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ON WEIGHTED HARDY INEQUALITIES FOR NON–INCREASING SEQUENCES

PENG GAO

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Abstract. A result of Bennett and Grosse-Erdmann characterizes the weights for which the corresponding weighted Hardy inequality holds on the cone of non-negative, non-increasing sequences and a bound for the best constant is given. In this paper, we improve the bound for $1 < p \leqslant 2$.

1. Introduction

Throughout this paper, we let $p \geqslant 1$. For $p \neq 1$ we let q be defined by $\frac{1}{p}+\frac{1}{q}=1$ and we set $1/q=0$ when $p=1$. Consider the following weighted Hardy inequality on the cone of non-negative, non-increasing sequences $\mathbf{x}=(x_n)_{n\geqslant 1}$:

$$\sum_{n=1}^{\infty} b_n\left(\sum_{k=1}^{n}\frac{x_k}{n}\right)^p \leqslant U_p\sum_{n=1}^{\infty} b_n x_n^p, \tag{1.1}$$

where $(b_n)_{n\geqslant 1}$ is a non-negative sequence, $U_p>0$ a constant independent of $\mathbf{x}$. In [2, Theorem 1], Bennett and Grosse-Erdmann gave a complete characterization on the sequence $(b_n)_{n\geqslant 1}$ such that U_p exists. They showed that this is the case if and only if there exists a constant $U'_p>0$ such that for all $n\geqslant 1$,

$$\sum_{k=n}^{\infty}\frac{b_k}{k^p} \leqslant \frac{U'_p}{n^p}\sum_{k=1}^{n} b_k.$$

Moreover, if the constants U_p, U'_p are chosen best possible, then

$$U'_p \leqslant U_p \leqslant p^p(U'_p+1)^p. \tag{1.2}$$

Integral inequalities analogous to (1.1) for non-increasing functions have been studied by Ariño and Muckenhoupt in [1]. They showed that if $p\geqslant 1$ and v is a non-negative measurable function on $(0,\infty)$ then there is a constant $V_p>0$ such that

$$\int_0^{\infty} v(x)\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant V_p\int_0^{\infty} v(x)f^p(x)dx$$

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holds for all non-negative non-increasing functions $f(x)$ if and only if there is a constant $V_p' > 0$ such that for all $x > 0$,

$$\int_x^\infty \frac{v(t)}{t^p} dt \leqslant \frac{V_p'}{x^p} \int_0^x v(t) dt.$$

The argument of Bennett and Grosse-Erdmann also works for the integral case and it implies that [2, (17)] if the constants V_p, V_p' are chosen best possible, then

$$V_p' \leqslant V_p \leqslant (V_p'+1)^p.$$

Comparing the above two results, we see that in the discrete case, the corresponding bounds for the best constants are not as good as what is given in the integral case. It is then natural to seek for an improvement on the bounds given in (1.2), which is the goal of this paper. Our result in this paper is the following generalization of the above mentioned result of Bennett and Grosse-Erdmann:

THEOREM 1.1. *Let $p \geqslant 1$ be fixed. Let $(b_n)_{n\geqslant 1}$ be a non-negative sequence and let $(\lambda_n)_{n\geqslant 1}$ be a non-negative, non-increasing sequence with $\lambda_1 > 0$. Let $\Lambda_n = \sum_{k=1}^n \lambda_k$. Then there is a constant $U_p > 0$ such that*

$$\sum_{n=1}^\infty b_n \left(\sum_{k=1}^n \frac{\lambda_k x_k}{\Lambda_n} \right)^p \leqslant U_p \sum_{n=1}^\infty b_n x_n^p \tag{1.3}$$

holds for all non-negative, non-increasing sequences $(x_n)_{n\geqslant 1}$ if and only if there is a constant $U_p' > 0$ such that for all $n \geqslant 1$,

$$\sum_{k=n}^\infty \frac{b_k}{\Lambda_k^p} \leqslant \frac{U_p'}{\Lambda_n^p} \sum_{k=1}^n b_k. \tag{1.4}$$

Moreover, if U_p and U_p' are chosen best-possible then we have

$$U_p' \leqslant U_p \leqslant \begin{cases} \left(pU_p'+1\right)^p, & 1 \leqslant p \leqslant 2; \\ p^p(U_p'+1)^p, & p > 2. \end{cases} \tag{1.5}$$

The case $\lambda_n = 1$ of Theorem 1.1 gives back the result of Bennett and Grosse-Erdmann except that instead of (1.5), the upper bound given for U_p in [2, Theorem 1] is given as in (1.2) for all $p \geqslant 1$. Theorem 1.1 therefore improves upon the result of Bennett and Grosse-Erdmann for $1 < p \leqslant 2$ in this sense. We point out here that this improvement comes from our refinement (see Lemma 2.5) on the so called "Power Rule" (Lemma 2.1 below), a key lemma used in the proof of [2, Theorem 1] by Bennett and Grosse-Erdmann.

2. Lemmas

LEMMA 2.1. ([2, Lemma 3]) *Let* $p \geqslant 1$. *Then for all non-negative sequences* $(a_k)_{k\geqslant 1}$, *any integer* $n \geqslant 1$,

$$\left(\sum_{k=1}^{n} a_k\right)^p \leqslant p\sum_{k=1}^{n} a_k \left(\sum_{i=1}^{k} a_i\right)^{p-1}.$$

LEMMA 2.2. ([2, Lemma 2]) *Let* $(u_n)_{n\geqslant 1}, (v_n)_{n\geqslant 1}$ *be two non-negative sequences satisfying for any integer* $n \geqslant 1$,

$$\sum_{i=1}^{n} u_i \leqslant \sum_{i=1}^{n} v_i,$$

then for all non-negative, non-increasing sequences $(a_n)_{n\geqslant 1}$,

$$\sum_{i=1}^{n} u_i a_i \leqslant \sum_{i=1}^{n} v_i a_i.$$

LEMMA 2.3. ([3, Lemma 3.1]) *Let* $(B_n)_{n\geqslant 1}$ *and* $(C_n)_{n\geqslant 1}$ *be strictly increasing positive sequences with* $B_1/B_2 \leqslant C_1/C_2$. *If for any integer* $n \geqslant 1$,

$$\frac{B_{n+1}-B_n}{B_{n+2}-B_{n+1}} \leqslant \frac{C_{n+1}-C_n}{C_{n+2}-C_{n+1}}.$$

Then $B_n/B_{n+1} \leqslant C_n/C_{n+1}$ *for any integer* $n \geqslant 1$.

LEMMA 2.4. *Let* $1 \leqslant p \leqslant 2$ *and let* $n \geqslant 1$ *be a fixed integer. Let* $\lambda = (\lambda_k)_{1\leqslant k\leqslant n}$ *be a non-negative, non-increasing sequence with* $\lambda_1 > 0$. *For* $1 \leqslant k \leqslant n$, *let* $\Lambda_k = \sum_{i=1}^{k} \lambda_i$ *and*

$$C_{k,p,\lambda} = \frac{\Lambda_k^p}{\sum_{i=1}^{k} \lambda_i \Lambda_i^{p-1}}.$$

Then the sequence $(C_{k,p,\lambda})_{1\leqslant k\leqslant n}$ *is increasing with respect to* k.

Proof. The assertion holds trivially when $p = 1$, so we may assume $p > 1$. We may assume $n \geqslant 2$ and $\lambda_k > 0$ for all $1 \leqslant k \leqslant n$. We extend the sequence λ to be indexed by all positive integers by defining $\lambda_i = \lambda_n/i$ for $i \geqslant n+1$. We define similarly $\Lambda_k, C_{k,p,\lambda}$ for $k > n$. It therefore suffices to show that $C_{k,p,\lambda} \leqslant C_{k+1,p,\lambda}$ for all $k \geqslant 1$. Applying Lemma 2.3 with $B_k = \Lambda_k^p$, $C_k = \sum_{i=1}^{k} \lambda_i \Lambda_i^{p-1}$, ones checks directly that $B_1/B_2 \leqslant C_1/C_2$. Thus, it remains to show for that all $k \geqslant 1$,

$$\frac{\Lambda_{k+1}^p - \Lambda_k^p}{\lambda_{k+1}\Lambda_{k+1}^{p-1}} \leqslant \frac{\Lambda_{k+2}^p - \Lambda_{k+1}^p}{\lambda_{k+2}\Lambda_{k+2}^{p-1}}.$$

When we regard λ_{k+2} as a variable with $0 \leqslant \lambda_{k+2} \leqslant \lambda_{k+1}$, then it is easy to see that the right-hand side expression above is a decreasing function of λ_{k+2} and hence it suffices to show that the above inequality holds with $\lambda_{k+2} = \lambda_{k+1}$. In this case, on setting $\lambda_{k+1} = x, \Lambda_k = y$ with $y \geqslant x$, we can recast the above inequality as

$$x - (x+y)^p(2x+y)^{1-p} + y^p(x+y)^{1-p} \geqslant 0.$$

We further set $z = x/y$ to recast the above inequality as

$$z - (1+z)^p(1+2z)^{1-p} + (1+z)^{1-p} \geqslant 0.$$

Upon dividing $1+z$ on both sides of the above inequality and setting $t = z/(1+z)$, we see that it suffices to show for $0 \leqslant t \leqslant 1/2$,

$$g(t) := t - (1+t)^{1-p} + (1-t)^p \geqslant 0.$$

It's easy to see that $g(0) = g'(0) = 0$ and $g''(t) = p(p-1)((1-t)^{p-2} - (1+t)^{-p-1}) \geqslant 0$ when $1 < p \leqslant 2$. This implies that $g(t)$ is an increasing function of $0 \leqslant t \leqslant 1/2$ which completes the proof. $\square$

LEMMA 2.5. *Let $p \geqslant 1$, $\lambda = (\lambda_k)_{k\geqslant 1}$ a non-negative, non-increasing sequence with $\lambda_1 > 0$. Then for all non-negative, non-increasing sequences $(a_k)_{k\geqslant 1}$, any integer $n \geqslant 1$,*

$$\left(\sum_{k=1}^n \lambda_k a_k\right)^p \leqslant C_{n,p,\lambda} \sum_{k=1}^n \lambda_k a_k \left(\sum_{i=1}^k \lambda_i a_i\right)^{p-1}, \tag{2.1}$$

where $C_{n,p,\lambda}$ is defined as in Lemma 2.4 when $1 \leqslant p \leqslant 2$ and $C_{n,p,\lambda} = p$ when $p > 2$. Moreover, when $1 \leqslant p \leqslant 2$, the constant $C_{n,p,\lambda}$ is best possible and equality in (2.1) holds when $1 < p \leqslant 2$ if and only if $a_1 = a_2 = \ldots = a_n$.

Proof. As inequality (2.1) follows from Lemma 2.1 when $p > 2$ and the assertion of the lemma follows trivially for $p = 1$, we only need to consider the case $1 < p \leqslant 2$. We define

$$f_n(x_1, x_2, \ldots, x_n) = \left(\sum_{k=1}^n \lambda_k x_k\right)^p - C_{n,p,\lambda} \sum_{k=1}^n \lambda_k x_k \left(\sum_{i=1}^k \lambda_i x_i\right)^{p-1}.$$

By homogeneity, it suffices to show $f_n \leqslant 0$ on the compact set $\{(x_1, \ldots, x_n) | 1 \geqslant x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0\}$. We may assume $\lambda_k > 0$ for all k here as discarding the zero terms and relabeling will not change the expression.

As $f_1 = 0$ holds trivially, we may assume $n \geqslant 2$ here. Assume the maximum of f_n is attained at some $\mathbf{x}_0 = ((\mathbf{x}_0)_1, (\mathbf{x}_0)_2, \ldots, (\mathbf{x}_0)_n)$ with $(\mathbf{x}_0)_1 \geqslant (\mathbf{x}_0)_2 \geqslant \ldots \geqslant (\mathbf{x}_0)_n$. If $(\mathbf{x}_0)_{m+1} = 0$ for some $1 \leqslant m < n$, then as $C_{m,p,\lambda} \leqslant C_{n,p,\lambda}$ by Lemma 2.4, it is easy to see that we are reduced to the consideration of $f_m \leqslant 0$. Thus, we may further assume $(\mathbf{x}_0)_n > 0$ here.

Suppose $(\mathbf{x}_0)_m > (\mathbf{x}_0)_{m+1} > 0$ for some $1 \leqslant m < n$. In this case we must have $\partial f_n/\partial x_m(\mathbf{x}_0) \geqslant 0$ since $\partial f_n/\partial x_m(\mathbf{x}_0) < 0$ means decreasing the value of $(\mathbf{x}_0)_m$ will increase the value of f_n, a contradiction. Similar argument implies that $\partial f_n/\partial x_{m+1}(\mathbf{x}_0) \leqslant 0$. Therefore, we conclude that we have

$$
\begin{aligned}
0 \leqslant & \frac{1}{\lambda_m}\frac{\partial f_n}{\partial x_m}(\mathbf{x}_0) - \frac{1}{\lambda_{m+1}}\frac{\partial f_n}{\partial x_{m+1}}(\mathbf{x}_0) \\
= & C_{n,p,\lambda}\left(\left(\sum_{i=1}^{m+1}\lambda_i\,(\mathbf{x}_0)_i\right)^{p-1} - \left(\sum_{i=1}^{m}\lambda_i\,(\mathbf{x}_0)_i\right)^{p-1}\right. \\
& \left. -(p-1)\lambda_m\,(\mathbf{x}_0)_m\left(\sum_{i=1}^{m}\lambda_i\,(\mathbf{x}_0)_i\right)^{p-2}\right).
\end{aligned}
$$

If $p=2$, this would imply $\lambda_{m+1}\,(\mathbf{x}_0)_{m+1} \geqslant \lambda_m\,(\mathbf{x}_0)_m$, a contradiction. If $1<p<2$, by the Mean Value Theorem, we have

$$
\begin{aligned}
& \left(\sum_{i=1}^{m+1}\lambda_i\,(\mathbf{x}_0)_i\right)^{p-1} - \left(\sum_{i=1}^{m}\lambda_i\,(\mathbf{x}_0)_i\right)^{p-1} \\
= & (p-1)\lambda_{m+1}\,(\mathbf{x}_0)_{m+1}\,\xi^{p-2} < (p-1)\lambda_m\,(\mathbf{x}_0)_m\left(\sum_{i=1}^{m}\lambda_i\,(\mathbf{x}_0)_i\right)^{p-2},
\end{aligned}
$$

as $\sum_{i=1}^{m}\lambda_i\,(\mathbf{x}_0)_i < \xi < \sum_{i=1}^{m+1}\lambda_i\,(\mathbf{x}_0)_i$. This again leads to a contradiction. Thus we must have $(\mathbf{x}_0)_1 = (\mathbf{x}_0)_2 = \ldots = (\mathbf{x}_0)_n$, which implies that $f_n(\mathbf{x}_0) = 0$ and the assertion of the lemma follows for $1 < p \leqslant 2$. $\square$

In what follows we make two remarks about Lemma 2.5. Throughout our remarks, we let $1 \leqslant p \leqslant 2$, $\lambda_k = 1$ for all k with the function f_n being defined as in the proof of Lemma 2.5 and $C_{n,p,\lambda}$ being defined as in Lemma 2.4.

REMARK 1. For any given $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, we let $\mathbf{x}'=(x_1,x_2,\ldots,x_{i+1},x_i,\ldots,x_n)$ by permuting two adjacent coordinates x_i,x_{i+1} of $\mathbf{x}$ for some $1 \leqslant i < n$, then we have

$$
\begin{aligned}
f_n(\mathbf{x}) - f_n(\mathbf{x}') = -C_{n,p,\lambda}\,\big(& x_i(a+x_i)^{p-1} + x_{i+1}(a+x_i+x_{i+1})^{p-1} \\
& -x_{i+1}(a+x_{i+1})^{p-1} - x_i(a+x_i+x_{i+1})^{p-1}\big),
\end{aligned}
$$

where we set (with empty sum being 0) $a = \sum_{k=1}^{i-1} x_k$.

It is easy to check that the function $S_r(x,y) = (x^r - y^r)/(x-y)$ is an increasing (respectively, decreasing) function of $0<y<x$ for fixed x when $r \geqslant 1$ (respectively, $0 < r \leqslant 1$). Apply this with $r = p-1$, $x = a+x_i+x_{i+1}$, $y = a+x_i$, $y' = a+x_{i+1}$, we see immediately that $f_n(\mathbf{x}) \geqslant f_n(\mathbf{x}')$ when $x_{i+1} \geqslant x_i \geqslant 0$ and $1 < p \leqslant 2$ or when $x_i \geqslant x_{i+1} \geqslant 0$ and $p \geqslant 2$.

It follows that when $p=2$ and $\lambda_k = 1$ for all k, the maximum of f_n on all non-negative sequences is the same as the maximum of f_n on all non-negative, non-increasing sequences. Thus, when $p=2, \lambda_k = 1$ for all k, the assertion of Lemma 2.5 holds for all non-negative sequences.

REMARK 2. Note that when $p > 2$, $n \geqslant 2$,

$$\frac{\partial f_n}{\partial x_n}((1,1,\ldots,1)) = n^{p-2}\left(np - C_{n,p,\lambda}(n+p-1)\right) < 0, \tag{2.2}$$

where the last inequality is equivalent to

$$\sum_{k=1}^{n} k^{p-1} < \frac{n^{p-1}}{p}(n+p-1), \quad n \geqslant 2,$$

which in turn can be easily established by induction.

Inequality (2.2) implies that in this case $0 = f_n((1,1,\ldots,1)) < f_n((1,1,\ldots,1-\varepsilon))$ for some $\varepsilon > 0$ small enough and this shows that inequality (2.1) does not hold for all non-negative, non-increasing sequences when $p > 2$.

3. Proof of Theorem 1.1

We now proceed to the proof of Theorem 1.1. Our approach here follows that of Bennett and Grosse-Erdmann in their proof of [2, Theorem 1]. By considering the sequences $(1,\ldots,1,0,0,\ldots)$, we see first that (1.4) is a necessary condition for the validity of inequality (1.3) and that $U'_p \leqslant U_p$. Conversely, assume that condition (1.4) holds. Note first that it follows from Lemma 2.1 and 2.5 that $C_{n,p,\lambda} \leqslant p$ where $C_{n,p,\lambda}$ is defined as in Lemma 2.5. Further note that for any integer $n \geqslant 1$,

$$\begin{aligned}\sum_{k=1}^{n} \lambda_k \Lambda_k^{p-1} \sum_{i=k}^{\infty} C_{i,p,\lambda} \frac{b_i}{\Lambda_i^p} &\leqslant \sum_{k=1}^{n} \lambda_k \Lambda_k^{p-1} \sum_{i=k}^{n} C_{i,p,\lambda} \frac{b_i}{\Lambda_i^p} + \sum_{k=1}^{n} \lambda_k \Lambda_k^{p-1} \sum_{i=n}^{\infty} C_{i,p,\lambda} \frac{b_i}{\Lambda_i^p} \\ &\leqslant \sum_{i=1}^{n} C_{i,p,\lambda} \frac{b_i}{\Lambda_i^p} \sum_{k=1}^{i} \lambda_k \Lambda_k^{p-1} + p \sum_{k=1}^{n} \lambda_k \Lambda_k^{p-1} \sum_{i=n}^{\infty} \frac{b_i}{\Lambda_i^p} \\ &\leqslant \sum_{i=1}^{n} C_{i,p,\lambda} \frac{b_i}{\Lambda_i^p} \sum_{k=1}^{i} \lambda_k \Lambda_k^{p-1} + pU'_p \frac{1}{\Lambda_n^p} \sum_{k=1}^{n} \lambda_k \Lambda_k^{p-1} \sum_{i=1}^{n} b_i \\ &\leqslant U''_p \sum_{i=1}^{n} b_i,\end{aligned} \tag{3.1}$$

where $U''_p = pU'_p + 1$, $1 \leqslant p \leqslant 2$, $U''_p = pU'_p + p$, $p > 2$ and we have used (1.4) in the third inequality above and the bound $\sum_{k=1}^{n} \lambda_k \Lambda_k^{p-1} \leqslant \sum_{k=1}^{n} \lambda_k \Lambda_n^{p-1} = \Lambda_n^p$ in the last inequality above.

Now by Lemma 2.5, we have, for any non-negative, non-increasing sequences $(x_n)_{n \geqslant 1}$,

$$\begin{aligned}\sum_{n=1}^{\infty} b_n \left(\sum_{k=1}^{n} \frac{\lambda_k x_k}{\Lambda_n}\right)^p &\leqslant \sum_{n=1}^{\infty} C_{n,p,\lambda} \frac{b_n}{\Lambda_n^p} \sum_{k=1}^{n} \lambda_k x_k \left(\sum_{i=1}^{k} \lambda_i x_i\right)^{p-1} \\ &= \sum_{k=1}^{\infty} \lambda_k x_k \left(\sum_{n=k}^{\infty} C_{n,p,\lambda} \frac{b_n}{\Lambda_n^p}\right) \left(\sum_{i=1}^{k} \lambda_i x_i\right)^{p-1}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \left(\lambda_k \Lambda_k^{p-1} \sum_{n=k}^{\infty} C_{n,p,\lambda} \frac{b_n}{\Lambda_n^p} \right) x_k \left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{\Lambda_k} \right)^{p-1} \\
&\leqslant U_p'' \sum_{k=1}^{\infty} b_k x_k \left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{\Lambda_k} \right)^{p-1} \\
&= U_p'' \sum_{k=1}^{\infty} b_k^{1/p} x_k b_k^{1/q} \left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{\Lambda_k} \right)^{p-1},
\end{aligned}
$$

where the second inequality above follows from Lemma 2.2 and (3.1), the sequence

$$
\left(x_k \left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{\Lambda_k} \right)^{p-1} \right)_{k \geqslant 1}
$$

being non-negative, non-increasing.

By Hölder's inequality, we then have

$$
\sum_{n=1}^{\infty} b_n \left(\sum_{k=1}^{n} \frac{\lambda_k x_k}{\Lambda_n} \right)^p \leqslant U_p'' \left(\sum_{n=1}^{\infty} b_n x_n^p \right)^{1/p} \left(\sum_{k=1}^{\infty} b_k \left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{\Lambda_k} \right)^p \right)^{1/q},
$$

which implies (1.3) with U_p being replaced by $U_p''^{p}$ and this completes the proof of Theorem 1.1. □

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Peng Gao
Department of Mathematics
School of Mathematics and Systems Science
Beihang University
P. R. China
e-mail: penggao@buaa.edu.cn