

A NOTE ON THE STRONG CONVERGENCE FOR WEIGHTED SUMS OF ρ^* -MIXING RANDOM VARIABLES

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Abstract. In this work, the authors investigate the strong convergence for weighted sums of ρ^* -mixing random variables and obtain an improved convergence theorem so called the complete moment convergence in some sense. The result archived not only generalizes the corresponding ones of Sung (*Stat. Papers* 52: 447–454 (2011)), Zhou et al. (*J. Inequal. Appl.* 2011, Article ID 157816 (2011)), Sung (*Stat. Papers* 54: 773–781 (2013)) and Wu et al. (*Lith. Math. J.* 54 (2): 220–228 (2014)), but also improves them, respectively.

1. Introduction

Sung [1] proved the following strong law of large numbers for weighted sums of identically distributed negatively associated random variables.

THEOREM A. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed negatively associated random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying*

$$\sum_{i=1}^n |a_{ni}|^\alpha = O(n), \quad (1.1)$$

for some $0 < \alpha \leq 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ for $1 < \alpha \leq 2$. If

$$\begin{aligned} E|X|^\alpha &< \infty, & \text{for } \alpha > \gamma, \\ E|X|^\alpha \log(1 + |X|) &< \infty, & \text{for } \alpha = \gamma, \\ E|X|^\gamma &< \infty, & \text{for } \alpha < \gamma. \end{aligned} \quad (1.2)$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{n} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon b_n \right) < \infty \quad \text{for all } \varepsilon > 0. \quad (1.3)$$

Recently, Zhou et al. [2] partially extended Theorem A for negatively associated random variable to ρ^* -mixing random variables and obtained the following theorem.

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THEOREM B. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed ρ^* -mixing random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying

$$\sum_{i=1}^n |a_{ni}|^{\max\{\alpha, \gamma\}} = O(n), \tag{1.4}$$

for some $0 < \alpha \leq 2$ and $\gamma > 0$ with $\alpha \neq \gamma$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. Assume further that $EX = 0$ for $1 < \alpha \leq 2$. If (1.2) is satisfied for the cases $\alpha > \gamma$ and $\alpha < \gamma$, then (1.3) holds.

Zhou et al. [2] left an open problem whether the case $\alpha = \gamma$ of Theorem A holds for ρ^* -mixing random variables. Sung [3] solved the open problem and obtained the following result.

THEOREM C. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed ρ^* -mixing random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying (1.4) for $\alpha = \gamma$ with some $0 < \alpha \leq 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$. Assume further that $EX = 0$ for $1 < \alpha \leq 2$. If (1.2) is satisfied for the case $\alpha = \gamma$, then (1.3) holds.

Sung [3] also presented an open problem whether the case $\alpha < \gamma$ of Theorem A holds for ρ^* -mixing random variables. Wu et al. [4] settled out this open problem.

It is interesting and meaningful to find the optimal moment conditions for (1.3). For the case $\alpha < \gamma$, the moment condition $E|X|^\gamma < \infty$ is optimal. Namely, if (1.3) satisfies for all array $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfying (1.1), then the moment condition $E|X|^\gamma < \infty$ can be obtained easily. It is not known whether the moment conditions for the cases $\alpha \geq \gamma$ are optimal. But for the case $\alpha > \gamma$, Li et al. [5] improved the moment condition from $E|X|^\alpha < \infty$ to $E|X|^\alpha / (\log(1 + |X|))^{\alpha/\gamma - 1} < \infty$.

Inspired by the results of Wu et al. [6], in this paper, we shall further study the strong convergence properties for weighted sums of ρ^* -mixing random variables without assumption of identical distribution, and obtain an improved convergence theorem so called the complete moment convergence in some sense. The theorem archived not only generalizes the corresponding ones of the above listed, but also improves them under the same conditions, respectively.

In the following, some definitions will be restated for easy reference.

DEFINITION 1.1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space (Ω, \mathcal{F}, P) . For any $S \subset \mathbb{N} = \{1, 2, \dots\}$, define $\mathcal{F}_S = \sigma(X_i, i \in S)$. Given two σ -algebra A and B in \mathcal{F} , put

$$\rho(A, B) = \sup \left\{ \frac{|EXY - EXEY|}{\sqrt{\text{Var}X} \cdot \sqrt{\text{Var}Y}}; X \in L_2(A), Y \in L_2(B) \right\}. \tag{1.5}$$

Define the ρ^* -mixing coefficients by

$$\rho^*(n) = \sup \{ \rho(F_S, F_T) : \text{finite subsets } S, T \subset \mathbb{N} \text{ with } \text{dist}(S, T) \geq n \}, \tag{1.6}$$

where $\text{dist}(S, T) = \inf \{|s - t|; s \in S, t \in T\}$. Then the sequence $\{X_n, n \geq 1\}$ is called ρ^* -mixing if there exists $n \in \mathbb{N}$ such that $\rho^*(n) < 1$. Obviously, $0 \leq \rho^*(n + 1) \leq \rho^*(n) \leq \rho^*(0) = 1$.

Since the concept of ρ^* -mixing random variables was firstly introduced by Bradley [7], many applications have been established. For more details, one can refer to [2–5], [7–16] among others.

DEFINITION 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant λ if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - \lambda| > \varepsilon) < \infty.$$

This notion was firstly introduced by Hsu and Robbins [17].

DEFINITION 1.3. Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If for all $\varepsilon \geq 0$,

$$\sum_{n=1}^{\infty} a_n E(b_n^{-1} |X_n| - \varepsilon)_+^q < \infty,$$

then $\{X_n, n \geq 1\}$ is called to converge in the sense of complete moment convergence. The concept of the complete moment convergence was introduced by Chow [18]. It is well known that the complete moment convergence can imply the complete convergence. Thus, the complete moment convergence is stronger than the complete convergence.

The definition of stochastic domination below will play an important role throughout this article.

DEFINITION 1.4. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| \geq x) \leq CP(|X| \geq x),$$

for all $x \geq 0$ and $n \geq 1$.

Throughout this paper, let $I(A)$ be the indicator function of the set A . The symbols $C, C_1, C_2 \dots$ denote positive constants, which may be different in various places, $a_n = O(b_n)$ stands for $a_n \leq Cb_n$ for all $n \geq 1$. Set $\log x = \ln \max(x, e)$ and $x^+ = xI(x \geq 0)$.

2. Main results and proofs

To prove the main results of this paper, we need the following lemmas.

LEMMA 2.1. (Utev and Peligrad [8]) *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^* -mixing random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p \geq 2$ and all $n \geq 1$. Then there exists a positive constant $C = C(p, n, \rho^*(n))$ depending only on p, n and $\rho^*(n)$ such that*

$$E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right|^p \right) \leq C \left(\sum_{i=1}^n E|X_i|^p + \left(\sum_{i=1}^n EX_i^2 \right)^{p/2} \right). \tag{2.1}$$

In particular, if $p = 2$,

$$E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j X_i \right|^2 \right) \leq C \sum_{i=1}^n EX_i^2. \tag{2.2}$$

LEMMA 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X . For all $\alpha > 0$ and $b > 0$, the following two statements hold:

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C_1 (E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)), \tag{2.3}$$

$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b), \tag{2.4}$$

where C_1 and C_2 are different positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$.

LEMMA 2.3. (Wu et al. [4]) Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real constants satisfying (1.1) for some $\alpha > 0$, and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^\alpha} \sum_{i=1}^n E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n) \leq \begin{cases} CE|X|^\alpha, & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1 + |X|), & \text{for } \alpha = \gamma, \\ CE|X|^\gamma, & \text{for } \alpha < \gamma. \end{cases} \tag{2.5}$$

LEMMA 2.4. (Wu et al. [4]) Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real constants satisfying (1.1) for some $\alpha > 0$, and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. If $q > \max\{\alpha, \gamma\}$, then

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^q} \sum_{i=1}^n E|a_{ni}X|^q I(|a_{ni}X| \leq b_n) \leq \begin{cases} CE|X|^\alpha, & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1 + |X|), & \text{for } \alpha = \gamma, \\ CE|X|^\gamma, & \text{for } \alpha < \gamma. \end{cases} \tag{2.6}$$

Now, we state and prove the main results of this paper.

THEOREM 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of ρ^* -mixing random variables which is stochastically dominated by a random variable X , and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying (1.1) for some $0 < \alpha \leq 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, assume that $EX_n = 0$ for $1 < \alpha \leq 2$. If (1.2) holds, then

$$\sum_{n=1}^{\infty} \frac{1}{n} E \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| - \varepsilon \right)_+^\alpha < \infty \quad \text{for all } \varepsilon > 0. \tag{2.7}$$

Proof of Theorem 2.1. For all $\varepsilon > 0$, noting that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n} E \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| - \varepsilon \right)_+^\alpha \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty P \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| - \varepsilon > t^{1/\alpha} \right) dt \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 P \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon + t^{1/\alpha} \right) dt \\ & \quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty P \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon + t^{1/\alpha} \right) dt \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=1}^{\infty} \frac{1}{n} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon b_n \right) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > b_n t^{1/\alpha} \right) dt \\ &\triangleq I + J. \end{aligned} \tag{2.8}$$

To prove (2.7), it suffices to prove $I < \infty$ and $J < \infty$. By the above corresponding results of Zhou et al. [2], Sung [3] and Wu et al. [4], we can obtain that $I < \infty$. Here omit the details.

Without loss of generality, assume that $a_{ni} \geq 0$ (otherwise, we will use a_{ni}^+ and a_{ni}^- instead of a_{ni} , and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). For any $t \geq 1$ and all $n \geq 1$, define

$$Y_{ni} = a_{ni} X_i I \left(|a_{ni} X_i| \leq b_n t^{1/\alpha} \right), i \geq 1;$$

$$A = \bigcap_{i=1}^n (Y_{ni} = a_{ni} X_i),$$

$$B = \bar{A} = \bigcup_{i=1}^n (Y_{ni} \neq a_{ni} X_i) = \bigcup_{i=1}^n \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right).$$

It is easy to check that for all $\varepsilon > 0$,

$$\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > b_n t^{1/\alpha} \right) \subset \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > b_n t^{1/\alpha} \right) \cup \left(\bigcup_{i=1}^n \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right) \right),$$

which implies that

$$\begin{aligned} &P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > b_n t^{1/\alpha} \right) \\ &\leq P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j Y_{ni} \right| > b_n t^{1/\alpha} \right) + P \left(\bigcup_{i=1}^n \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right) \right) \\ &\leq P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right| > b_n t^{1/\alpha} - \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| \right) \\ &\quad + \sum_{i=1}^n P \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right). \end{aligned} \tag{2.9}$$

Firstly, we will show that

$$\sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.10}$$

For $0 < \alpha \leq 1$, it follows from (2.3) of Lemma 2.2, the Markov inequality and (1.2) that

$$\begin{aligned} \sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j EY_{ni} \right| &\leq C \sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^n |EY_{ni}| \\ &= C \sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^n E |a_{ni} X_i| I \left(|a_{ni} X_i| \leq b_n t^{1/\alpha} \right) \end{aligned}$$

$$\begin{aligned}
&\leq C \sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^n E |a_{ni} X| I \left(|a_{ni} X| \leq b_n t^{1/\alpha} \right) \\
&\quad + C \sup_{t \geq 1} \sum_{i=1}^n P \left(|a_{ni} X| > b_n t^{1/\alpha} \right) \\
&\leq C \sup_{t \geq 1} \frac{1}{b_n^\alpha t} \sum_{i=1}^n a_{ni}^\alpha E |X|^\alpha I \left(|a_{ni} X| \leq b_n t^{1/\alpha} \right) \\
&\quad + C \sup_{t \geq 1} \frac{1}{b_n^\alpha t} \sum_{i=1}^n a_{ni}^\alpha E |X|^\alpha I \left(|a_{ni} X| > b_n t^{1/\alpha} \right) \\
&\leq C (\log n)^{-\alpha/\gamma} E |X|^\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.11}
\end{aligned}$$

For $1 < \alpha \leq 2$, it follows from $EX_n = 0$, (2.4) of Lemma 2.2, the c_r inequality and (1.2) again that

$$\begin{aligned}
\sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E Y_{ni} \right| &= \sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E |a_{ni} X_i| I \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right) \right| \\
&\leq C \sup_{t \geq 1} \frac{1}{b_n t^{1/\alpha}} \sum_{i=1}^n E |a_{ni} X| I \left(|a_{ni} X| > b_n t^{1/\alpha} \right) \\
&\leq C \sup_{t \geq 1} \frac{1}{b_n^\alpha t} \sum_{i=1}^n a_{ni}^\alpha E |X|^\alpha I \left(|a_{ni} X| > b_n t^{1/\alpha} \right) \\
&\leq C (\log n)^{-\alpha/\gamma} E |X|^\alpha \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.12}
\end{aligned}$$

By (2.11) and (2.12), we can obtain (2.10) immediately. Hence, for n large enough,

$$\begin{aligned}
P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > b_n t^{1/\alpha} \right) &\leq P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - E Y_{ni}) \right| > \frac{b_n t^{1/\alpha}}{2} \right) \\
&\quad + \sum_{i=1}^n P \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right).
\end{aligned}$$

holds uniformly for all $t \geq 1$.

To prove $J < \infty$, it suffices to show that

$$J_1 \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - E Y_{ni}) \right| > \frac{b_n t^{1/\alpha}}{2} \right) dt < \infty, \tag{2.13}$$

$$J_2 \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^n P \left(|a_{ni} X_i| > b_n t^{1/\alpha} \right) dt < \infty. \tag{2.14}$$

For J_2 , by some standard computation, it follows from Lemma 2.3 and (1.2) that

$$\begin{aligned}
J_2 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^n P \left(|a_{ni} X| > b_n t^{1/\alpha} \right) dt \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^n \int_1^{\infty} P \left(|a_{ni} X|^\alpha > b_n^\alpha t \right) dt \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n b_n^\alpha} \sum_{i=1}^n E |a_{ni} X|^\alpha I \left(|a_{ni} X| > b_n \right) < \infty. \tag{2.15}
\end{aligned}$$

From the Markov inequality (for $q > \max\{2, \frac{2\gamma}{\alpha}\}$) and (2.1) of Lemma 2.1, it follows that

$$\begin{aligned}
 J_1 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \frac{1}{b_n^q t^{q/\alpha}} E \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j (Y_{ni} - EY_{ni}) \right|^q \right) dt \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \left(\sum_{i=1}^n E|a_{ni}X_i|^2 I(|a_{ni}X_i| \leq b_n t^{1/\alpha}) \right)^{q/2} dt \\
 &\quad + C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^n E|a_{ni}X_i|^q I(|a_{ni}X_i| \leq b_n t^{1/\alpha}) dt \\
 &\triangleq J_3 + J_4.
 \end{aligned} \tag{2.16}$$

For J_3 , by $0 < \alpha \leq 2$ and $q > \max\{2, 2\gamma/\alpha\}$, it follows from (2.3) of Lemma 2.2 that

$$\begin{aligned}
 J_3 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \left(\sum_{i=1}^n \left(E|a_{ni}X|^2 I(|a_{ni}X| \leq b_n t^{1/\alpha}) \right. \right. \\
 &\quad \left. \left. + b_n^2 t^{2/\alpha} P(|a_{ni}X| > b_n t^{1/\alpha}) \right) \right)^{q/2} dt \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \left(\frac{1}{b_n^\alpha t} \sum_{i=1}^n \left(E|a_{ni}X|^\alpha I(|a_{ni}X| \leq b_n t^{1/\alpha}) \right. \right. \\
 &\quad \left. \left. + E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n t^{1/\alpha}) \right) \right)^{q/2} dt \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} t^{-q/2} \left(\frac{1}{b_n^\alpha} \sum_{i=1}^n (E|a_{ni}X|^\alpha) \right)^{q/2} dt \\
 &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\alpha q/(2\gamma)} (E|X|^\alpha)^{q/2} < \infty.
 \end{aligned} \tag{2.17}$$

For J_4 , it follows from (2.3) of Lemma 2.2 again that

$$\begin{aligned}
 J_4 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^n E|a_{ni}X|^q I(|a_{ni}X| \leq b_n t^{1/\alpha}) dt \\
 &\quad + C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^n P(|a_{ni}X| > b_n t^{1/\alpha}) dt \\
 &= C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^n E|a_{ni}X|^q I(|a_{ni}X| \leq b_n) dt \\
 &\quad + C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^n E|a_{ni}X|^q I(b_n < |a_{ni}X| \leq b_n t^{1/\alpha}) dt
 \end{aligned}$$

$$\begin{aligned}
 &+C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^n P\left(|a_{ni}X| > b_n t^{1/\alpha}\right) dt \\
 &\triangleq J_5 + J_6 + J_7.
 \end{aligned} \tag{2.18}$$

For J_5 , by $q > 2 \geq \alpha$ and Lemma 2.4, it follows that $J_5 < \infty$. Here omit the details.

For J_7 , by the proof of $J_2 < \infty$, i.e. (2.15), we can easily obtain $J_7 < \infty$.

For J_6 , by $q > 2 \geq \alpha$, Lemma 2.3 and (1.2), it follows that

$$\begin{aligned}
 J_6 &= C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \int_1^{\infty} \frac{1}{t^{q/\alpha}} \sum_{i=1}^n E|a_{ni}X|^q I\left(b_n < |a_{ni}X| \leq b_n t^{1/\alpha}\right) dt \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \sum_{i=1}^n \sum_{m=1}^{\infty} \int_m^{m+1} \frac{1}{t^{q/\alpha}} E|a_{ni}X|^q I\left(b_n < |a_{ni}X| \leq b_n t^{1/\alpha}\right) dt \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \sum_{i=1}^n \sum_{m=1}^{\infty} \frac{1}{m^{q/\alpha}} E|a_{ni}X|^q I\left(b_n < |a_{ni}X| \leq b_n(m+1)^{1/\alpha}\right) \\
 &= C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \sum_{i=1}^n \sum_{m=1}^{\infty} \sum_{s=1}^m m^{-q/\alpha} E|a_{ni}X|^q I\left(b_n s^{1/\alpha} < |a_{ni}X| \leq b_n(s+1)^{1/\alpha}\right) \\
 &= C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \sum_{i=1}^n \sum_{s=1}^{\infty} E|a_{ni}X|^q I\left(b_n s^{1/\alpha} < |a_{ni}X| \leq b_n(s+1)^{1/\alpha}\right) \sum_{m=s}^{\infty} m^{-q/\alpha} \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^q} \sum_{i=1}^n \sum_{s=1}^{\infty} E|a_{ni}X|^q I\left(b_n s^{1/\alpha} < |a_{ni}X| \leq b_n(s+1)^{1/\alpha}\right) s^{1-(q/\alpha)} \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^\alpha} \sum_{i=1}^n \sum_{s=1}^{\infty} E|a_{ni}X|^\alpha I\left(b_n s^{1/\alpha} < |a_{ni}X| \leq b_n(s+1)^{1/\alpha}\right) \\
 &\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^\alpha} \sum_{i=1}^n E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n) < \infty.
 \end{aligned} \tag{2.19}$$

The proof of Theorem 2.1 is completed. \square

REMARK 2.1. Under the conditions of Theorem 2.1, noting that

$$\begin{aligned}
 \infty &> \sum_{n=1}^{\infty} \frac{1}{n} E \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| - \varepsilon \right)_+^\alpha \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty P \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| - \varepsilon > t^{1/\alpha} \right) dt \\
 &\geq C \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\varepsilon^\alpha} P \left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon + t^{1/\alpha} \right) dt \\
 &\geq C \sum_{n=1}^{\infty} \frac{1}{n} P \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > 2\varepsilon b_n \right) \quad \text{for all } \varepsilon > 0.
 \end{aligned} \tag{2.20}$$

Since $\varepsilon > 0$ is arbitrary, hence by (2.20), we can easily know that the complete moment convergence implies the complete convergence. Compared with the results of

Zhou et al. [2], Sung [3] and Wu et al. [4], it is worth pointing out that our main result is much stronger under the same moment conditions. So, Theorem 2.1 is an extension and improvement of the corresponding ones of Zhou et al. [2], Sung [3] and Wu et al. [4].

REMARK 2.2. As Sung [3] and Wu et al. [4] pointed out that the essential tool of the proof of Theorem 2.1 is the Rosenthal-type inequality for maximum partial sums of ρ^* -mixing random variables. For negatively associated random variable, the Rosenthal-type inequality for maximum partial sums also holds (see [19]). Hence, Theorem 2.1 also holds for negatively associated random variables.

REMARK 2.3. To our knowledge, Li et al. [5] investigated the complete convergence for weighted sums of identically distributed ρ^* -mixing random variables for the case $\alpha > \gamma > 0$. Under the almost optimal moment condition $E|X|^\alpha / (\log(1 + |X|))^{\alpha/\gamma-1} < \infty$, they obtained (1.3) for a sequence of identically distributed ρ^* -mixing random variables and an array $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ of real constants satisfying (1.1) for some $0 < \alpha \leq 2$. It is still an open question whether (2.7) for the case $\alpha > \gamma$ also holds for weighted sums of ρ^* -mixing random variables under the moment condition $E|X|^\alpha / (\log(1 + |X|))^{\alpha/\gamma-1} < \infty$.

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