequality and (1.2) that

$$\sup_{t \geqslant 1} \frac{1}{b_n t^{1/\alpha}} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} EY_{ni} \right| \leqslant C \sup_{t \geqslant 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^{n} |EY_{ni}|$$

$$= C \sup_{t \geqslant 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^{n} E |a_{ni} X_i| I \left(|a_{ni} X_i| \leqslant b_n t^{1/\alpha} \right)$$

$$\leqslant C\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\,|a_{ni}X|\,I\left(|a_{ni}X|\leqslant b_n t^{1/\alpha}\right)$$

$$+C\sup_{t\geqslant 1}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)$$

$$\leqslant C\sup_{t\geqslant 1}\frac{1}{b_n^{\alpha}t}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha}I\left(|a_{ni}X|\leqslant b_n t^{1/\alpha}\right)$$

$$+C\sup_{t\geqslant 1}\frac{1}{b_n^{\alpha}t}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha}I\left(|a_{ni}X|>b_n t^{1/\alpha}\right)$$

$$\leqslant C(\log n)^{-\alpha/\gamma}E|X|^{\alpha}\to 0 \quad \text{as } n\to\infty. \qquad (2.11)$$

For $1<\alpha\leqslant 2$, it follows from $EX_n=0$, (2.4) of Lemma 2.2, the c_r inequality and (1.2) again that

$$\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right|=\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}E\,|a_{ni}X_i|\,I\left(|a_{ni}X_i|>b_n t^{1/\alpha}\right)\right|$$

$$\leqslant C\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\,|a_{ni}X|\,I\left(|a_{ni}X|>b_n t^{1/\alpha}\right)$$

$$\leqslant C\sup_{t\geqslant 1}\frac{1}{b_n^{\alpha}t}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha}I\left(|a_{ni}X|>b_n t^{1/\alpha}\right)$$

$$\leqslant C(\log n)^{-\alpha/\gamma}E|X|^{\alpha}\to 0 \quad \text{as } n\to\infty. \qquad (2.12)$$

By (2.11) and (2.12), we can obtain (2.10) immediately. Hence, for n large enough,

$$P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>b_n t^{1/\alpha}\right)\leqslant P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|>\frac{b_n t^{1/\alpha}}{2}\right)$$

$$+\sum_{i=1}^{n}P\left(|a_{ni}X_i|>b_n t^{1/\alpha}\right).$$

holds uniformly for all $t\geqslant 1$.

To prove $J<\infty$, it suffices to show that

$$J_1\triangleq\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|>\frac{b_n t^{1/\alpha}}{2}\right)dt<\infty, \qquad (2.13)$$

$$J_2\triangleq\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X_i|>b_n t^{1/\alpha}\right)dt<\infty. \qquad (2.14)$$

For J_2, by some standard computation, it follows from Lemma 2.3 and (1.2) that

$$J_2\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)dt$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{1}^{\infty}P\left(|a_{ni}X|^{\alpha}>b_n^{\alpha}t\right)dt$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|a_{ni}X|>b_n\right)<\infty. \qquad (2.15)$$

From the Markov inequality (for $q > \max\left\{2, \frac{2\gamma}{\alpha}\right\}$) and (2.1) of Lemma 2.1, it follows that

$$
\begin{aligned}
J_1 &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \frac{1}{b_n^q t^{q/\alpha}} E\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right|^q \right) dt \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I\left(|a_{ni}X_i| \leqslant b_n t^{1/\alpha} \right) \right)^{q/2} dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^{n} E|a_{ni}X_i|^q I\left(|a_{ni}X_i| \leqslant b_n t^{1/\alpha} \right) dt \\
&\triangleq J_3 + J_4.
\end{aligned}
\tag{2.16}
$$

For J_3, by $0 < \alpha \leqslant 2$ and $q > \max\{2, 2\gamma/\alpha\}$, it follows from (2.3) of Lemma 2.2 that

$$
\begin{aligned}
J_3 &\leqslant C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \left(\sum_{i=1}^{n} \left(E|a_{ni}X|^2 I\left(|a_{ni}X| \leqslant b_n t^{1/\alpha} \right) \right. \right. \\
&\qquad \left. \left. + b_n^2 t^{2/\alpha} P\left(|a_{ni}X| > b_n t^{1/\alpha} \right) \right) \right)^{q/2} dt \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \left(\frac{1}{b_n^\alpha t} \sum_{i=1}^{n} \left(E|a_{ni}X|^\alpha I\left(|a_{ni}X| \leqslant b_n t^{1/\alpha} \right) \right. \right. \\
&\qquad \left. \left. + E|a_{ni}X|^\alpha I\left(|a_{ni}X| > b_n t^{1/\alpha} \right) \right) \right)^{q/2} dt \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} t^{-q/2} \left(\frac{1}{b_n^\alpha} \sum_{i=1}^{n} \left(E|a_{ni}X|^\alpha \right) \right)^{q/2} dt \\
&\leqslant C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\alpha q/(2\gamma)} \left(E|X|^\alpha \right)^{q/2} < \infty.
\end{aligned}
\tag{2.17}
$$

For J_4, it follows from (2.3) of Lemma 2.2 again that

$$
\begin{aligned}
J_4 &\leqslant C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^{n} E|a_{ni}X|^q I\left(|a_{ni}X| \leqslant b_n t^{1/\alpha} \right) dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\left(|a_{ni}X| > b_n t^{1/\alpha} \right) dt \\
&= C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^{n} E|a_{ni}X|^q I\left(|a_{ni}X| \leqslant b_n \right) dt \\
&\quad + C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^{n} E|a_{ni}X|^q I\left(b_n < |a_{ni}X| \leqslant b_n t^{1/\alpha} \right) dt
\end{aligned}
$$

$$+C\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_{n}t^{1/\alpha}\right)dt$$

$$\triangleq J_5+J_6+J_7. \tag{2.18}$$

For J_5, by $q>2\geqslant\alpha$ and Lemma 2.4, it follows that $J_5<\infty$. Here omit the details.

For J_7, by the proof of $J_2<\infty$, i.e. (2.15), we can easily obtain $J_7<\infty$.

For J_6, by $q>2\geqslant\alpha$, Lemma 2.3 and (1.2), it follows that

$$J_6=C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\int_1^{\infty}\frac{1}{t^{q/\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^q I\left(b_n<|a_{ni}X|\leqslant b_n t^{1/\alpha}\right)dt$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\int_m^{m+1}\frac{1}{t^{q/\alpha}}E|a_{ni}X|^q I\left(b_n<|a_{ni}X|\leqslant b_n t^{1/\alpha}\right)dt$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\frac{1}{m^{q/\alpha}}E|a_{ni}X|^q I\left(b_n<|a_{ni}X|\leqslant b_n(m+1)^{1/\alpha}\right)$$

$$=C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{s=1}^{m}m^{-q/\alpha}E|a_{ni}X|^q I\left(b_n s^{1/\alpha}<|a_{ni}X|\leqslant b_n(s+1)^{1/\alpha}\right)$$

$$=C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^q I\left(b_n s^{1/\alpha}<|a_{ni}X|\leqslant b_n(s+1)^{1/\alpha}\right)\sum_{m=s}^{\infty}m^{-q/\alpha}$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^q I\left(b_n s^{1/\alpha}<|a_{ni}X|\leqslant b_n(s+1)^{1/\alpha}\right)s^{1-(q/\alpha)}$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^{\alpha}I\left(b_n s^{1/\alpha}<|a_{ni}X|\leqslant b_n(s+1)^{1/\alpha}\right)$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|a_{ni}X|>b_n\right)<\infty. \tag{2.19}$$

The proof of Theorem 2.1 is completed. $\square$

REMARK 2.1. Under the conditions of Theorem 2.1, noting that

$$\infty>\sum_{n=1}^{\infty}\frac{1}{n}E\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon\right)_+^{\alpha}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}P\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon>t^{1/\alpha}\right)dt$$

$$\geqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\varepsilon^{\alpha}}P\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon+t^{1/\alpha}\right)dt$$

$$\geqslant C\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>2\varepsilon b_n\right)\quad\text{for all }\varepsilon>0. \tag{2.20}$$

Since $\varepsilon>0$ is arbitrary, hence by (2.20), we can easily know that the complete moment convergence implies the complete convergence. Compared with the results of

Zhou et al. [2], Sung [3] and Wu et al. [4], it is worth pointing out that our main result is much stronger under the same moment conditions. So, Theorem 2.1 is an extension and improvement of the corresponding ones of Zhou et al. [2], Sung [3] and Wu et al. [4].

REMARK 2.2. As Sung [3] and Wu et al. [4] pointed out that the essential tool of the proof of Theorem 2.1 is the Rosenthal-type inequality for maximum partial sums of ρ^*-mixing random variables. For negatively associated random variable, the Rosenthal-type inequality for maximum partial sums also holds (see [19]). Hence, Theorem 2.1 also holds for negatively associated random variables.

REMARK 2.3. To our knowledge, Li et al. [5] investigated the complete convergence for weighted sums of identically distributed ρ^*-mixing random variables for the case $\alpha > \gamma > 0$. Under the almost optimal moment condition $E|X|^\alpha / (\log(1+|X|))^{\alpha/\gamma - 1} < \infty$, they obtained (1.3) for a sequence of identically distributed ρ^*-mixing random variables and an array $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of real constants satisfying (1.1) for some $0 < \alpha \leqslant 2$. It is still an open question whether (2.7) for the case $\alpha > \gamma$ also holds for weighted sums of ρ^*-mixing random variables under the moment condition $E|X|^\alpha / (\log(1+|X|))^{\alpha/\gamma - 1} < \infty$.

REFERENCES

[1] S. H. SUNG, *On the strong convergence for weighted sums of random variables*, Stat. Papers **52**: 447–454, 2011.

[2] X. C. ZHOU, C. C. TAN AND J. G. LIN, *On the strong laws for weighted sums of ρ^*-mixing random variables*, J. Inequal. Appl. **2011**, Article ID 157816, `doi:10.1155/2011/157816`.

[3] S. H. SUNG, *On the strong convergence for weighted sums of ρ^*-mixing random variables*, Stat. Papers **54**: 773–781, 2013.

[4] Y. F. WU, S. H. SUNG AND A. VOLODIN, *A note on the rates of convergence for weighted sums of ρ^*-mixing random variables*, Lith., Math. J. **54** (2): 220–228, 2014.

[5] W. LI, P. Y. CHEN AND S. H SUNG, *Remark on convergence rate for weighted sums of ρ^*-mixing random variables*, RACSAM **2016**, `doi:10.1007/s13398-016-0314-2`.

[6] Y. F. WU, T. C. HU AND A. VOLODIN, *Complete convergence and complete moment convergence for weighted sums of m-NA random variables*, J. Inequal. Appl. **2015**, `doi:10.1186/s13660-015-0717-1`.

[7] R. C. BRADLEY, *On the spectral density and asymptotic normality of weakly dependent random fields*, J. Theor. Probab. **5** (2): 355–373, 1992.

[8] S. UTEV AND M. PELIGRAD, *Maximal inequalities and an invariance principle for a class of weakly dependent random variables*, J. Theor. Probab. **16** (1): 101–115, 2003.

[9] Q. Y. WU AND Y. Y. JIANG, *Some strong limit theorems for $\tilde{\rho}$-mixing sequences of random variables*, Stat. Probab. Lett. **78** (8): 1017–1023, 2008.

[10] X. J. WANG, X. Q. LI, W. Z. YANG AND S. H. HU, *On complete convergence for arrays of rowwise weakly dependent random variables*, Appl. Math. Lett. **25**: 1916–1920, 2012.

[11] M. L. GUO AND D. J. ZHU, *Equivalent conditions of complete moment convergence of weighted sums for ρ^*-mixing sequence of random variables*, Stat. Probab. Lett. **83**: 13–20, 2013.

[12] Y. F. WU AND J. Y. PENG, *Strong convergence for weighted sums of $\tilde{\rho}$-mixing random variables*, Glasnik Matematicki **49** (1): 221–234, 2014.

[13] X. J. WANG, X. DENG, F. X. XIA AND S. H. HU, *The consistency for the estimators of semiparametric regression model based on weakly dependent errors*, Stat. Papers **58** (2): 303–318, 2017.

[14] Y. WU, X. J. WANG AND S. H. HU, *Complete moment convergence for weighted sums of weakly dependent random variables and its application in nonparametric regression model*, Stat. Probab. Lett. **127**: 56–66, 2017.

[15] Y. WU AND X. J. WANG, *A note on the consistency for the estimators of semiparametric regression model*, Stat. Papers, in press, 2016. `doi:10.1007/s00362-016-0807-2`.

[16] A. T. SHEN, H. Y. ZHU AND Y. ZHANG, *Exponential inequality for $\tilde{\rho}$-mixing sequences and its applications*, Filomat **28** (4): 859–870, 2014.

[17] P. L. HSU AND H. ROBBINS, *Complete convergence and the law of large numbers*, Proc. Natl. Acad. Sci. USA, **33** (2): 25–31, 1947.

[18] Y. S. CHOW, *On the rate of moment complete convergence of sample sums and extremes*, Bull. Inst. Math. Acad. Sin. **16**: 177–201, 1988.

[19] Q. M. SHAO, *A comparison theorem on moment inequalities between negatively associated and independent random variables*, J. Theor. Probab. **13** (2): 343–356, 2000.

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