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OZAKI'S INEQUALITY AND UMEZAWA'S CONDITION FOR MULTIVALENT FUNCTIONS

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In memory Prof. Shigeo Ozaki (1906 – 1983)
and Prof. Toshio Umezawa (1914 – 2015)

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Abstract. Let $f(z)$ be analytic in $|z|<R$, continuous on $|z|=R$ and $f'(z)\neq 0$ on $|z|=R$. Then holds Ozaki's inequality that the total variation of $\arg\{f(z)\}$ on $|z|=R$ is not more than the total variation of $\arg\{\mathrm{d}f(z)\}$ on $|z|=R$. Here we consider also Umezawa's condition that

$$-\frac{\alpha}{2\alpha-3}<1+\mathfrak{Re}\frac{zf''(z)}{f'(z)}<\alpha \quad |z|<1$$

follows the univalence of $f(z)$ in $|z|<1$. In this paper we extended these results for multivalent functions.

1. Introduction

Let $\mathscr{A}(p)$ be the class of functions of the form :

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n, \quad (p\in\mathbb{N}=\{1,2,\ldots\}). \tag{1.1}$$

which are analytic in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)$ which is analytic in a domain $D\in\mathbb{C}$ is called p-valent in D if for every complex number w, the equation $f(z)=w$ have at most p roots in D and there will be a complex number w_0 such that the equation $f(z)=w_0$, has exactly p roots in D. A function $f(z)$ is called univalent in D if it is 1-valent in D. Recall that the well known Noshiro-Warschawski univalence condition, (see [3] and [16]) says that if $f(z)$ is analytic in a convex domain $D\subset\mathbb{C}$ and

$$\mathfrak{Re}\{e^{i\theta}f'(z)\}>0 \quad (z\in D), \tag{1.2}$$

for some real θ, then $f(z)$ is univalent in D. In [11] S. Ozaki extended the above result by showing that if $f(z)$ of the form (1.1) is analytic in a convex domain D and for some real θ we have

$$\mathfrak{Re}\{e^{i\theta}f^{(p)}(z)\}>0 \quad (z\in D),$$

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then $f(z)$ is at most p-valent in D. Applying Ozaki's theorem for $D=\mathbb{D}$, we find that if $f(z)\in\mathscr{A}(p)$ and

$$\mathfrak{Re}\{e^{i\theta}f^{(p)}(z)\}>0 \quad (z\in\mathbb{D}), \tag{1.3}$$

then $f(z)$ is at most p-valent in $\mathbb{D}$. In [5] it was proved that if $f(z)\in\mathscr{A}(p)$, $p\geqslant 2$, and

$$|\arg\{f^{(p)}(z)\}|<\frac{3\pi}{4} \quad (z\in\mathbb{D}), \tag{1.4}$$

then $f(z)$ is at most p-valent in $\mathbb{D}$.

Umezawa [14] proved the following theorem.

THEOREM 1.1. *Let $f(z)\in\mathscr{A}(1)$ be analytic in $\mathbb{D}$. If $f(z)$ satisfies the following condition*

$$-\frac{\alpha}{2\alpha-3}<1+\mathfrak{Re}\frac{zf''(z)}{f'(z)}<\alpha, \quad (z\in\mathbb{D}), \tag{1.5}$$

where α is an arbitrary real number not less than $3/2$, then $f(z)$ is univalent in $\mathbb{D}$. Moreover, $f(z)$ maps $|z|=r$ for every r, $0<r<1$ into a curve which is convex in one direction and

$$|a_n|\leqslant n \ \ for\ \ all\ \ n\in\mathbb{N}. \tag{1.6}$$

Now then, let us define the functions convex of order p in one direction as the following:

Let $f(z)$ be analytic in $\mathbb{D}$, $f'(z)\neq 0$ on $|z|=1$ and $f(z)$ is continuous on $|z|=1$. Let C be the image curve of $|z|=1$. If every straight-line parallel to a direction cuts C in not more than $2p$ points and there exists at least such straight-line which cuts C in $2p$ points. Then we call that $f(z)$ is convex of order p in one direction.

Putting $\alpha=\infty$, $\alpha=3/2$, $\alpha=2$ and $\alpha=3$, Umezawa [14] obtained the following theorem.

THEOREM 1.2. *Let $f(z)\in\mathscr{A}(1)$. If $f(z)$ satisfies in $\mathbb{D}$ one of the following conditions*

$$(i)\ \ 1+\mathfrak{Re}\frac{zf''(z)}{f'(z)}>-\frac{1}{2}, \quad (ii)\quad 1+\mathfrak{Re}\frac{zf''(z)}{f'(z)}<\frac{3}{2},$$
$$(iii)\ \left|1+\mathfrak{Re}\frac{zf''(z)}{f'(z)}\right|<2, \quad (iv)\quad \left|\mathfrak{Re}\frac{zf''(z)}{f'(z)}\right|<2,$$

then $f(z)$ is univalent in $\mathbb{D}$.

Theorem 1.2 was initially obtained by Ozaki [11]. Umezawa in [15] proved that

$$\left|\frac{f''(z)}{f'(z)}\right|\leqslant\sqrt{6} \quad (|z|\leqslant 1), \tag{1.7}$$

implies the univalence of $f(z)$ in $|z|\leqslant 1$. Notice also here that in [12] Ozaki proved that if $f(z)=z+a_2z^2+a_3z^3+\cdots$ is analytic in $\mathbb{D}$, with $f(z)f'(z)/z\neq 0$ there, and if either (i) or (ii) holds throughout $\mathbb{D}$, then f is univalent and convex in at least one direction in $\mathbb{D}$. It has been generalized in [10], [13]. The number $\sqrt{6}$ in (1.7), was improved to $3.05\ldots$ in [1]. In [7] it was also proved that the condition

$$\left|1+\frac{zf''(z)}{f'(z)}\right| < \sqrt{7} \quad (z \in \mathbb{D}),$$

implies the univalence of $f \in \mathscr{A}(1)$. Notice that the condition

$$\left|\frac{f''(z)}{f'(z)}\right| \leqslant 2, \quad (z \in \mathbb{D}),$$

implies that $f(z)$ is starlike in $\mathbb{D}$, [2, Th.4]. In [11] Ozaki proved also the following fundamental inequality.

LEMMA 1.3. *Let $f(z)$ be analytic in $|z|<R$, continuous on $|z|=R$ and $f'(z) \neq 0$ on $|z|=R$. Then the total variation of* $\arg\{f(z)\}$ *on $|z|=R$ is not more than the total variation of* $\arg\{\mathrm{d}f(z)\}$ *on $|z|=R$, namely*

$$\int_{|z|=R} |\mathrm{d}\arg\{f(z)\}| \leqslant \int_{|z|=R} |\mathrm{d}\arg\{\mathrm{d}f(z)\}| \tag{1.8}$$

or by a modification of the above inequality, we have

$$\int_0^{2\pi} \left|\Re\mathfrak{e}\frac{zf'(z)}{f(z)}\right| \mathrm{d}\theta \leqslant \int_0^{2\pi} \left|1+\Re\mathfrak{e}\frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta, \tag{1.9}$$

where $z=Re^{i\theta}$ and $0 \leqslant \theta \leqslant 2\pi$.

Applying Lemma 1.3, we have the following inequalities.

LEMMA 1.4. *Let $f(z) \in \mathscr{A}(p)$, $f^{(k)}(z)$ continuous on $|z|=R \leqslant 1$ and $f^{(k)}(z) \neq 0$ on $|z|=R$ for $k=0,1,2,\ldots,p$. Then we have*

$$\begin{aligned}
&\int_{|z|=R} |\mathrm{d}\arg\{f(z)\}| \leqslant \int_{|z|=R} |\mathrm{d}\arg\{\mathrm{d}f(z)\}| \\
\leqslant &\int_{|z|=R} \left|\mathrm{d}\arg\{\mathrm{d}^2 f(z)\}\right| \leqslant \int_{|z|=R} \left|\mathrm{d}\arg\{\mathrm{d}^3 f(z)\}\right| \\
\leqslant &\quad \cdots \\
\leqslant &\int_{|z|=R} \left|\mathrm{d}\arg\{\mathrm{d}^{p-1} f(z)\}\right| \leqslant \int_{|z|=R} |\mathrm{d}\arg\{\mathrm{d}^p f(z)\}|.
\end{aligned} \tag{1.10}$$

Equality in (1.10) holds for $f(z)=z^p$.

Proof. Applying Lemma 1.3, we have

$$\begin{aligned}
&\int_{|z|=R} |\mathrm{d}\arg\{f(z)\}| = \int_{|z|=R} \left|\frac{\mathrm{d}\arg\{f(z)\}}{\mathrm{d}\theta}\right| \mathrm{d}\theta = \int_{|z|=R} \left|\mathfrak{Re}\frac{zf'(z)}{f(z)}\right| \mathrm{d}\theta \\
&\leqslant \int_{|z|=R} |\mathrm{d}\arg\{\mathrm{d}f(z)\}| = \int_{|z|=R} \left|\mathrm{d}\arg\left\{\left(\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right)\mathrm{d}z\right\}\right| \\
&= \int_{|z|=R} \left|\mathrm{d}\arg\left\{f'(z)\right\} + \mathrm{d}\arg\{\mathrm{d}z\}\right| = \int_{|z|=R} \left|\frac{\mathrm{d}\arg\{f'(z)\}}{\mathrm{d}\theta} + \frac{\mathrm{d}\arg\{\mathrm{d}z\}}{\mathrm{d}\theta}\right| \mathrm{d}\theta \\
&= \int_{|z|=R} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta = \int_{|z|=R} |\mathrm{d}\arg\{\mathrm{d}(\mathrm{d}f(z))\}| \\
&= \int_{|z|=R} \left|\mathrm{d}\arg\left\{\mathrm{d}^2 f(z)\right\}\right| \leqslant \int_{|z|=R} \left|\mathrm{d}\arg\left\{\mathrm{d}(\mathrm{d}^2 f(z))\right\}\right| \\
&= \int_{|z|=R} \left|\mathrm{d}\arg\left\{\mathrm{d}^3 f(z)\right\}\right| \leqslant \int_{|z|=R} \left|\mathrm{d}\arg\left\{\mathrm{d}(\mathrm{d}^3 f(z))\right\}\right| \\
&\leqslant \quad \ldots \\
&\leqslant \int_{|z|=R} \left|\mathrm{d}\arg\left\{\mathrm{d}^{p-1} f(z)\right\}\right| = \int_{|z|=R} \left|\mathrm{d}\arg\left\{\left(\frac{\mathrm{d}^{p-1} f(z)}{(\mathrm{d}z)^{p-1}}\right)(\mathrm{d}z)^{p-1}\right\}\right| \\
&= \int_{|z|=R} \left|\mathrm{d}\arg\left\{f^{(p-1)}(z)\right\} + (p-1)\mathrm{d}\arg\{\mathrm{d}z\}\right| \\
&= \int_{|z|=R} \left|\frac{\mathrm{d}\arg\left\{f^{(p-1)}(z)\right\}}{\mathrm{d}\theta} + (p-1)\frac{\mathrm{d}\arg\{\mathrm{d}z\}}{\mathrm{d}\theta}\right| \mathrm{d}\theta \\
&= \int_{|z|=R} \left|p - 1 + \mathfrak{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| \mathrm{d}\theta \leqslant \int_{|z|=R} \left|\frac{\mathrm{d}\arg\{\mathrm{d}^p f(z)\}}{\mathrm{d}\theta}\right| \mathrm{d}\theta \\
&= \int_{|z|=R} \left|p + \mathfrak{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| \mathrm{d}\theta,
\end{aligned}$$

where $z = Re^{i\theta}$ and $0 \leqslant \theta \leqslant 2\pi$. It completes the proof of Lemma 1.4 $\square$

In [14] Umezawa obtained the following theorem.

LEMMA 1.5. *Let $f(z)$ be analytic in $|z| \leqslant 1$ and continuous and $f'(z) \neq 0$ on $|z| = 1$. If $f(z)$ satisfies the condition*

$$\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta < 2(p+1)\pi, \tag{1.11}$$

where $z = e^{i\theta}$ and $0 \leqslant \theta \leqslant 2\pi$, then $f(z)$ is at most p-valent in $|z| \leqslant 1$.

COROLLARY 1.6. *Let $f(z)$ be analytic in $|z| \leqslant 1$ and continuous and $f'(z) \neq 0$ on $|z| = 1$. If $f(z)$ satisfies the condition*

$$\int_0^{2\pi} \left|p + \mathfrak{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| \mathrm{d}\theta < 2(p+1)\pi, \tag{1.12}$$

where $z = e^{i\theta}$ *and* $0 \leqslant \theta \leqslant 2\pi$, *then* $f(z)$ *is at most* p*-valent in* $|z| \leqslant 1$.

Proof. From Lemma 1.4 with $R = 1$, we have

$$\int_0^{2\pi} \left| 1 + \mathfrak{Re} \frac{zf''(z)}{f'(z)} \right| d\theta \leqslant \int_0^{2\pi} \left| p + \mathfrak{Re} \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right| d\theta,$$

where $z = e^{i\theta}$ and $0 \leqslant \theta \leqslant 2\pi$, so Lemma 1.5 and (1.12) follow that $f(z)$ is at most p-valent in D. $\square$

Applying the above lemmas, we have the following theorem.

THEOREM 1.7. *Let* $f(z) \in \mathscr{A}(p)$ *and suppose that* $f^{(k)}(z) \neq 0$ *in* $\mathbb{D} \setminus \{0\}$ *for* $k = 0, 1, 2, \ldots, p$ *and suppose that have*

$$-\frac{\alpha}{2\alpha - (2p+1)} < p + \mathfrak{Re} \left\{ \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right\} < \alpha \quad (z \in \mathbb{D}), \tag{1.13}$$

where α *is an arbitrary real number,* $\alpha \geqslant p + 1/2$. *Then* $f(z)$ *is at most* p*-valent in* $\mathbb{D}$.

Proof. Let us put

$$\arg\{d^p f(z)\} = \Theta, \quad z = e^{i\theta} \quad \text{and} \quad 0 \leqslant \theta \leqslant 2\pi,$$

then we have

$$\frac{d \arg\{d^p f(z)\}}{d\theta} = p + \mathfrak{Re} \left\{ \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right\} = \frac{d\Theta}{d\theta} \quad \text{on} \quad |z| = r < 1. \tag{1.14}$$

From the hypothesis, we have

$$-\frac{\alpha}{2\alpha - (2p+1)} < \frac{d\Theta}{d\theta} < \alpha \quad \text{in} \quad |z| < 1.$$

Now, let us put by C_1 the part of $|z| = r$ on which

$$\frac{d\Theta}{d\theta} > 0 \quad \text{and} \quad \text{put} \quad \int_{C_1} d\arg\{z\} = x$$

and by C_2 the part of $|z| = r$ on which

$$\frac{d\Theta}{d\theta} \leqslant 0 \quad \text{and} \quad \text{so} \quad \int_{C_2} d\arg\{z\} = 2\pi - x. \tag{1.15}$$

Putting

$$y_1 = \int_{C_1} d\Theta, \quad -y_2 = \int_{C_2} d\Theta$$

then from (1.14), we have

$$\int_C \mathrm{d}\Theta = y_1 - y_2 = 2p\pi. \tag{1.16}$$

On the other hand, we have

$$y_1 = \int_{C_1} \frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\mathrm{d}\theta < \alpha x \tag{1.17}$$

and

$$y_2 = \int_{C_2} \left(-\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)\mathrm{d}\theta < (2\pi - x)\frac{\alpha}{2\alpha-(2p+1)} = \frac{2\pi\alpha - \alpha x}{2\alpha-(2p+1)}. \tag{1.18}$$

Now, we want to prove that $y_1 < (2p+1)\pi$ and so, if we suppose

$$y_1 \geqslant (2p+1)\pi,$$

then we must have

$$y_2 \geqslant \pi, \tag{1.19}$$

and by (1.17), we have

$$\alpha x > (2p+1)\pi. \tag{1.20}$$

Then from (1.18) and (1.20), we have

$$y_2 < \frac{2\pi\alpha-(2p+1)\pi}{2\alpha-(2p+1)} = \pi\left(\frac{2\alpha-(2p+1)}{2\alpha-(2p+1)}\right) = \pi.$$

This contradicts (1.19). Therefore, we must have

$$y_1 < (2p+1)\pi.$$

This shows that

$$\int_C \left| p + \mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}\right| \mathrm{d}\theta < 2(p+1)\pi$$

where $|z| = r$ and $0 < r < 1$. Applying Nunokawa's result, which is a generalization of (1.11), [4, Th.3], or from Lemma 1.5, we complete the proof of Theorem 1.7 □

Notice that it was proved earlier [4], that if $f(z) \in \mathscr{A}(p)$ and if it satisfies

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad z \in \mathbb{D},$$

then $f(z)$ is p-valently starlike in $\mathbb{D}$ and

$$\mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0, \quad z \in \mathbb{D},$$

for $k=1,2,\ldots,(p-1)$. Furthermore, in [6] it was proved that if $f(z)\in\mathscr{A}(p)$, $p\geqslant 3$, and $f(z)$ is typically real in $\mathbb{D}$, $f^{(p-1)}(z)/z\neq 0$ and that

$$\left|\mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}\right|<\frac{1}{2}\left\{\alpha+\frac{2}{\pi}\tan^{-1}(\alpha)\right\}$$

in the unit disc $\mathbb{D}$, where $\alpha=1.2951672353\ldots$, then

$$\mathfrak{Re}\left\{\frac{f^{(p-2)}(z)}{z^2}\right\}>0,\quad z\in\mathbb{D},$$

and $f(z)$ is p-valent in $\mathbb{D}$. Note here that in [8] it was proved a related result of the form: if $f(z)\in\mathscr{A}(p)$, $p\geqslant 2$, $0<\alpha<1$, and

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}>-\frac{2\alpha^2-\alpha+1}{2(1-\alpha)},\quad (z\in\mathbb{D}),\tag{1.21}$$

then

$$\mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\}>1+\alpha,\quad (z\in\mathbb{D})$$

or $f(z)$ is at most p-valent in $\mathbb{D}$.

Putting $\alpha=\infty$, $\alpha=p+1/2$, $\alpha=p+1$ and $\alpha=2p+1$ in Theorem 1.7 gives the following result.

COROLLARY 1.8. *Let $f(z)\in\mathscr{A}(p)$ and suppose that $f^{(k)}(z)\neq 0$ in $\mathbb{D}\setminus\{0\}$ for $k=0,1,2,\ldots,p$. If $f(z)$ satisfies in $\mathbb{D}$ one of the following conditions*

$$(i)\ \ p+\mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}>-\frac{1}{2},\qquad (ii)\quad p+\mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}<p+\frac{1}{2},$$

$$(iii)\ \left|p+\mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}\right|<p+1,\qquad (iv)\quad \left|\mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}\right|<p+1,$$

then $f(z)$ is at most p-valent in $\mathbb{D}$.

For $p=2$ Theorem 1.7 becomes the following corollary.

COROLLARY 1.9. *Let $f(z)\in\mathscr{A}(2)$ and suppose that $f^{(k)}(z)\neq 0$ in $\mathbb{D}\setminus\{0\}$ for $k=0,1,2$ and suppose that have*

$$-\frac{\alpha}{2\alpha-5}<2+\mathfrak{Re}\left\{\frac{zf'''(z)}{f''(z)}\right\}<\alpha\quad (z\in\mathbb{D}),\tag{1.22}$$

where α is an arbitrary real number, $\alpha>5/2$. Then $f(z)$ is at most 2-valent in $\mathbb{D}$.

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