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CIRCULAR REARRANGEMENT INEQUALITY

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Abstract. This paper presents an analogue of the rearrangement inequality, namely the circular rearrangement inequality. It holds for any finite sequence of real numbers. A volume-invariant packing problem and a combinatorial isoperimetric problem are addressed, as the geometric interpretation of the inequality.

1. Introduction

For two real sequences $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$, the rearrangement inequality (RI) can be stated as [9]

$$\sum_{i=1}^{n} a_i b_{n-i+1} \leqslant \sum_{i=1}^{n} a_i b_{r(i)} \leqslant \sum_{i=1}^{n} a_i b_i \tag{1}$$

where r denotes a permutation of $(1,2,\cdots,n)$.

This inequality is easy to learn and yet a powerful tool. For example, many fundamental inequalities, e.g. the AM-GM-HM inequality, the Cauchy-Schwarz inequality, and the Chebyshev's Sum inequality, can be generated from the RI [10]. Actually, the RI has been a result of fundamental importance in mathematics [5, 11] and in practice [6, 7, 8].

Therefore, it is of interest to study every aspect of the inequality. The obtained results split into three categories. The first derives the reverse of the RI, where worth noting the Kantorovich inequality [13], the Cassels inequality [3], and the Bourin inequality [1]. The second expands the RI to vectors or matrices, notably including the von Neumann's trace inequality [17], the Richter inequality [14], the Mirsky inequality [12], among some others [4, 16]. The third applies the idea of rearrangement into the manipulation of function space, e.g. the Hardy-Littlewood inequality [9] and the Riesz rearrangement inequality [15], thus providing a powerful tool for analysis [2, 5].

Nevertheless, the prosperity of the RI makes it natural to look for something analogous. The beauty of the RI, due to the author, resides in its symmetry and universality. It rearranges a sequence of numbers along the line of the other sequence (see Eq. (1)),

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and then calculates their inner product. An analogue of the inequality might exist if we conduct the rearrangement and the product in a different way.

In particular, we introduce and consider the *circular product* of two simultaneously rearranged n-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\langle \boldsymbol{a}(r), \boldsymbol{b}(r) \rangle \overset{def}{=} a_{r(1)}b_{r(2)} + a_{r(2)}b_{r(3)} + \cdots + a_{r(n-1)}b_{r(n)} + a_{r(n)}b_{r(1)} \tag{2}$$

where r is a permutation of $(1,2,\cdots,n)$. We will denote by $\langle \boldsymbol{a}(\mathbf{r}) \rangle$ the circular product of $\boldsymbol{a}(r)$ and itself.

This paper shows that there are fixed permutations for which $\langle \boldsymbol{a}(r) \rangle$ attains its minimum and maximum, respectively. Hence, the result is named as the circular rearrangement inequality (CRI). The permutations for which extreme is achieved are completely determined by the ordering of $\boldsymbol{a}$'s entries. For example, let $n = 9$ and denote the entries of $\boldsymbol{a}$ by round disks in different diameters - the greater the entry, the larger the disk is. Then the CRI can be illustrated by Fig. 1.

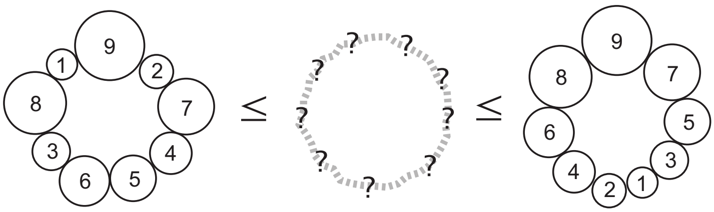

Figure 1: *An illustration of the CRI*

This paper consists of four parts. In Section 2, the CRI is presented, proved, and then applied to the sequence of natural numbers. The geometric interpretation of the CRI is addressed in Section 3, along with an insight into the case when the vectors involved in the circular product are not identical. Then, the paper is concluded in Section 4.

2. The CRI

We will state the CRI in two theorems and then prove them. The numbers and the vectors considered in this paper are all real. Besides, we need the following conventions.

DEFINITION 1. For a given n-vector $\boldsymbol{a}$, we call a permutation r the maximal or minimal circular permutation (maxCP or minCP) of $\boldsymbol{a}$ if $\langle \boldsymbol{a}(r) \rangle$ attains its maximum or minimum over all possible permutations.

DEFINITION 2. Given a permutation $r = (r_1, r_2, \cdots, r_i, r_{i+1}, \cdots, r_{n-1}, r_n)$ of $(1,2,\cdots,n)$, we call the permutation $s = (r_{i+1}, \cdots, r_{n-1}, r_n, r_1, r_2, \cdots, r_i)$ a circular shift of r, for some $1 \leqslant i \leqslant n-1$. Similarly, we call $\boldsymbol{a}(s)$ a circular shift of $\boldsymbol{a}(r)$. In addition, we denote by $r' = (r_n, r_{n-1}, \cdots, r_2, r_1)$ the reverse of r, and, in consequence, call $\boldsymbol{a}(r')$ the reverse of $\boldsymbol{a}(r)$.

REMARK 1. $\langle \boldsymbol{a}(r), \boldsymbol{b}(r) \rangle = \langle \boldsymbol{a}(s), \boldsymbol{b}(s) \rangle$ if s is a circular shift of r.

REMARK 2. $\langle \boldsymbol{a}(r') \rangle \equiv \langle \boldsymbol{a}(r) \rangle$.

REMARK 3. For two permutations r and s and two n-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $b_i \equiv a_i + \alpha$ for $1 \leqslant i \leqslant n$ and $\alpha \in \mathbb{R}$, if $\langle \boldsymbol{a}(r) \rangle \geqslant \langle \boldsymbol{a}(s) \rangle$, then $\langle \boldsymbol{b}(r) \rangle \geqslant \langle \boldsymbol{b}(s) \rangle$.

DEFINITION 3. Given a permutation $r = (r_1, \cdots, r_{i-1}, r_i, r_{i+1}, \cdots, r_{j-1}, r_j, r_{j+1}, \cdots, r_n)$ of $(1, 2, \cdots, n)$, we call the permutation $s = (r_1, \cdots, r_{i-1}, r_j, r_{j-1}, \cdots, r_{i+1}, r_i, r_{j+1}, \cdots, r_n)$ a turnover of r, for some $1 \leqslant i < j \leqslant n$. Similarly, we call $\boldsymbol{a}(s)$ a turnover of $\boldsymbol{a}(r)$.

REMARK 4. As $\langle \boldsymbol{a}(r) \rangle$ is invariant upon circular shifts and the reverse of r, it suffices to consider only one permutation out of the whole class. On the contrary, a partial turnover of r usually alters the value of $\langle \boldsymbol{a}(r) \rangle$. For example, let $r = (1, \cdots, p, 2, \cdots, q)$ and $s = (1, \cdots, p, q, \cdots, 2)$ be a turnover of r by $(2, \cdots, q)$. Then

$$\langle \boldsymbol{a}(s) \rangle = \langle \boldsymbol{a}(r) \rangle + a_1 a_2 + a_p a_q - a_1 a_q - a_2 a_p \tag{3}$$

2.1. The maxCP

THEOREM 1. *Suppose* $\boldsymbol{a}$ *is an* n*-vector with* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$, $n \geqslant 3$, *then* $(1, 3, 5, \cdots, n, \cdots, 6, 4, 2)$ *is* $\boldsymbol{a}$*'s maxCP.*

Proof. The mathematical induction is adopted. It is trivial to verify Theorem 1 for $n = 3$ and $n = 4$. Assume it is valid for $n \leqslant k$. Now let $n = k + 1$ and $\boldsymbol{a}$ be a $(k+1)$-vector with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant a_{k+1}$.

Firstly, we will prove that $(2, 1, 3)$ must exist in $\boldsymbol{a}$'s maxCP. Otherwise, without loss of generality, suppose $r = (1, t, \cdots, 3, q, \cdots, p, 2, \cdots, s)$ is $\boldsymbol{a}$'s maxCP (see Fig. 2).

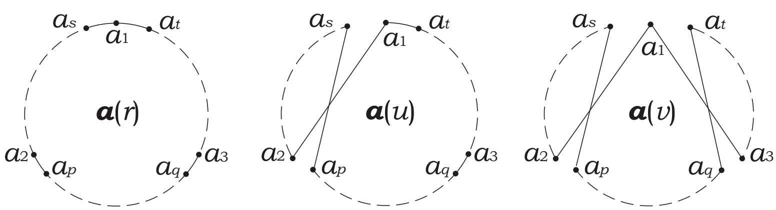

Figure 2: *Turnovers that outperform* $\boldsymbol{a}(r)$

For $u = (1, t, \cdots, 3, q, \cdots, p, s, \cdots, 2)$ and $v = (1, 3, \cdots, t, q, \cdots, p, s, \cdots, 2)$, it holds

$$\langle \boldsymbol{a}(u) \rangle - \langle \boldsymbol{a}(r) \rangle = a_1 a_2 + a_s a_p - a_1 a_s - a_2 a_p = (a_1 - a_p)(a_2 - a_s) \geqslant 0 \tag{4}$$

$$\langle \boldsymbol{a}(v) \rangle - \langle \boldsymbol{a}(u) \rangle = a_1 a_3 + a_t a_q - a_1 a_t - a_3 a_q = (a_1 - a_q)(a_3 - a_t) \geqslant 0 \tag{5}$$

This contradicts that r is the maxCP. Hence, $\boldsymbol{a}$'s maxCP must contain $(2,1,3)$.

As $(2,1,3)$ exists in $\boldsymbol{a}$'s maxCP, we now focus on the maxCP of $\boldsymbol{b} = ([a_2,a_1,a_3],a_4,\cdots,a_{k+1})$, where $[.]$ imposes a_2, a_1, and a_3 in order $(2,1,3)$ or $(3,1,2)$.

Denote by $\boldsymbol{c} = ([a_2,a_3],a_4,\cdots,a_{k+1})$, where $[.]$ imposes a_2 and a_3 in order $(2,3)$ or $(3,2)$. The maximum of $\langle \boldsymbol{c}(.)\rangle$ is no greater than that of $\boldsymbol{d} = (a_2,a_3,a_4,\cdots,a_{k+1})$. By induction hypothesis, $\boldsymbol{d}$'s maxCP is $w = (2,4,\cdots,k+1,\cdots,5,3)$. In w, a_2 and a_3 coincide to be in order $(2,3)$. Thus w is also $\boldsymbol{c}$'s maxCP.

On the other hand, for any permutation x of $([2,3],4,\cdots,k+1)$ and $y = ([2,1,3], x(4,\cdots,k+1))$, it holds $\langle \boldsymbol{b}(y)\rangle \equiv \langle \boldsymbol{c}(x)\rangle + (a_2a_1 + a_1a_3)$. That is, they attain maximum simultaneously. Since $\boldsymbol{c}$'s maxCP is w, $\boldsymbol{b}$'s maxCP (i.e. $\boldsymbol{a}$'s maxCP), have to be $(1,3,5,\cdots,k+1,\cdots,6,4,2)$. $\square$

2.2. The minCP

THEOREM 2. *Suppose $\boldsymbol{a}$ is an n-vector with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$, $n \geqslant 3$, then $(1,n-1,3,n-3,5,n-5,\cdots,n-6,6,n-4,4,n-2,2,n)$ is $\boldsymbol{a}$'s minCP.*

Proof. According to Remark 3, we suppose $a_n > 0$.

The mathematical induction is adopted. It is trivial to verify Theorem 2 for $n=3$ and $n=4$. Assume it is valid for $n \leqslant k$. Now let $n = k+1$ and $\boldsymbol{a}$ be a $(k+1)$-vector such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant a_{k+1} > 0$.

Firstly, we will prove that $\boldsymbol{a}$'s minCP must contain $(k+1,1,k)$. Otherwise, without lost of generality, suppose $\boldsymbol{a}$'s minCP is $r = (1,t,\cdots,k,q,\cdots,p,k+1,\cdots,s)$ (see Fig. 3).

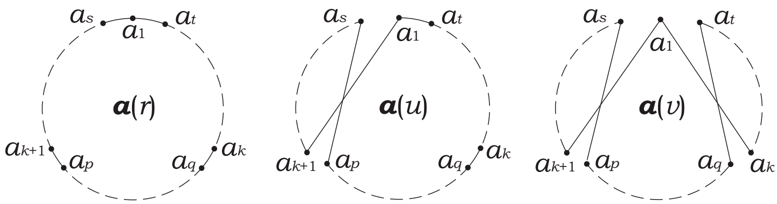

Figure 3: *Turnovers that outperform $\boldsymbol{a}(r)$*

For $u = (1,t,\cdots,k,q,\cdots,p,s,\cdots,k+1)$ and $v = (1,k,\cdots,t,q,\cdots,p,s,\cdots,k+1)$, it holds

$$\langle \boldsymbol{a}(u)\rangle - \langle \boldsymbol{a}(r)\rangle = a_1a_{k+1} + a_sa_p - a_1a_s - a_{k+1}a_p = (a_1-a_p)(a_{k+1}-a_s) \leqslant 0 \quad (6)$$

$$\langle \boldsymbol{a}(v)\rangle - \langle \boldsymbol{a}(u)\rangle = a_1a_k + a_ta_q - a_1a_t - a_ka_q = (a_1-a_q)(a_k-a_t) \leqslant 0 \quad (7)$$

This contradicts that r is the minCP. Therefore, $\boldsymbol{a}$'s minCP must contain $(k+1,1,k)$.

Secondly, we will prove that $\boldsymbol{a}$'s minCP must contain $(2,k+1,1,k)$. Otherwise, without lost of generality, suppose $\boldsymbol{a}$'s minCP is $w = (k+1,1,k,\cdots,p,2,\cdots,t)$. Then for $x = (k+1,1,k,\cdots,p,t,\cdots,2)$, it holds

$$\langle \boldsymbol{a}(x)\rangle - \langle \boldsymbol{a}(w)\rangle = a_2a_{k+1} + a_ta_p - a_2a_p - a_{k+1}a_t = (a_2-a_t)(a_{k+1}-a_p) \leqslant 0 \quad (8)$$

This contradicts that w is the minCP. Thus, $\boldsymbol{a}$'s minCP must have $(2,k+1,1,k)$.

Now let $\boldsymbol{b} = ([a_2,a_k],a_3,\cdots,a_{k-1})$, where $[.]$ imposes a_2 and a_k in order $(2,k)$ or $(k,2)$. The minimum of $\langle \boldsymbol{b}(.)\rangle$ is no less than that of $\boldsymbol{c} = (a_2,a_3,\cdots,a_k)$. By induction hypothesis, $\boldsymbol{c}$'s minCP is $y = (k,3,k-2,5,k-4,\cdots,k-3,4,k-1,2)$. In y, a_2 and a_k coincide to be in order $(2,k)$. Hence, y is also $\boldsymbol{c}$'s minCP.

On the other hand, for any permutation z of $([2,k],3,\cdots,k-1)$ and $f = ([2,k+1,1,k],z(3,\cdots,k-1))$, it holds $\langle \boldsymbol{b}(f)\rangle \equiv \langle \boldsymbol{c}(z)\rangle + (a_2a_{k+1}+a_{k+1}a_1+a_1a_k)$. That is, they attain maximum simultaneously. Since $\boldsymbol{c}$'s minCP is y, $\boldsymbol{b}$'s minCP (i.e. $\boldsymbol{a}$'s minCP), must be $(1,k,3,k-2,5,k-4,\cdots,k-3,4,k-1,2,k+1)$. $\square$

2.3. The CRI and the RI

For an n-vector $\boldsymbol{a}$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$, we denote its maxCP and minCP by

$$r^U = (1,3,5,\cdots,n,\cdots,6,4,2) \tag{9}$$

$$r^L = (1,n-1,3,n-3,\cdots,n-4,4,n-2,2,n) \tag{10}$$

Then for any permutation r of $(1,2,\cdots,n)$, the CRI holds

$$\langle \boldsymbol{a}(r^L)\rangle \leqslant \langle \boldsymbol{a}(r)\rangle \leqslant \langle \boldsymbol{a}(r^U)\rangle \tag{11}$$

Nevertheless, a chain inequality is naturally induced by combining the RI and the CRI (i.e. combining Eq. (1) and Eq. (11)):

$$\sum_{i=1}^{n} a_i a_{n-i+1} \leqslant \langle \boldsymbol{a}(r^L)\rangle \leqslant \langle \boldsymbol{a}(r)\rangle \leqslant \langle \boldsymbol{a}(r^U)\rangle \leqslant \sum_{i=1}^{n} a_i^2 \tag{12}$$

Applying this to the sequence of natural numbers, we obtain the following.

COROLLARY 1. *Denote by* $\boldsymbol{n} = (1,2,\cdots,n)$ *and by* $|\boldsymbol{n}| = \sqrt{\sum_{k=1}^{n} k^2}$. *It holds*

$$\left(\frac{1}{2}+\frac{3}{4n+2}\right) \leqslant \left(\frac{1}{2}+\frac{3}{4n+2}\right) + \frac{n-2+mod(n,2)}{2|\boldsymbol{n}|^2} \leqslant \frac{\langle \boldsymbol{n}(r)\rangle}{|\boldsymbol{n}|^2} \leqslant 1 - \frac{2n-3}{|\boldsymbol{n}|^2} \leqslant 1 \tag{13}$$

Proof. Denote by $F_n = \sum_{k=1}^{n} k(n-k+1)$, $P_n = \langle \boldsymbol{n}(r^L)\rangle$, $Q_n = \langle \boldsymbol{n}(r^U)\rangle$, and $G_n = |\boldsymbol{n}|^2 = \dfrac{n(n+1)(2n+1)}{6}$. It is easy to see that $F_n = \dfrac{n(n+1)(n+2)}{6} = \dfrac{|\boldsymbol{n}|^2}{2} + \dfrac{n(n+1)}{4}$. In addition, no matter n is odd or even, we always have $Q_{n+1} - Q_n = (n^2+2n-1)$. Hence $Q_n = |\boldsymbol{n}|^2 - (2n-3)$.

To derive P_n, we need to clarify the middle terms of $\boldsymbol{n}(r^L) = (1,n-1,3,n-3,\cdots,n-4,4,n-2,2,n)$. This can be done by arguing on n. It follows that, for $k \geqslant 1$

$$\boldsymbol{n}(r^L) = \begin{cases} (1,4k-1,\cdots,2k-1,2k+1,2k,\cdots,4k-2,2,4k), & n = 4k \\ (1,4k,\cdots,2k+1,2k,\cdots,4k-1,2,4k+1), & n = 4k+1 \\ (1,4k+1,\cdots,2k+1,2k+2,2k,\cdots,4k,2,4k+2), & n = 4k+2 \\ (1,4k+2,\cdots,2k+1,2k+2,\cdots,4k+1,2,4k+3), & n = 4k+3 \end{cases} \tag{14}$$

Consequently, we see

$$\begin{cases} P_{4k+1} - P_{4k} & = 8k^2 + 6k + 2 = (4k+1)(4k+2)/2 + 1 \\ P_{4k+2} - P_{4k+1} & = 8k^2 + 10k + 3 = (4k+2)(4k+3)/2 \\ P_{4k+3} - P_{4k+2} & = 8k^2 + 14k + 7 = (4k+3)(4k+4)/2 + 1 \\ P_{4k+4} - P_{4k+3} & = 8k^2 + 18k + 10 = (4k+4)(4k+5)/2 \end{cases} \tag{15}$$

which implies

$$P_n - P_{n-1} = n(n+1)/2 + mod(n,2) \tag{16}$$

In view of $P_4 = 21$, we obtain

$$P_n = \frac{n(n+1)(n+2)}{6} + \frac{n-2+mod(n,2)}{2} \tag{17}$$

According to Eq. (12), we have

$$F_n \leqslant P_n \leqslant \langle \boldsymbol{n}(r) \rangle \leqslant Q_n \leqslant G_n \tag{18}$$

Then Eq. (13) follows from Eq. (18) by dividing by G_n. □

3. Geometric interpretation

3.1. A volume-invariant packing problem

For a 3-vector $\boldsymbol{a}$, it yields $\langle \boldsymbol{a}(r) \rangle \equiv a_1a_2 + a_2a_3 + a_3a_1$. Intuitively, it is the total shadowed areas of the cuboid in Fig. 4.

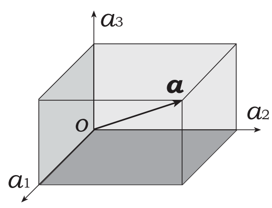

Figure 4: $\langle \boldsymbol{a}(r) \rangle$ *equals to half the surface area of the cuboid*

In general, for an n-vector $\boldsymbol{v}$, its circular product is half of the surface area of a hyper-rectangle that holds $\boldsymbol{v}$ as its diagonal line. The circular product of the 3-vector $\boldsymbol{a}$ is invariant upon all permutations. However, the circular product of the n-vector $\boldsymbol{v}$ for $n \geqslant 4$ depends on permutations. It is the permutation, which circularly sequences the numbers, that determines the underlying hyper-rectangle. Consequently, the surface area of the hyper-rectangle varies (upon permutations), whilst its volume and the length of its diagonal line are invariant. The CRI has specified the extreme hyper-rectangles that pack up a vector in fixed length.

3.2. A combinatorial isoperimetric problem

Still, we may add the square of the euclidean norm of the vector to each side of the CRI (see Eq. (11))

$$\langle \boldsymbol{a}(r^L)\rangle + \sum_{i=1}^{n} a_i^2 \leqslant \langle \boldsymbol{a}(r)\rangle + \sum_{i=1}^{n} a_i^2 \leqslant \langle \boldsymbol{a}(r^U)\rangle + \sum_{i=1}^{n} a_i^2 \tag{19}$$

Equivalently,

$$\left\{\begin{array}{l} \cdots + (a_2+a_n)^2 + (a_n+a_1)^2 + (a_1+a_{n-1})^2 + (a_{n-1}+a_3)^2 + \cdots \\ \leqslant (a_{r(1)}+a_{r(2)})^2 + (a_{r(2)}+a_{r(3)})^2 + \cdots + (a_{r(n)}+a_{r(1)})^2 \\ \leqslant \cdots + (a_4+a_2)^2 + (a_2+a_1)^2 + (a_1+a_3)^2 + (a_3+a_5)^2 + \cdots \end{array}\right. \tag{20}$$

Multiple by $\pi = 3.1415926\cdots$ to each term of the above inequality. View each entry of $\boldsymbol{a}(r)$ as the radius of a circle (see Fig. 5) and arrange these circles as circularly circumscribed. The CRI has specified the minimal and maximal total areas of the spotted round disks, the radii of which correspond to the according center distances of externally tangent circles.

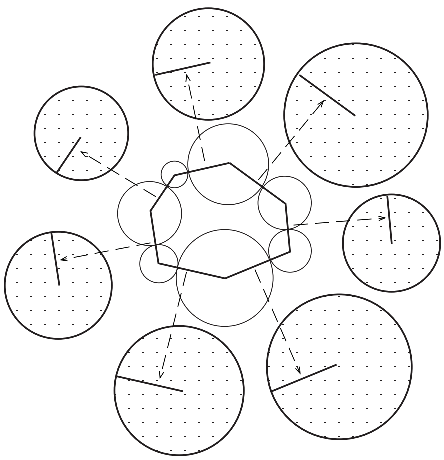

Figure 5: *An underlying combinatorial isoperimetric problem*

Apparently, the perimeter of the polygon is $2(\sum_{i=1}^{n} a_i)$, which is invariant upon the circular order of the circles, whilst the total area of the spotted round disks varies. Therefore, the CRI is the solution of a combinatorial isoperimetric problem.

3.3. The general case

For two 3-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, their circular product is the total areas of shadowed rectangles in Fig. 6.

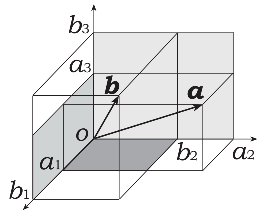

Figure 6: $2\langle \boldsymbol{a}(r), \boldsymbol{b}(r)\rangle$ *doesn't equal to the surface area of neither cuboids*

The surface area of neither the cuboid holding $\boldsymbol{a}$ nor the one holding $\boldsymbol{b}$, equals to $2\langle \boldsymbol{a}(r), \boldsymbol{b}(r)\rangle$. Indeed, the maxCP or minCP of $\langle \boldsymbol{a}(r), \boldsymbol{b}(r)\rangle$ seems much more complicated than that of $\langle \boldsymbol{a}(r)\rangle$. For instance, let $\boldsymbol{a} = (1,2,3,4,5)$ and $\boldsymbol{b} = \sin(\boldsymbol{a})$. The minCP of $\langle \boldsymbol{a}(r), \boldsymbol{b}(r)\rangle$ is $(1,2,3,5,4)$. Take another vector $\boldsymbol{c}$ all identical to $\boldsymbol{b}$ except the third entry being -0.5. Although the ordering of $\boldsymbol{c}$'s entries is identical to that of $\boldsymbol{b}$, the minCP of $\langle \boldsymbol{a}(r), \boldsymbol{c}(r)\rangle$ is $(1,2,4,5,3)$. This phenomenon indicates there might not exist an inequality for the circular product of different vectors.

4. Conclusion

This paper obtained an analogue of the classical rearrangement inequality, namely, whilst the circular rearrangement inequality. The RI considers the inner product of vectors, the CRI conducts the circular product. Similar to the RI, the extreme permutations of the CRI are explicit and solely determined by the ordering of the vector's entries. The geometric observation on the CRI reveals two combinationally invariant problems. In the meantime, finding the extreme permutation(s) for the circular product of two different vectors seems to be quite involved.

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