Journal of
Mathematical
Inequalities
Volume 12, Number 2 (2018), 545–550 doi:10.7153/jmi-2018-12-40

CONVERSE JENSEN INEQUALITY FOR STRONGLY CONVEX SET–VALUED MAPS

MILICA KLARIČIĆ BAKULA AND KAZIMIERZ NIKODEM

(*Communicated by A. Gilányi*)

Abstract. Integral and discrete counterparts of the converse Jensen inequality for strongly convex set-valued maps are presented.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval. Following Polyak [12] a function $f : I \to \mathbb{R}$ is called *strongly convex with modulus* $c > 0$ if

$$f(px_1 + (1-p)x_2) \leqslant pf(x_1) + (1-p)f(x_2) - cp(1-p)(x_1 - x_2)^2 \tag{1}$$

for all $x_1, x_2 \in I$ and $p \in [0,1]$.

Many properties and applications of strongly convex functions can be found in the literature (see, for instance, [7], [8], [9], [11], [13], [14], [15]). It is known, in particular, that if $f : I \to \mathbb{R}$ is strongly convex with modulus c, then the following stronger version of the classical Jensen inequality holds (see [7]):

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) \leqslant \sum_{i=1}^{n} p_i f(x_i) - c\sum_{i=1}^{n} p_i (x_i - \bar{x})^2, \tag{2}$$

for all $x_1, \ldots, x_n \in I$, $p_1, \ldots, p_n > 0$ with $p_1 + \cdots + p_n = 1$ and $\bar{x} = p_1x_1 + \cdots + p_nx_n$.

Recently, a counterpart of the converse Jensen inequality for strongly convex functions was also proved (see [4]):

$$\sum_{i=1}^{n} p_i f(x_i) - c\sum_{i=1}^{n} p_i (x_i - \bar{x})^2 \leqslant \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M) - c(M - \bar{x})(\bar{x} - m), \tag{3}$$

for all $x_1, \ldots, x_n \in [m, M] \subset I$, $p_1, \ldots, p_n > 0$ with $p_1 + \cdots + p_n = 1$ and $\bar{x} = p_1x_1 + \cdots + p_nx_n$.

In 2010 H. Huang [3], extended the definition (1) of strongly convex function to set-valued maps in the following way. Let $(Y, \|\cdot\|)$ be a Banach space. Suppose B

Mathematics subject classification (2010): Primary 26A51, Secondary 39B62, 54C60.
Keywords and phrases: Strongly convex functions, strongly convex set-valued map, Jensen inequality, converse Jensen inequality.

© ELEMENT, Zagreb
Paper JMI-12-40

is a closed unit ball in Y and $I \subset \mathbb{R}$ is an interval. Denote by $n(Y)$ the family of all nonempty subsets of Y and by $cl(Y)$ the family of all closed nonempty subsets of Y. A set-valued map $F : I \to n(Y)$ is called *strongly convex with modulus* $c > 0$ if

$$pF(x_1) + (1-p)F(x_2) + cp(1-p)(x_1 - x_2)^2 B \subset F(px_1 + (1-p)x_2) \tag{4}$$

for all $x_1, x_2 \in I$ and $p \in [0,1]$.

Clearly, the above definition of strongly convex set-valued maps is motivated by that of strongly convex functions (1). Note also that the usual notion of convex set-valued maps corresponds to relation (4) with $c = 0$.

Strongly convex set-valued maps were used by Huang [3] to investigate error bounds for some inclusion problems with set constraints. Some further properties of such maps can be found in [5], [10]. In particular, the following integral and discrete counterparts of the Jensen-type inequality (2) for strongly convex set-valued maps are proved in [10] (see also [6] where similar results for convex set-valued maps are given).

THEOREM 1. *Let (T, Σ, P) be a probability measure space and $I \subset \mathbb{R}$ an open interval. If $F : I \to cl(Y)$ is strongly convex with modulus c, then for each square-integrable function $\varphi : T \to I$*

$$\int_T F(\varphi(t))dP + c \int_T (\varphi(t) - \mu)^2 dPB \subset F\Big(\int_T \varphi(t)dP\Big),$$

where $\mu = \int_T \varphi(t)dP$.

THEOREM 2. *Let $I \subset \mathbb{R}$ be an open interval. If $F : I \to cl(Y)$ is strongly convex with modulus c, then*

$$\sum_{i=1}^{n} p_i F(x_i) + c \sum_{i=1}^{n} p_i (x_i - \bar{x})^2 B \subset F\Big(\sum_{i=1}^{n} p_i x_i\Big),$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in I$, $p_1, \ldots, p_n > 0$ with $p_1 + \cdots + p_n = 1$ and $\bar{x} = p_1 x_1 + \cdots + p_n x_n$.

The aim of this paper is to present integral and discrete versions of the converse Jensen inequality (3) for strongly convex set-valued maps.

2. Results

The next theorem gives an integral counterpart of the converse Jensen inequality for set-valued maps. Throughout this paper the integral of a set-valued map $F : T \to n(Y)$ is understood in the sense of Aumann, i.e. it is the set of integrals of all integrable (in the sense of Bochner) selections of this map (cf. [1]):

$$\int_T F(t)dP = \Big\{ \int_T f(t)dP \ : \ f : T \to Y \text{ is integrable and } f(t) \in F(t),\ t \in T \Big\}.$$

THEOREM 3. *Let* (T,Σ,P) *be a probability measure space. Suppose* I *is an interval containing* $[m,M]$ *and* $\varphi : T \to [m,M]$ *is a Lebesgue integrable function. If a set-valued map* $F : I \to n(Y)$ *is strongly convex with modulus* c*, then*

$$\frac{M-\mu}{M-m}F(m)+\frac{\mu-m}{M-m}F(M)+c\int_T (M-\varphi(t))(\varphi(t)-m)dP\,B \subset \int_T F(\varphi(t))dP, \quad (5)$$

where $\mu = \int_T \varphi(t)dP$.

Proof. Take an integrable function $\varphi : T \to [m,M]$ and denote $\mu = \int_T \varphi(t)dP$. Fix arbitrary points $y_1 \in F(m)$, $y_2 \in F(M)$, $b \in B$ and put

$$z = \frac{M-\mu}{M-m}y_1+\frac{\mu-m}{M-m}y_2+c\int_T (M-\varphi(t))(\varphi(t)-m)dP\,b.$$

To prove that $z \in \int_T F(\varphi(t))dP$, we will show that there exists an integrable selection g of $F\circ\varphi$ such that $z = \int_T g(t)dP$. Define

$$f(x) = \frac{M-x}{M-m}y_1+\frac{x-m}{M-m}y_2+c(M-x)(x-m)b, \;\; x \in [m,M].$$

By the definition of strong convexity (4), we have

$$\begin{aligned} f(x) &\in \frac{M-x}{M-m}F(m)+\frac{x-m}{M-m}F(M)+c(M-x)(x-m)B \\ &\subset F\Big(\frac{M-x}{M-m}m+\frac{x-m}{M-m}M\Big) = F(x), \end{aligned}$$

which shows that f is a selection of F. Consequently, $g = f\circ\varphi$ is a selection of $F\circ\varphi$. Since

$$\int_T \|g(t)\|dP \leqslant \frac{M-\mu}{M-m}\|y_1\|+\frac{\mu-m}{M-m}\|y_2\|+c(M-m)^2\|b\| < \infty,$$

g is Bochner integrable (see [2], Theorem 2, p. 45) and

$$\begin{aligned} \int_T g(t)dP &= \int_T f(\varphi(t))dP \\ &= \int_T \Big(\frac{M-\varphi(t)}{M-m}y_1+\frac{\varphi(t)-m}{M-m}y_2+c(M-\varphi(t))(\varphi(t)-m)b\Big)dP \\ &= \frac{M-\mu}{M-m}y_1+\frac{\mu-m}{M-m}y_2+c\Big(\int_T (M-\varphi(t))(\varphi(t)-m)dP\Big)b = z. \end{aligned}$$

This finishes the proof. □

Consider now the set-valued map $Q : [m,M] \to n(Y)$ defined by

$$Q(x) = \frac{M-x}{M-m}F(m)+\frac{x-m}{M-m}F(M)+c(M-x)(x-m)B, \;\; x \in [m,M]. \quad (6)$$

Combining Theorem 3 with the Jensen-type inclusion for strongly convex set-valued maps formulated in Theorem 1, we obtain the following result.

COROLLARY 4. *Let* (T,Σ,P) *be a probability measure space. Suppose* I *is an open interval containing* $[m,M]$, $\varphi : T \to [m,M]$ *is a Lebesgue integrable function and* $\mu = \int_T \varphi(t)dP$. *If a set-valued map* $F : I \to cl(Y)$ *is strongly convex with modulus* c, *then*

$$Q(\mu) \subset \int_T F(\varphi(t))dP + \int_T (\varphi(t)-\mu)^2 dPB \subset F(\mu). \tag{7}$$

Proof. To prove the first inclusion note that the term $\int_T (M-\varphi(t))(\varphi(t)-m)dP$ in (5) can be written in the form

$$\int_T (M-\varphi(t))(\varphi(t)-m)dP = (M-\mu)(\mu-m) - \int_T (\varphi(t)-\mu)^2 dP. \tag{8}$$

From here $(M-\mu)(\mu-m) - \int_T (\varphi(t)-\mu)^2 dP \geqslant 0$. Therefore, by the convexity of B, we have

$$\Big((M-\mu)(\mu-m) - \int_T (\varphi(t)-\mu)^2 dP\Big)B + \int_T (\varphi(t)-\mu)^2 dPB = (M-\mu)(\mu-m)B. \tag{9}$$

Now, substituting equality (8) in (5), adding the term $c\int_T (\varphi(t)-\mu)^2 dPB$ to both sides of (5) and using (9), we obtain the first inclusion in (7). The second inclusion follows by Theorem 1 (the assumptions that I is an open interval and F has closed values are needed here). □

As a consequence of Theorem 3, we get also the following discrete version of the converse Jensen inequality for strongly convex set-valued maps.

THEOREM 5. *If a set-valued map* $F : I \to n(Y)$ *is strongly convex with modulus* c, *then*

$$\frac{M-\bar{x}}{M-m}F(m) + \frac{\bar{x}-m}{M-m}F(M) + c\sum_{i=1}^{n} p_i(M-x_i)(x_i-m)B \subset \sum_{i=1}^{n} p_i F(x_i), \tag{10}$$

for all $n \in \mathbb{N}$, $x_1,\dots,x_n \in [m,M] \subset I$, $p_1,\dots,p_n > 0$ *with* $p_1+\dots+p_n = 1$ *and* $\bar{x} = p_1x_1+\dots+p_nx_n$.

Proof. Assume that $T = I$, $\varphi(x) = x$ for $x \in I$, and $x_1,\dots,x_n \in [m,M]$ are distinct points. Moreover, assume that P is a probability measure on I concentrate at $x_1,\dots,x_n$, that is $P(x_i) = p_i > 0$, $i = 1,\dots,n$ and $p_1+\dots+p_n = 1$. Then

$$\mu = \int_I \varphi(x)dP = \sum_{i=1}^{n} p_i x_i,$$

$$\int_I (M-\varphi(x))(\varphi(x)-m)dP = \sum_{i=1}^{n} p_i(M-x_i)(x_i-m)$$

and

$$\int_I F(\varphi(x))dP = \sum_{i=1}^{n} p_i F(x_i).$$

Now, using Theorem 3, we get (10). □

Using the set-valued map Q defined by (6) and combining the above Theorem 5 with Theorem 2 we obtain the following result.

COROLLARY 6. *Let $I \subset \mathbb{R}$ be an open interval. If $F : I \to cl(Y)$ is strongly convex with modulus c, then*

$$Q(\bar{x}) \subset \sum_{i=1}^{n} p_i F(x_i) + c \sum_{i=1}^{n} p_i (x_i - \bar{x})^2 B \subset F(\bar{x}),$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in I$, $p_1, \ldots, p_n > 0$ with $p_1 + \cdots + p_n = 1$ and $\bar{x} = p_1 x_1 + \cdots + p_n x_n$.

Proof. To prove the first inclusion add the term $c \sum_{i=1}^{n} p_i (x_i - \bar{x})^2 B$ to both sides of (10) and use the equality

$$\sum_{i=1}^{n} p_i (M - x_i)(x_i - m) + \sum_{i=1}^{n} p_i (x_i - \bar{x})^2 = (M - \bar{x})(\bar{x} - m).$$

The second inclusion is given in Theorem 2. □

REFERENCES

[1] J.-P. AUBIN, H. FRANKOWSKA, *Set-Valued Analysis*, Birkhäuser, Boston-Basel-Berlin, 1990.

[2] J. DIESTEL, J. J. UHL, JR., *Vector Measures*, Amer. Math. Soc., Providence, 1977.

[3] H. HUANG, *Global error bounds with exponents for multifunctions with set constraints*, Communications in Contemporary Math. **12** (2010), 417–435.

[4] M. KLARIČIĆ BAKULA, K. NIKODEM, *On the converse Jensen inequality for strongly convex functions*, J. Math. Anal. Appl. **434** (2016), 516–522.

[5] H. LEIVA, N. MERENTES, K. NIKODEM AND J. L. SÁNCHEZ, *Strongly convex set-valued maps*, J. Glob. Optim. **57** (2013), 695–705.

[6] J. MATKOWSKI, K. NIKODEM, *An integral Jensen inequality for convex multifunctions*, Results Math. **26** (1994), 348–353.

[7] N. MERENTES AND K. NIKODEM, *Remarks on strongly convex functions*, Aequationes Math. **80** (2010), 193–199.

[8] K. NIKODEM, *Strongly Convex Functions and Related Classes of Functions* in: Th. M. Rassias (Ed.) Handbook of Functional Equations. Functional Inequalities, Springer Optimizations and Its Applications, vol. 95, 2015, 365–405.

[9] K. NIKODEM AND ZS. PÁLES, *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal. **5** (2011), no. 1, 83–87.

[10] K. NIKODEM, J. L. SÁNCHEZ, L. SÁNCHEZ, *Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps*, Math. Aeterna **4** (2014), 979–987.

[11] E. S. POLOVINKIN, *Strongly convex analysis*, Sbornik Mathematics **187/2** (1996), 103–130.

[12] B. T. POLYAK, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. **7** (1966), 72–75.

[13] T. RAJBA, SZ. WĄSOWICZ, *Probabilistic characterization of strong convexity*, Opuscula Math. **31** (2011), 97–103.

[14] R. SCHULTZ, *Strong convexity in stochastic programs with complete recourse*, J. Comput. Appl. Math. **56** (1994), 3–22.

[15] J. P. VIAL, *Strong convexity of sets and functions*, J. Math. Economy **9** (1982), 187–205.

(Received October 20, 2016)

M. Klaričić Bakula
Faculty of Science, University of Split
Rudera Boskovica 33, 21000 Split, Croatia
e-mail: milica@pmfs.hr

Kazimierz Nikodem
Department of Mathematics
University of Bielsko-Biala
ul. Willowa 2, 43-309 Bielsko-Biała, Poland
e-mail: knikodem@ath.bielsko.pl