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TIME SCALE VERSIONS OF THE OSTROWSKI–GRÜSS TYPE INEQUALITY WITH A PARAMETER FUNCTION

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Abstract. In this paper, we obtain two Ostrowski–Grüss type inequalities on time scales for bounded differentiable mappings with a parameter function. Our result generalizes some known results in this direction, for example, a result due to Ngô and Liu [11]. In addition, we consider the special cases where the time scale is chosen to be the set of real numbers and the set of integers.

1. Introduction

In 1997, Dragomir and Wang [5] proved that if $f:[a,b]\to\mathbb{R}$ is differentiable with bounded derivative, then for all $t\in[a,b]$,

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(s)ds - \frac{f(b)-f(a)}{b-a}\left(t-\frac{a+b}{2}\right)\right| \leqslant \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{1}$$

where $\gamma := \inf_{t\in[a,b]} f'(t)$ and $\Gamma := \sup_{t\in[a,b]} f'(t)$. The above inequality is known in the literature as the Ostrowski–Grüss type inequality.

Ngô and Liu [11] proved the following Ostrowski–Grüss type inequality for time scales (see Section 2 for definition) – which is a combination of both Grüss inequality and Ostrowski inequality on time scales due to Bohner and Matthews [1, 2]. For more results in this direction, see [6, 8, 9, 10, 7, 13, 12].

THEOREM 1. *Let $a,b,s,t\in\mathbb{T}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. If f^{Δ} is rd-continuous and $\gamma\leqslant f^{\Delta}(s)\leqslant\Gamma$ for all $s\in[a,b]$, then we have*

$$\begin{aligned}&\left| f(t) - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s - \frac{f(b)-f(a)}{(b-a)^2}\Big[h_2(t,a)-h_2(t,b)\Big]\right| \\ &\leqslant \frac{\Gamma-\gamma}{2(b-a)}\int_a^b \left| p(s,t) - \frac{h_2(t,a)-h_2(t,b)}{b-a}\right|\Delta s,\end{aligned} \tag{2}$$

where $h_2(t,s)$ is given in item (a) of Remark 1 and

$$p(s,t) = \begin{cases} s-a, & s\in[a,t), \\ s-b, & s\in[t,b]. \end{cases} \tag{3}$$

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In 2010, Tuna and Daghan [14] used a different version of (3) to obtain some generalizations of the Ostrowski–Grüss type inequality. Our aim in this paper is to give two new Ostrowski–Grüss type inequalities using a different generalization of (3) given in [15]. Our first result generalizes Theorem 1, which in turn, unifies the continuous and discrete known results.

This paper is organized as follows. In Section 2, we recall necessary results and definitions in time scale theory. We outline the required lemmas, which will be used in the proof of the main results, in Section 3. Finally, our results are formulated and justified in Section 4.

2. Preliminaries

In this section, we only collect results and definitions that will play a crucial role in what follows. For more on the theory of time scales, we refer the reader to the books [3, 4]. We start with the following definition.

DEFINITION 1. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward *jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$, and $\sigma(\sup \mathbb{T}) := \sup \mathbb{T}$. Clearly, we see that $\sigma(t) \geqslant t$ for all $t \in \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leqslant b$, we define the interval $[a,b]$ in $\mathbb{T}$ by $[a,b] = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. Open intervals and half-open intervals are defined in the same manner.

DEFINITION 2. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that for any given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)\,[\sigma(t) - s] \right| \leqslant \varepsilon\, |\sigma(t) - s|, \quad \forall s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^k$.

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = \frac{df(t)}{dt}$. In the case $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$, which is the usual forward difference operator.

THEOREM 2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f(\sigma(t)) g^\Delta(t).$$

DEFINITION 3. The function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined as $f^\sigma(t) = f(\sigma(t))$.

DEFINITION 4. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous on $\mathbb{T}$ provided it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

DEFINITION 5. Let f be a rd-continuous function. Then $g:\mathbb{T}\to\mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\Delta}(t)=f(t)$ for any $t\in\mathbb{T}^k$. In this case, we have

$$\int_a^b f(s)\Delta s = g(b)-g(a).$$

Next, we state the following known properties of the $\Delta-$integral.

THEOREM 3. *If $a,b,c\in\mathbb{T}$ with $a<c<b$, $\alpha\in\mathbb{R}$ and f,g are rd-continuous, then*

(i) $\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$.

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha\int_a^b f(t)\Delta t$.

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$.

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$.

(v) $\left|\int_a^b f(t)\Delta t\right| \leqslant \int_a^b |f(t)|\Delta t$.

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t$.

DEFINITION 6. The polynomials $h_k:\mathbb{T}^2\to\mathbb{R},\ k\in\mathbb{N}$ are functions that are recursively defined as

$$h_0(t,s)=1$$

and

$$h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\Delta\tau,\ \ for\ all\ \ s,\ \ t\in\mathbb{T}.$$

In view of the above definition, we make the following remarks (see [3, Example 1.102]) that will come handy in the sequel.

REMARK 1. (a). Using the fact that for all $s,t\in\mathbb{T}$, $h_1(t,s)=t-s$, we get that

$$h_2(t,s)=\int_s^t(\tau-s)\Delta\tau.$$

(b). When $\mathbb{T}=\mathbb{R}$, then for all $s,t\in\mathbb{T}$,

$$h_k(t,s)=\frac{(t-s)^k}{k!}.$$

(c). When $\mathbb{T}=\mathbb{Z}$, then for all $s,t\in\mathbb{T}$,

$$h_k(t,s)=\binom{t-s}{k}=\prod_{i=1}^{k}\frac{t-s+1-i}{i}.$$

3. Lemmas

The following lemmas will be used in the proof of the main results. The first lemma is due to Xu and Fang [15].

LEMMA 1. (Generalized Montgomery Identity) *Suppose that* $a,b,s,t \in \mathbb{T}$, $a < b$, $f:[a,b] \to \mathbb{R}$ *is differentiable, and* ψ *is a function of* $[0,1]$ *into* $[0,1]$. *Then*

$$\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}$$
$$=\frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s+\frac{1}{b-a}\int_a^b K(s,t)f^{\Delta}(s)\Delta s,$$

where

$$K(s,t)=\begin{cases} s-(a+\psi(\lambda)\frac{b-a}{2}), & s\in[a,t),\\ s-(a+(1+\psi(1-\lambda))\frac{b-a}{2}), & s\in[t,b]. \end{cases} \tag{4}$$

It is important to note that Relation (4) boils down to (3) if we take $\psi(\lambda)=\lambda$ and thereafter, in particular, choose $\lambda=0$.

LEMMA 2. *Let* ψ *and* $K(\cdot,\cdot)$ *be given as in Lemma 1 above. Then for all* $\lambda\in[0,1]$ *such that* $a+\psi(\lambda)\frac{b-a}{2}$ *and* $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ *are in* $\mathbb{T}$, *and* $t\in\left[a+\psi(\lambda)\frac{b-a}{2},a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$, *we have*

(1).

$$\int_a^b |K(s,t)|\Delta s = h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)$$
$$+h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right).$$

(2).

$$\int_a^b K(s,t)\Delta s = h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)$$
$$+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right).$$

Proof. The proof of item 1 is somewhat embedded in the proof of Theorem 1 in [15]. For the sake of completeness, we present the proof here. Using Theorem 3, one gets

$$
\begin{aligned}
&\int_a^b |K(s,t)|\Delta s\\
&= \int_a^t |K(s,t)|\Delta s + \int_t^b |K(s,t)|\Delta s\\
&= \int_a^t \left|s - \left(a + \psi(\lambda)\frac{b-a}{2}\right)\right|\Delta s + \int_t^b \left|s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2}\right)\right|\Delta s\\
&= \int_a^{a+\psi(\lambda)\frac{b-a}{2}} \left|s - \left(a + \psi(\lambda)\frac{b-a}{2}\right)\right|\Delta s + \int_{a+\psi(\lambda)\frac{b-a}{2}}^t \left|s - \left(a + \psi(\lambda)\frac{b-a}{2}\right)\right|\Delta s\\
&\quad + \int_t^{a+(1+\psi(1-\lambda))\frac{b-a}{2}} \left|s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2}\right)\right|\Delta s\\
&\quad + \int_{a+(1+\psi(1-\lambda))\frac{b-a}{2}}^b \left|s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2}\right)\right|\Delta s\\
&= \int_{a+\psi(\lambda)\frac{b-a}{2}}^a \left[s - \left(a + \psi(\lambda)\frac{b-a}{2}\right)\right]\Delta s + \int_{a+\psi(\lambda)\frac{b-a}{2}}^t \left[s - \left(a + \psi(\lambda)\frac{b-a}{2}\right)\right]\Delta s\\
&\quad + \int_{a+(1+\psi(1-\lambda))\frac{b-a}{2}}^t \left[s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\Delta s\\
&\quad + \int_{a+(1+\psi(1-\lambda))\frac{b-a}{2}}^b \left[s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\Delta s\\
&= h_2\left(a, a+\psi(\lambda)\frac{b-a}{2}\right) + h_2\left(t, a+\psi(\lambda)\frac{b-a}{2}\right)\\
&\quad + h_2\left(t, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) + h_2\left(b, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right).
\end{aligned}
$$

The proof of item 2 follows from the same line of reasoning. □

LEMMA 3. ([11]) *Let $a,b,s \in \mathbb{T}$, f,g rd-continuous and $f,g:[a,b] \to \mathbb{R}$. If there exist $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \leqslant g(s) \leqslant \Gamma$ for all $s \in [a,b]$, then we have*

$$
\left|\int_a^b f(s)g(s)\Delta s - \frac{1}{b-a}\int_a^b f(s)\Delta s \int_a^b g(s)\Delta s\right| \leqslant \frac{\Gamma-\gamma}{2}\int_a^b \left|f(s) - \frac{1}{b-a}\int_a^b f(\tau)\Delta\tau\right|\Delta s.
$$

4. Statement and proof of the main results

Inspired by the papers [11, 14], we formulate and prove the following results.

THEOREM 4. *Let $a,b,s,t \in \mathbb{T}$, $a<b$, $f:[a,b] \to \mathbb{R}$ be differentiable, and ψ a function of $[0,1]$ into $[0,1]$. If f^{Δ} is rd-continuous and there exist $\gamma, \Gamma \in \mathbb{R}$ such that*

$\gamma \leqslant f^{\Delta}(t) \leqslant \Gamma$ for all $t \in [a,b]$, then the following inequality holds

$$\begin{aligned}
&\Bigg|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s\\
&\quad-\frac{f(b)-f(a)}{(b-a)^2}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\Bigg|\\
&\leqslant\frac{\Gamma-\gamma}{2(b-a)}\int_a^b\Bigg|K(s,t)-\frac{1}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\Bigg|\Delta s \qquad (5)
\end{aligned}$$

for all $\lambda \in [0,1]$ such that $a+\psi(\lambda)\frac{b-a}{2}$ and $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$, and $t \in \left[a+\psi(\lambda)\frac{b-a}{2}, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$.

Proof. We now present the proof of Theorem 4. For this, we define the function

$$K(s,t)=\begin{cases} s-\left(a+\psi(\lambda)\frac{b-a}{2}\right), & s\in[a,t),\\ s-\left(a+(1+\psi(1-\lambda))\frac{b-a}{2}\right), & s\in[t,b],\end{cases}$$

and then put $f(t)=K(s,t)$ and $g(t)=f^{\Delta}(t)$ in Lemma 3 to obtain

$$\begin{aligned}
&\left|\int_a^b K(s,t)f^{\Delta}(s)\Delta s-\frac{1}{b-a}\int_a^b K(s,t)\Delta s\int_a^b f^{\Delta}(s)\Delta s\right|\\
&\qquad\leqslant\frac{\Gamma-\gamma}{2}\int_a^b\left|K(s,t)-\frac{1}{b-a}\int_a^b K(\tau,t)\Delta\tau\right|\Delta s. \qquad (6)
\end{aligned}$$

From item 2 of Lemma 2, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b K(s,t)\Delta s&=\frac{1}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]. \qquad (7)
\end{aligned}$$

Also, from Definition 5, we have

$$\int_a^b f^{\Delta}(s)\Delta s=f(b)-f(a). \qquad (8)$$

Substituting Equation (7) in the right hand side of Equation (6), we get

$$
\begin{aligned}
&\frac{\Gamma-\gamma}{2}\int_a^b\left|K(s,t)-\frac{1}{b-a}\int_a^b K(\tau,t)\Delta\tau\right|\Delta s\\
&=\frac{\Gamma-\gamma}{2}\int_a^b\left|K(s,t)-\frac{1}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\right.\\
&\quad\left.\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\right|\Delta s. \quad (9)
\end{aligned}
$$

Relation (6) becomes

$$
\begin{aligned}
&\left|\int_a^b K(s,t)f^{\Delta}(s)\Delta s-\frac{1}{b-a}\int_a^b K(s,t)\Delta s\int_a^b f^{\Delta}(s)\Delta s\right|\\
&\leqslant\frac{\Gamma-\gamma}{2}\int_a^b\left|K(s,t)-\frac{1}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\right.\\
&\quad\left.\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\right|\Delta s. \quad (10)
\end{aligned}
$$

From Lemma 1, we have

$$
\begin{aligned}
\int_a^b K(s,t)f^{\Delta}(s)\Delta s&=\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}(b-a)f(t)\\
&\quad+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}(b-a)-\int_a^b f^{\sigma}(s)\Delta s. \quad (11)
\end{aligned}
$$

Substituting Equations (7), (8) and (11) into (10), we arrive at

$$
\begin{aligned}
&\left|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}(b-a)f(t)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}(b-a)\right.\\
&\quad-\int_a^b f^{\sigma}(s)\Delta s-\frac{f(b)-f(a)}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a\quad+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\right|\\
&\leqslant\frac{\Gamma-\gamma}{2}\int_a^b\left|K(s,t)-\frac{1}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a\quad+\psi(\lambda)\frac{b-a}{2}\right)\right.\right.\\
&\quad\left.\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\right|\Delta s. \quad (12)
\end{aligned}
$$

Hence,

$$\begin{aligned}
&\Bigg|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\\
&\quad-\frac{f(b)-f(a)}{(b-a)^2}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\Bigg|\\
&\leqslant\frac{\Gamma-\gamma}{2(b-a)}\int_a^b\Bigg|K(s,t)-\frac{1}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\Bigg|\Delta s. \qquad (13)
\end{aligned}$$

That completes the proof. □

COROLLARY 1. *Under the assumption of Theorem 4 with* $\psi(\lambda)=\lambda^2$, *we have that*

$$\begin{aligned}
&\Bigg|(1-\lambda)f(t)+\frac{\lambda^2f(a)+(2\lambda-\lambda^2)f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\\
&\quad-\frac{f(b)-f(a)}{(b-a)^2}\left[h_2\left(t,a+\lambda^2\frac{b-a}{2}\right)-h_2\left(a,a+\lambda^2\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(2-2\lambda+\lambda^2)\frac{b-a}{2}\right)-h_2\left(t,a+(2-2\lambda+\lambda^2)\frac{b-a}{2}\right)\right]\Bigg|\\
&\leqslant\frac{\Gamma-\gamma}{2(b-a)}\int_a^b\Bigg|K(s,t)-\frac{1}{b-a}\left[h_2\left(t,a+\lambda^2\frac{b-a}{2}\right)-h_2\left(a,a+\lambda^2\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(2-2\lambda+\lambda^2)\frac{b-a}{2}\right)-h_2\left(t,a+(2-2\lambda+\lambda^2)\frac{b-a}{2}\right)\right]\Bigg|\Delta s \qquad (14)
\end{aligned}$$

for all $\lambda\in[0,1]$ *such that* $a+\lambda^2\frac{b-a}{2}$ *and* $a+(2-2\lambda+\lambda^2)\frac{b-a}{2}$ *are in* $\mathbb{T}$, *and* $t\in\left[a+\lambda^2\frac{b-a}{2},a+(2-2\lambda+\lambda^2)\frac{b-a}{2}\right]$.

Corollary 14 generalizes Theorem 1. This is evident by substituting $\lambda=0$ in the above inequality.

COROLLARY 2. *Under the assumption of Corollary 1 with* $\lambda=1$ *such that* $\frac{a+b}{2}\in$

$\mathbb{T}$, *we have that*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s-\frac{f(b)-f(a)}{(b-a)^2}\left[h_2\left(b,\frac{a+b}{2}\right)-h_2\left(a,\frac{a+b}{2}\right)\right]\right|$$
$$\leqslant\frac{\Gamma-\gamma}{2(b-a)}\int_a^b\left|K\left(s,\frac{a+b}{2}\right)-\frac{1}{b-a}\left[h_2\left(b,\frac{a+b}{2}\right)-h_2\left(a,\frac{a+b}{2}\right)\right]\right|\Delta s. \tag{15}$$

We now state and prove our second result.

THEOREM 5. *Let $a,b,s,t\in\mathbb{T}$, $a<b$, $f:[a,b]\to\mathbb{R}$ be differentiable, and ψ a function of $[0,1]$ into $[0,1]$. If f^Δ is rd-continuous and there exist $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leqslant f^\Delta(t)\leqslant\Gamma$ for all $t\in[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right.\\
&\quad-\frac{\Gamma-\gamma}{2(b-a)}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.\left.+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)-h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\right|\\
&\leqslant\frac{\Gamma-\gamma}{2(b-a)}\left[h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\end{aligned}$$

for all $\lambda\in[0,1]$ such that $a+\psi(\lambda)\frac{b-a}{2}$ and $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$, and $t\in\left[a+\psi(\lambda)\frac{b-a}{2},a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$.

Proof. We start by letting $\Theta=\dfrac{\Gamma+\gamma}{2}$. Since, $\gamma\leqslant f^\Delta(s)\leqslant\Gamma$ for all $s\in[a,b]$, it implies that $\gamma-\Theta\leqslant f^\Delta(s)-\Theta\leqslant\Gamma-\Theta$ for all $s\in[a,b]$. This further implies that $|f^\Delta(s)-\Theta|\leqslant\dfrac{\Gamma-\gamma}{2}$ for all $s\in[a,b]$. Hence,

$$\max_{s\in[a,b]}|f^\Delta(s)-\Theta|\leqslant\frac{\Gamma-\gamma}{2}. \tag{16}$$

Now, recall the following function used in Theorem 4.

$$K(s,t)=\begin{cases}s-(a+\psi(\lambda)\frac{b-a}{2}), & s\in[a,t),\\ s-(a+(1+\psi(1-\lambda))\frac{b-a}{2}), & s\in[t,b].\end{cases}$$

Using Lemma 1, we have

$$\frac{1}{b-a}\int_a^b K(s,t)f^{\Delta}(s)\Delta s = \frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2} - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s. \tag{17}$$

Also, from item 2 of Lemma 2, we have

$$\frac{1}{b-a}\int_a^b K(s,t)\Delta s = \frac{1}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right) - h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right) + h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) - h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]. \tag{18}$$

From Equations (17) and (18), we get

$$\begin{aligned} &\frac{1}{b-a}\int_a^b K(s,t)\big(f^{\Delta}(s)-\Theta\big)\Delta s \\ &= \frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2} \\ &\quad - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s - \frac{\Theta}{b-a}\left[h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right)\right. \\ &\quad - h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right) + h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) \\ &\quad \left. - h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]. \end{aligned} \tag{19}$$

The absolute value of the left hand side of (19) is estimated as follows

$$\left|\frac{1}{b-a}\int_a^b K(s,t)\big(f^{\Delta}(s)-\Theta\big)\Delta s\right| \leqslant \max_{s\in[a,b]}|f^{\Delta}(s)-\Theta|\frac{1}{b-a}\int_a^b |K(s,t)|\Delta s. \tag{20}$$

Also, from item 1 of Lemma 2, we get

$$\frac{1}{b-a}\int_a^b |K(s,t)|\Delta s = \frac{1}{b-a}\left[h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right) + h_2\left(t,a+\psi(\lambda)\frac{b-a}{2}\right) + h_2\left(t,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) + h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]. \tag{21}$$

Using Relation (16) in (20), gives

$$\left|\frac{1}{b-a}\int_a^b K(s,t)\big(f^{\Delta}(s)-\Theta\big)\Delta s\right| \leqslant \frac{\Gamma-\gamma}{2}\frac{1}{b-a}\int_a^b |K(s,t)|\Delta s. \tag{22}$$

Substituting Relations (19) and (21) into (22) amounts to the desired result. □

Applying Theorem 5 to the continuous and discrete case, we obtain

COROLLARY 3. (Continuous case) *Let* $\mathbb{T}=\mathbb{R}$.

$$\Bigg|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)ds$$
$$-\frac{\Gamma-\gamma}{2(b-a)}\Bigg[\frac{\left(t-(a+\psi(\lambda)\frac{b-a}{2})\right)^2}{2}-\frac{\psi^2(\lambda)(b-a)^2}{8}$$
$$+\frac{\left(b-\left(a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right)^2}{2}-\frac{\left(t-\left(a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right)^2}{2}\Bigg]\Bigg|$$
$$\leqslant\frac{\Gamma-\gamma}{2(b-a)}\Bigg[\frac{\psi^2(\lambda)(b-a)^2}{8}+\frac{\left(t-(a+\psi(\lambda)\frac{b-a}{2})\right)^2+\left(t-(a+(1+\psi(1-\lambda))\frac{b-a}{2})\right)^2}{2}$$
$$+\frac{\left(b-\left(a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right)^2}{2}\Bigg]$$

for all $\lambda\in[0,1]$ *such that* $a+\psi(\lambda)\frac{b-a}{2}$ *and* $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ *are in* $\mathbb{R}$, *and* $t\in\left[a+\psi(\lambda)\frac{b-a}{2},a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$.

Proof. This follows from the fact that $\sigma(s)=s$, $f^{\Delta}=f'$ and $h_2(t,s)=\frac{(t-s)^2}{2}$ (from item (b) of Remark 1). □

COROLLARY 4. (Discrete case) *Let* $\mathbb{T}=\mathbb{Z}$, $a=0$, $b=n$, $s=j$, $t=i$, *and* $f(k)=x_k$. *Then*

$$\Bigg|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}x_i+\frac{\psi(\lambda)x_0+(1-\psi(1-\lambda))x_n}{2}-\frac{1}{n}\sum_{j=1}^{n}x_j$$
$$-\frac{\Gamma-\gamma}{2n}\Bigg[h_2\left(i,\frac{n\psi(\lambda)}{2}\right)-h_2\left(0,\frac{n\psi(\lambda)}{2}\right)$$
$$+h_2\left(n,\frac{n(1+\psi(1-\lambda))}{2}\right)-h_2\left(i,\frac{n(1+\psi(1-\lambda))}{2}\right)\Bigg]\Bigg|$$
$$\leqslant\frac{\Gamma-\gamma}{2n}\Bigg[h_2\left(0,\frac{n\psi(\lambda)}{2}\right)+h_2\left(i,\frac{n\psi(\lambda)}{2}\right)+h_2\left(i,\frac{n(1+\psi(1-\lambda))}{2}\right)$$
$$+h_2\left(n,\frac{n(1+\psi(1-\lambda))}{2}\right)\Bigg]$$

for all $\lambda \in [0,1]$ *such that* $\frac{n\psi(\lambda)}{2}$ *and* $\frac{n(1+\psi(1-\lambda))}{2}$ *are in* $\mathbb{Z}$, *and* $i \in \left[\frac{n\psi(\lambda)}{2}, \frac{n(1+\psi(1-\lambda))}{2}\right]$, *where*

$$h_2\left(0, \frac{\psi(\lambda)n}{2}\right) = \frac{1}{2}\left[\frac{\psi^2(\lambda)n^2}{4} + \frac{\psi(\lambda)n}{2}\right],$$

$$h_2\left(i, \frac{\psi(\lambda)n}{2}\right) = \frac{1}{2}\left[i^2 - i(n\psi(\lambda)+1) + \frac{\psi^2(\lambda)n^2}{4} + \frac{\psi(\lambda)n}{2}\right],$$

$$\begin{aligned} h_2\left(i, \frac{(1+\psi(1-\lambda))n}{2}\right) = \frac{1}{2}\Big[& i^2 - i\Big(n(1+\psi(1-\lambda))+1\Big) \\ & + \frac{(1+\psi(1-\lambda))^2n^2}{4} + \frac{(1+\psi(1-\lambda))n}{2}\Big], \end{aligned}$$

and

$$\begin{aligned} h_2\left(n, \frac{(1+\psi(1-\lambda))n}{2}\right) = \frac{1}{2}\Big[& n^2 - \Big(n^2(1+\psi(1-\lambda)\Big) - n \\ & + \frac{(1+\psi(1-\lambda))^2n^2}{4} + \frac{(1+\psi(1-\lambda))n}{2}\Big]. \end{aligned}$$

Proof. This follows from the fact that $\sigma(s) = s+1$. Also, from item (c) of Remark 1, we have $h_2(t,s) = \frac{1}{2}(t-s)(t-s-1)$. □

REMARK 2. More results can be obtained, from Theorem 5, by choosing appropriate $\psi(\lambda)$ and considering some specific values of λ, for example, $\lambda = 0$, $\frac{1}{2}$ and 1 or by taking a different time scale like $\mathbb{T} = q^{\mathbb{N}_0}$, with $q > 1$. For this case,

$$\sigma(t) = qt,$$

$$\int_a^b f^\sigma(s)\Delta s = (q-1)\sum_{j=\log_q(a)}^{\log_q(b)-1} q^j f(q^{j+1}),$$

$$h_2(t,s) = \frac{(t-s)(t-qs)}{q+1},$$

and

$$f^\Delta(t) = D_q f(t) := \frac{f(qt)-f(t)}{(q-1)t}.$$

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