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CERTAIN APPROXIMATION PROPERTIES OF SRIVASTAVA–GUPTA OPERATORS

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Abstract. In this paper we consider a modification of the well known sequence of summation integral operators, the Srivastava-Gupta operators, in order to achieve faster convergence for our operators over the classical ones. Also, the order of approximation for our proposed operators via Peetre K-functional and weighted approximation properties are studied. Some numerical considerations regarding the approximation properties, are considered.

1. Introduction and preliminaries

In 2003, Srivastava and Gupta [16], introduced and investigated a new sequence of linear and positive operators $G_{n,c}$, which includes some well-known operators as Phillips operators [15], Baskakov-Durrmeyer type operators [9] or Bernstein-Durrmeyer operators [8]. The rate of convergence for the functions having the derivatives of bounded variation represents an active issue of study for many researchers [1, 11, 13]. In 2005, Ispir and Yuksel [7] gave the Bézier variant of Srivastava-Gupta operators and estimated the rate of convergence of function of bounded variation. Pandey and Mishra [14] established some direct estimates in simultaneous approximation and a quantitative Voronovskaja type asymptotic formula. In 2015, Malik [12] obtained important results regarding the class of operators $G_{n,c}$ in ordinary approximation and studied weighted approximation for the case $c\in\mathbb{N}\cup\{0\}$. In a recent paper, Acar et al. [2], introduced a new Stancu type generalization of Srivastava-Gupta operators to approximate integrable function on the interval $(0,\infty)$ and estimate the rate of convergence.

Let $C_\gamma[0,\infty)=\{f\in C[0,\infty):|f(t)|\leqslant M(1+t^\gamma)$ for some $M>0,\gamma>0\}$, endowed with the norm

$$\|f\|_\gamma:=\sup_{x\geqslant 0}\frac{|f(x)|}{1+x^\gamma}.$$

For $f\in C_\gamma[0,\infty)$, the $G_{n,c}$ operators are defined as follows

$$G_{n,c}(f;x)=n\sum_{k=1}^{\infty}p_{n,k}(x;c)\int_0^\infty p_{n+c,k-1}(t;c)f(t)+p_{n,0}(x;c)f(0), \tag{1}$$

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where

$$p_{n,k}(x;c)=\frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}(x)=\begin{cases} e^{-nx}, & c=0\\ (1+cx)^{-n/c}, & c=1,2,\dots,\end{cases}$$

where $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions defined on the closed interval $[0,b], b>0$, that satisfy the following properties for every $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$:

i) $\phi_{n,c}\in C^{\infty}([a,b])\ (b>a\geqslant 0)$;

ii) $\phi_{n,c}(0)=1$;

iii) $\phi_{n,c}(x)$ is completely monotone so that $(-1)^k\phi_{n,c}^{(k)}\geqslant 0\ (0\leqslant x\leqslant b)$;

iv) there exists an integer c such that $\phi_{n,c}^{(k+1)}(x)=-n\phi_{n+c,c}^{(k)}(x)$, $n>\max\{0,-c\}$, $x\in[0,b]$.

As we mentioned above, special cases of the operators $G_{n,c}$ occur when $c=0$ and $c=1$. For $c=0$, the Phillips operators are obtained as follows:

$$G_{n,0}(f;x)=n\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k-1}(t)f(t)dt+e^{-nx}f(0),$$

with $p_{n,k}(x)=p_{n,k}(x;0)=e^{-nx}\frac{(nx)^k}{k!}$, $0\leqslant x<\infty$. For $c=1$, the $G_{n,c}$ operators are reduced to the following operators considered by Gupta et al. [9]:

$$G_{n,1}(f;x)=n\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n+1,k-1}(t)f(t)dt+(1+x)^{-n}f(0),\ \ (0\leqslant x<\infty),$$

where

$$p_{n,k}(x)=p_{n,k}(x;1)=\binom{n+k-1}{k}x^k(1+x)^{-n-k},\ \ 0\leqslant x<\infty.$$

In the case of $c=-1$, on the closed interval $[0,1]$ the operators $G_{n,c}$ reduce to the operators introduced by Gupta and Maheshwari [10]:

$$G_{n,-1}(f;x)=n\sum_{k=1}^{n}p_{n,k}(x)\int_0^{1}p_{n-1,k-1}(t)f(t)dt+(1-x)^{-n}f(0),$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$.

Deo [4], proposed a modified form of the operators $G_{n,c}$, which reproduce the linear functions, in order to obtain a faster rate of convergence as follows:

$$\overline{G}_{n,c}(f;x)=n\sum_{k=1}^{\infty}p_{n,k}(r_n(x);c)\int_0^{\infty}p_{n+c,k-1}(t;c)f(t)dt+p_{n,0}(r_n(x);c)f(0),\qquad(2)$$

where $r_n(x)=\dfrac{(n-c)x}{n}$, $f\in C_{\gamma}[0,\infty)$, $x\geqslant 0$.

2. Construction of the operators and auxiliary results

The goal of this article is to construct and investigate a variant of Srivastava-Gupta operators [16], which preserve the test functions e_0 and e_2 in order to achive faster convergence for our operators over the original ones. Important result regarding a general class of linear and positive operators that preserve the test functions e_0 and e_2 are obtained by Agratini [3]. Also, we compare our result for the rate of convergence with the one of the operators introduced by Deo [4] which reproduce linear functions.

For the purpose of our study the following lemmas, which follows from [16] are mentioned below:

LEMMA 2.1. *Let $e_i(x)=x^i,\ i=0,1,2$. Then for each $x\geqslant 0$ and $n>2c$ (or n sufficiently large), we have:*

i) $G_{n,c}(e_0;x)=1$;

ii) $G_{n,c}(e_1;x)=\dfrac{nx}{n-c}$;

iii) $G_{n,c}(e_2;x)=\dfrac{nx\left((n+c)x+2\right)}{(n-c)(n-2c)}$.

LEMMA 2.2. *For each $x\geqslant 0$, $n>2c$ and $\phi_x(t)=t-x$, we have*

i) $G_{n,c}(\phi_x;x)=\dfrac{cx}{n-c}$;

ii) $G_{n,c}(\phi_x^2;x)=\dfrac{2c(n+c)x^2+2nx}{(n-c)(n-2c)}$;

iii) $G_{n,c}(\phi_x^m;x)=O\left(n^{-[(m+1)/2]}\right)$.

Our next construction of operator is obtained by modifying the $G_{n,c}$ operators in order to preserve the function e_2. By defining the function

$$v_n(x)=\frac{1}{n+c}\left(\sqrt{1+\frac{(n^2-c^2)(n-2c)}{n}x^2}-1\right),\quad x\geqslant 0, \tag{3}$$

we replace x by $v_n(x)$ in (1) and the modified version of $G_{n,c}$ is obtained:

$$G^*_{n,c}(f;x)=n\sum_{k=1}^{\infty}p_{n,k}(v_n(x);c)\int_0^{\infty}p_{n+c,k-1}(t;c)dt+p_{n,0}(v_n(x);c)f(0), \tag{4}$$

where $f\in C_\gamma[0,\infty)$.

LEMMA 2.3. *For each $n\in\mathbb{N}$, $n>2c$, the function v_n, given by (3) verifies*

$$0\leqslant v_n(x)\leqslant x,\quad x\in\mathbb{R}_+.$$

Proof. From the expression of $v_n(x)$, we observe that $\frac{d}{dx}v_n(x) \geqslant 0$, and this implies the inequalities from Lemma. □

EXAMPLE 1. *Let $n = 200$ and $c = 2$. The convergence of the operators $G_{n,c}$ and $G^*_{n,c}$ to the function f is illustrated in Figure 1 and Figure 2, for $f(x) = -5x^4 + 7x^3 - 8x + 2$, $x \in [0,5]$ and $f(x) = x^4 - x^2 + 6$, $x \in [0,1]$, respectively. We remark that the modified operator $G^*_{n,c}$ presents a order of approximation better than $G_{n,c}$.*

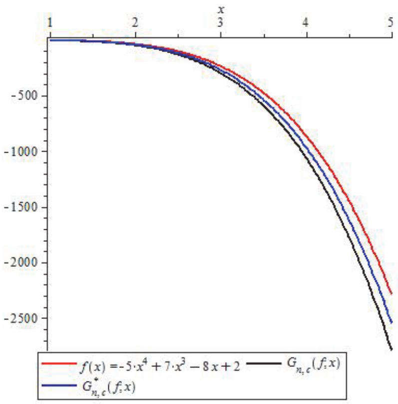

Figure 1: *The convergence of $G_{n,c}$ and $G^*_{n,c}$ to f*

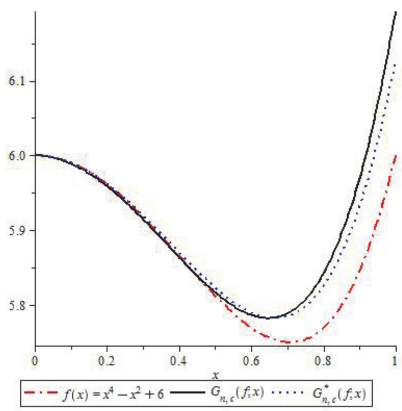

Figure 2: *The convergence of $G_{n,c}$ and $G^*_{n,c}$ to f*

EXAMPLE 2. *Let $f(x) = x^5 - 4x^3 + 2x^2 - 5$, $n = 100$, $c = 2$, $x \in [1,5]$. The approximation to the function f by the modified operators $G^*_{n,c}$, the modified operator introduced by Deo [4] and the classical ones is illustrated in the Figure 3. We note that in this case, the modified operator $G^*_{n,c}$ presents a order of approximation better than $\overline{G}_{n,c}$ and $G_{n,c}$.*

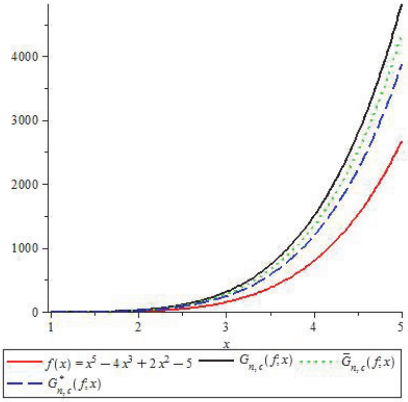

Figure 3: *The convergence of* $G_{n,c}$, $\overline{G}_{n,c}$ *and* $G^*_{n,c}$ *to* f

Also, for $f(x) = x^4 - x^2 + 6$, $n = 500$, $c = 2$, *in the Table 1 we computed the error of approximation for these operators at certain points.*

TABLE 1. *Error of approximation for* $G_{n,c}$, $\overline{G}_{n,c}$ *and* $G^*_{n,c}$

x	$\lvert G_{n,c}(f;x) - f(x)\rvert$	$\lvert \overline{G}_{n,c}(f;x) - f(x)\rvert$	$\lvert G^*_{n,c}(f;x) - f(x)\rvert$
1.0	0.071458953	0.062327130	0.049010148
1.2	0.153119462	0.129134357	0.095941904
1.4	0.286772991	0.236721136	0.170224798
1.6	0.490164775	0.398585571	0.280779126
1.8	0.783583356	0.630117911	0.437776472
2.0	1.189860597	0.948600515	0.652639752
2.2	1.734371643	1.373207911	0.938043134
2.4	2.445034978	1.925006761	1.307912260
2.6	3.352312397	2.626955762	1.777423860

By simple calculation we achieve the following results for the moments:

LEMMA 2.4. *The operators defined by (4) verify the identities:*

i) $G^*_{n,c}(e_0;x) = 1$;

ii) $G^*_{n,c}(e_1;x) = \dfrac{n}{n-c} v_n(x)$;

iii) $G^*_{n,c}(e_2;x) = x^2$.

LEMMA 2.5. *For each* $x \geqslant 0$, $n > 2c$ *and* $\phi_x(t) = t - x$, *we have*

i) $G^*_{n,c}(\phi_x;x) = \dfrac{n}{n-c} v_n(x) - x$,

ii) $G^*_{n,c}(\phi_x^2;x) = 2x\left(x - \frac{n}{n-c} v_n(x)\right)$.

Proof. From Lemma 2.4, the relations i) and ii) are obvious. □

LEMMA 2.6. *For n sufficiently large, the following inequalities*

i) $G^*_{n,c}(\phi_x;x) \leqslant G_{n,c}(\phi_x;x)$,

ii) $G^*_{n,c}(\phi_x^2;x) \leqslant G_{n,c}(\phi_x^2;x)$,

iii) $G^*_{n,c}(\phi_x^2;x) \leqslant \overline{G}_{n,c}(\phi_x^2;x)$

hold.

Proof. Since $v_n(x) < x$, we obtain $G^*_{n,c}(\phi_x;x) = \dfrac{n}{n-c}v_n(x) - x \leqslant G_{n,c}(\phi_x;x)$. The inequality $G_{n,c}(\phi_x^2;x) - G^*_{n,c}(\phi_x^2;x) \geqslant 0$ leads to the following relation

$$(n+c)(n+2xcn-xc^2)(n^2+4xcn^2-5nc-13xc^2n+xc^3) \geqslant 0,$$

that is true for n sufficiently large. In a similar way can be proved the last inequality. □

REMARK 2.1. We define the m^{th}-order moment as follows:

$$\mu_{n,c}^{(m)}(x) := G^*_{n,c}(t^m;x).$$

Using [12, Lemma1], the following recurrence relation holds

$$(n-c-mc)\mu_{n,c}^{(m+1)}(x) = x(1+cx)\frac{d}{dx}\mu_{n,c}^{(m)}(x) + (m+nx)\mu_{n,c}^{(m)}(x).$$

3. Approximation properties

Let $C_B[0,\infty)$ be the space of real-valued uniformly continous and bounded functions defined on the interval $[0,\infty)$, endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined as follows

$$\omega(f,\delta) = \sup_{0<h\leqslant\delta}\ \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Further, the K-functional is defined by

$$K_2(f,\delta) = \inf_{g\in W^2}\left\{\|f-g\| + \delta\|g''\|\right\},$$

where $\delta > 0$ and $W^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.

Let

$$\omega_2(f,\delta) = \sup_{0<h\leqslant\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

be the second order of smoothness of $f \in C_B[0,\infty)$. It is well known [5, Theorem 2.4] that there exists a constant $C > 0$ such that

$$K_2(f,\delta) \leqslant C\omega_2(f,\sqrt{\delta}). \tag{5}$$

THEOREM 3.1. *Let $f \in C_B[0,\infty)$, then we have*

$$\left|G^*_{n,c}(f;x) - f(x)\right| \leqslant M\omega_2\left(f, \sqrt{\alpha_{n,c}(x)}\right) + \omega\left(f, \left|x - \frac{n}{n-c}v_n(x)\right|\right),$$

for every $x \in [0,\infty)$, where $\alpha_{n,c}(x) = \frac{1}{4}\left(x - \frac{n}{n-c}v_n(x)\right)\left(3x - \frac{n}{n-c}v_n(x)\right)$ and $M > 0$ is a constant.

Proof. We will consider the auxiliary operator

$$\hat{G}_{n,c}(f;x) = G^*_{n,c}(f;x) + f(x) - f\left(\frac{n}{n-c}v_n(x)\right) \tag{6}$$

The operators $\hat{G}_{n,c}$ verify

$$\hat{G}_{n,c}(1;x) = G^*_{n,c}(1;x) = 1; \tag{7}$$

$$\hat{G}_{n,c}(t;x) = G^*_{n,c}(t;x) + x - \frac{n}{n-c}v_n(x) = x. \tag{8}$$

Let $g \in W^2[0,\infty)$. Using Taylor's expansion of g yields that

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad t > 0.$$

Using (7) and (8) we have

$$\hat{G}_{n,c}(g;x) - g(x) = \hat{G}_{n,c}\left(\int_x^t |t-u|g''(u)du; x\right). \tag{9}$$

By (6) we can write

$$\begin{aligned}
\left|\hat{G}_{n,c}(g;x) - g(x)\right| &\leqslant \left|G^*_{n,c}\left(\int_x^t (t-u)g''(u)du; x\right)\right| + \int_x^{\frac{n}{n-c}v_n(x)} \left|\frac{n}{n-c}v_n(x) - u\right| g''(u)du \\
&\leqslant \|g''\| \left(G^*_{n,c}\left((t-x)^2; x\right) + \left(\frac{n}{n-c}v_n(x) - x\right)^2\right) \\
&\leqslant 4\|g''\|\alpha_{n,c}(x).
\end{aligned}$$

Also, we obtain

$$|\hat{G}_{n,c}(f;x)| \leqslant |G^*_{n,c}(f;x)| + 2\|f\| \leqslant \|f\|G^*_{n,c}(1;x) + 2\|f\| \leqslant 3\|f\|.$$

But,

$$\begin{aligned}
\left|G^*_{n,c}(f;x) - f(x)\right| &\leqslant \left|\hat{G}_{n,c}(f-g;x) - (f-g)(x)\right| + \left|\hat{G}_{n,c}(g;x) - g(x)\right| \\
&\quad + \left|f(x) - f\left(\frac{n}{n-c}v_n(x)\right)\right| \\
&\leqslant 4\|f-g\| + 4\|g''\|\alpha_{n,c}(x) + \omega\left(f, \left|x - \frac{n}{n-c}v_n(x)\right|\right)
\end{aligned}$$

Taking infimum on the right hand side over all $g \in W^2$, we have

$$\left|G_{n,c}^{*}(f;x)-f(x)\right| \leqslant 4K_2\left(f,\alpha_{n,c}(x)\right)+\omega\left(f,\left|x-\frac{n}{n-c}v_n(x)\right|\right).$$

Using relation (5), we get

$$\left|G_{n,c}^{*}(f;x)-f(x)\right| \leqslant M\omega_2\left(f,\sqrt{\alpha_{n,c}(x)}\right)+\omega\left(f,\left|x-\frac{n}{n-c}v_n(x)\right|\right). \quad \square$$

REMARK 3.1. In order to show that the new modified operators $G_{n,c}^{*}$ present a order of approximation better than the operators $\overline{G}_{n,c}$ and $G_{n,c}$, we consider the following well-known Korovkin type inequalities:

$$\left|G_{n,c}^{*}(f;x)-f(x)\right| \leqslant \left(1+\frac{1}{\delta}G_{n,c}^{*}(\phi_x^2;x)\right)\omega(f;\delta),$$
$$\left|\overline{G}_{n,c}(f;x)-f(x)\right| \leqslant \left(1+\frac{1}{\delta}\overline{G}_{n,c}(\phi_x^2;x)\right)\omega(f;\delta),$$
$$\left|G_{n,c}(f;x)-f(x)\right| \leqslant \left(1+\frac{1}{\delta}G_{n,c}(\phi_x^2;x)\right)\omega(f;\delta),$$

where f is a uniformly continuous function on $[0,\infty)$. Using Lemma 2.6 it follows that the sequence of linear positive operators $G_{n,c}^{*}$ present a rate of convergence better than the operators $\overline{G}_{n,c}$ and $G_{n,c}$.

4. Voronovskaja type theorem

In this section we establish Voronvoskaja type theorem for the modified Srivastava-Gupta type operators.

LEMMA 4.1. *For the modified Srivastava-Gupta operators (4) hold:*

i) $\lim\limits_{n\to\infty} nG_{n,c}^{*}(\phi_x;x)=-cx-1$;

ii) $\lim\limits_{n\to\infty} nG_{n,c}^{*}(\phi_x^2;x)=2x(1+cx)$;

iii) $\lim\limits_{n\to\infty} n^2G_{n,c}^{*}(\phi_x^4;x)=12x^2(1+cx)^2$.

Proof. These results are obtained by simple calculations. $\square$

THEOREM 4.1. *Let $f \in C_2[0,\infty)$ and ω_a be its modulus of continuity on the finite interval $[0,a], a>0$. Then*

$$||G_{n,c}^{*}(f)-f||_{C[0,a]} \leqslant 4M(1+a^2)\eta_n(a)+2\omega_{a+1}(f,\sqrt{\eta_n(a)}),$$

where $\eta_n(a)=\max\limits_{x\in[0,a]} G_{n,c}^{*}\left((t-x)^2;x\right)$ *and* $||\cdot||_{C[0,a]}$ *denotes the sup norm on* $C[0,a]$.

Proof. For any $\delta > 0$, $x \in [0,a]$ and $t \geqslant 0$, the following inequality

$$|f(t) - f(x)| \leqslant 4M(1+x^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta), \tag{10}$$

holds (see [6]). Applying the operator $G^*_{n,c}(.;x)$ and using Cauchy-Schwarz inequality we get

$$\begin{aligned}|G^*_{n,c}(f;x) - f(x)| &\leqslant 4M(1+a^2)G^*_{n,c}((t-x)^2;x) + \left(1 + \frac{1}{\delta}G^*_{n,c}(|t-x|;x)\right)\omega_{a+1}(f,\delta)\\ &\leqslant 4M(1+a^2)\eta_n(a) + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\eta_n(a)}\right).\end{aligned}$$

Choosing $\delta = \sqrt{\eta_n(a)}$, the desired result follows. □

THEOREM 4.2. (Voronovskaya type theorem) *For every $f \in C_2[0,\infty)$ such that $f', f'' \in C_2[0,\infty)$, we have*

$$\lim_{n\to\infty} n[G^*_{n,c}(f;x) - f(x)] = (-cx-1)f'(x) + x(1+cx)f''(x),$$

uniformly with respect to $x \in [0,a]$, $a > 0$.

Proof. Using the classical Taylor's expansion of f yields

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2, \tag{11}$$

where $\varepsilon(t,x) \in C_2[0,\infty)$ and $\lim_{t\to x}\varepsilon(t,x) = 0$.

Applying the operator $G^*_{n,c}(\cdot,x)$ on both sides of (11), we have

$$\begin{aligned}\lim_{n\to\infty} n\left[G^*_{n,c}(f;x) - f(x)\right] &= \lim_{n\to\infty} nG^*_{n,c}(\phi_x;x)f'(x) + \lim_{n\to\infty} nG^*_{n,c}(\phi_x^2;x)\frac{f''(x)}{2}\\ &\quad + \lim_{n\to\infty} nG^*_{n,c}(\varepsilon(t,x)(t-x)^2;x).\end{aligned} \tag{12}$$

Using the Cauchy-Schwarz inequality in the last term of the right side of (12), we get

$$nG^*_{n,c}(\varepsilon(t,x)(t-x)^2;x) \leqslant \sqrt{G^*_{n,c}(\varepsilon^2(t,x);x)}\sqrt{n^2G_{n,c}(\phi_x^4;x)}.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, it follows from Theorem 4.1 that

$$\lim_{n\to\infty} G^*_{n,c}\left(\varepsilon^2(t,x);x\right) = \varepsilon^2(x,x) = 0,$$

uniformly with respect to $x \in [0,a]$. Now, from (12) and Lemma 4.1, we have the required result. □

5. Weighted approximation

As it is known, the uniform norm is not valid to compute the rate of converge in the case of boundless function defined on the noncompact interval $[0,\infty)$. This motivated us that in this section to study the approximation properties of the Srivastava-Gupta operators in the weighted spaces of continuous and unbounded functions defined on the internal $[0,\infty)$.

Let $C_2^*[0,\infty)=\{f\in C_2[0,\infty): \lim\limits_{x\to\infty}\frac{f(x)}{1+x^2}$ exists$\}$. For $f\in C_2^*[0,\infty)$, the weighted modulus of continuity (see [17]) is given by

$$\Omega(f,\delta)=\sup_{x\geqslant 0}\sup_{0<|h|\leqslant\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

The weighted modulus of continuity $\Omega(f,\delta)$ satisfies the following properties:

LEMMA 5.1. [17] *If $f\in C_2^*[0,\infty)$, then*

1. *$\Omega(f,\delta)$ is a monotone increasing function of δ;*
2. *$\lim\limits_{\delta\to 0^+}\Omega(f,\delta)=0$;*
3. *for any $\lambda\in[0,\infty),\Omega(f,\lambda\delta)\leqslant(1+\lambda)\Omega(f,\delta)$.*

THEOREM 5.1. *For each $f\in C_2^*[0,\infty)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|G_{n,c}^*(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}=0.$$

Proof. Let $x_0\in[0,\infty)$ be arbitrary but fixed. Then,

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|G_{n,c}^*(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}&\leqslant\sup_{x\leqslant x_0}\frac{|G_{n,c}^*(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x>x_0}\frac{|G_{n,c}^*(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}\\&\leqslant||G_{n,c}^*(f)-f||_{C[0,x_0]}+||f||_2\sup_{x>x_0}\frac{|G_{n,c}^*(1+t^2;x)|}{(1+x^2)^{1+\alpha}}\\&\quad+\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}.\end{aligned}\tag{13}$$

Since $|f(x)|\leqslant||f||_2(1+x^2)$, we have $\sup\limits_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}\leqslant\frac{||f||_2}{(1+x_0^2)^{\alpha}}$.

Let $\varepsilon>0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{||f||_2}{(1+x_0^2)^{\alpha}}<\frac{\varepsilon}{6}.\tag{14}$$

Since $\lim\limits_{n\to\infty}\sup\limits_{x>x_0}\dfrac{G^*_{n,c}(1+t^2;x)}{1+x^2}=1$, it follows that

$$\sup_{x>x_0}\frac{G^*_{n,c}(1+t^2;x)}{1+x^2}\leqslant\frac{(1+x_0^2)^a}{\|f\|_2}\frac{\varepsilon}{3}+1,$$

for sufficiently large n. Therefore,

$$\|f\|_2\sup_{x>x_0}\frac{G^*_{n,c}(1+t^2;x)}{(1+x^2)^{\alpha+1}}\leqslant\frac{\|f\|_2}{(1+x_0^2)^{\alpha}}\sup_{x>x_0}\frac{G^*_{n,c}(1+t^2;x)}{(1+x^2)}\leqslant\frac{\varepsilon}{3}+\frac{\|f\|_2}{(1+x_0^2)^{\alpha}}. \tag{15}$$

Applying Theorem 4.1, we can find for sufficiently large n

$$||G^*_{n,c}(f;x)-f(x)||_{C[0,x_0]}<\frac{\varepsilon}{3}. \tag{16}$$

Combining (14)–(16), we obtain

$$\sup_{x\in[0,\infty)}\frac{\left|G^*_{n,c}(f;x)-f(x)\right|}{(1+x^2)^{1+\alpha}}<\varepsilon. \quad \square$$

EXAMPLE 3. *The convergence of the operators $G^*_{n,c}(f;x)$ is illustrated in Figure 4, where $f(x)=x^4-3x^2+6$, $n=50$, $n=100$ and $n=500$, respectively. We can see that when the values of n are increasing, the graph of operators $G^*_{n,c}(f;x)$ are going to the graph of the function f.*

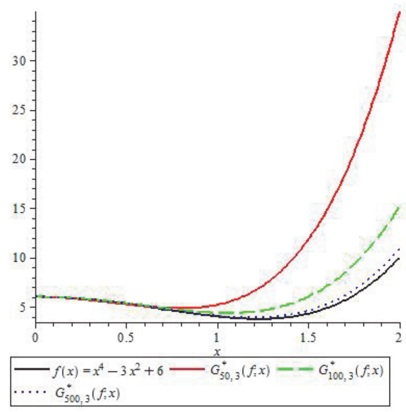

Figure 4: *The convergence of $G^*_{n,c}(f;x)$ to $f(x)$*

THEOREM 5.2. *Let $f\in C_2^*[0,\infty)$. Then for sufficiently large n, we have*

$$\sup_{x\in[0,\infty)}\frac{\left|G^*_{n,c}(f;x)-f(x)\right|}{(1+x^2)^{\frac{5}{2}}}\leqslant\tilde{C}\Omega\left(f;\frac{1}{\sqrt{n}}\right), \tag{17}$$

where $\tilde{C}$ is a positive constant.

Proof. For $x \in (0,\infty)$ and $\delta > 0$, by the definition of the weighted modulus of continuity and Lemma 5.1, we get

$$\begin{aligned}|f(t)-f(x)| &\leqslant \left(1+(x+|x-t|)^2\right)\Omega(f;|t-x|)\\ &\leqslant 2(1+x^2)(1+(t-x)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta).\end{aligned}$$

Applying $G^*_{n,c}(\cdot;x)$ both sides, we can write

$$\begin{aligned}&|G^*_{n,c}(f;x)-f(x)| \\ &\leqslant 2(1+x^2)\Omega(f;\delta)\left(1+G^*_{n,c}((t-x)^2;x)+G^*_{n,c}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\right).\end{aligned} \tag{18}$$

From Lemma 4.1, for sufficiently large n, it follows

$$nG^*_{n,c}(\phi_x^2;x) \leqslant C(1+x^2) \text{ and } n^2G^*_{n,c}(\phi_x^4;x) \leqslant C(1+x^2)^2, \tag{19}$$

where C is a positive constant.

Applying the Cauchy-Schwarz inequality in the last term of (18), we get

$$\begin{aligned}&G^*_{n,c}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right) \\ &\leqslant \frac{1}{\delta}\left(G^*_{n,c}(\phi_x^2;x)\right)^{1/2}+\frac{1}{\delta}\left(G^*_{n,c}(\phi_x^4;x)\right)^{1/2}\left(G^*_{n,c}(\phi_x^2(x);x)\right)^{1/2}.\end{aligned} \tag{20}$$

Combining the estimates (18)–(20) and taking

$$\tilde{C}=2\left(1+\sqrt{C}+2C\right) \text{ and } \delta=\frac{1}{\sqrt{n}},$$

we reach the required result. □

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