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EXTENDED NORMALIZED JENSEN FUNCTIONAL RELATED TO CONVEXITY, 1–QUASICONVEXITY AND SUPERQUADRACITY

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Abstract. In this paper we extend results related to Normalized Jensen Functional in several directions. We compare a specific Jensen functional with a sum of other functionals for convex functions, and we also extend these results for 1-quasiconvex functions and for Superquadratic functions.

1. Introduction

In this paper we extend and refine Jensen type inequalities appeared in [1], [2], [4], [5] and [8] related to the *Jensen functional*

$$J_n(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

We start with some theorems, definitions and notations that appeared in these papers.

THEOREM 1. [5] *Consider the normalized Jensen functional where $f : C \longrightarrow \mathbb{R}$ is a convex function on the convex set C in a real linear space, $\mathbf{x} = (x_1,\ldots,x_n) \in C^n$, and $\mathbf{p} = (p_1,\ldots,p_n)$, $\mathbf{q} = (q_1,\ldots,q_n)$ are non-negative n-tuples satisfying $\sum_{i=1}^{n} p_i = 1$, $\sum_{i=1}^{n} q_i = 1$, $q_i > 0$, $i = 1,\ldots,n$. Then*

$$MJ_n(f,\mathbf{x},\mathbf{q}) \geqslant J_n(f,\mathbf{x},\mathbf{p}) \geqslant mJ_n(f,\mathbf{x},\mathbf{q}),$$

provided

$$m = \min_{1\leqslant i\leqslant n}\left(\frac{p_i}{q_i}\right), \quad M = \max_{1\leqslant i\leqslant n}\left(\frac{p_i}{q_i}\right).$$

In [2] and in [4] a similar result is proved when f is a convex function on an interval on the real line, while $\mathbf{p}$ and $\mathbf{q}$ satisfy the conditions for Jensen-Steffensen inequality.

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DEFINITION 1. [3] A function $f:[0,b)\to\mathbb{R}$, $0<b\leqslant\infty$, is superquadratic provided that for all $0\leqslant x<b$ there exists a constant $C(x)\in\mathbb{R}$ such that

$$f(y)-f(x)-f(|y-x|)\geqslant C(x)(y-x)$$

for all $0\leqslant y<b$.

COROLLARY 1. [3] *Suppose that f is superquadratic. Let $0\leqslant x_i<b$, $i=1,\ldots,n$ and let $\overline{x}=\sum_{i=1}^{n}a_ix_i$, where $a_i\geqslant 0$, $i=1,\ldots,n$ and $\sum_{i=1}^{n}a_i=1$. Then*

$$\sum_{i=1}^{n}a_if(x_i)-f(\overline{x})\geqslant\sum_{i=1}^{n}a_if(|x_i-\overline{x}|).$$

If f is non-negative, it is also convex and the inequality refines Jensen's inequality.

THEOREM 2. [2, Theorem 3] *Under the same conditions and definitions on $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, m and M as in Theorem 1, if $f:[0,b)\to\mathbb{R}$, $0<b\leqslant\infty$, is a superquadratic function, $\sum_{j=1}^{n}p_jx_j=\overline{x}_p$ and $\sum_{j=1}^{n}q_jx_j=\overline{x}_q$, $\mathbf{x}\in[0,b)^n$, then the following inequlities hold:*

$$J_n(f,\mathbf{x},\mathbf{p})-mJ_n(f,\mathbf{x},\mathbf{q})\geqslant mf\left(\left|\overline{x}_q-\overline{x}_p\right|\right)+\sum_{i=1}^{n}(p_i-mq_i)f\left(\left|x_i-\overline{x}_p\right|\right),$$

and

$$J_n(f,\mathbf{x},\mathbf{p})-MJ_n(f,\mathbf{x},\mathbf{q})\leqslant-\sum_{i=1}^{n}(Mq_i-p_i)f\left(\left|x_i-\overline{x}_q\right|\right)-f\left(\left|\overline{x}_q-\overline{x}_p\right|\right).$$

DEFINITION 2. [1] A real-valued function f defined on an interval $[0,b)$ with $0<b\leqslant\infty$ is called γ-quasiconvex if it can be represented as the product of a comvex function and the power function x^{γ}. For $\gamma=1$, f is called 1-quasiconvex function.

COROLLARY 2. [1, Theorem 1] *Let $\varphi:[a,b)\to\mathbb{R}$, $a\geqslant 0$ be convex differentiable function, and let $\psi_1(x)$ be a 1-quasiconvex function where $\psi_1(x)=x\varphi(x)$. Let $p_i\geqslant 0$, $x_i\in[a,b)$, $i=1,\ldots,n$, $\sum_{i=1}^{n}p_i=1$, $\overline{x}=\sum_{i=1}^{n}p_ix_i$. Then a Jensen's type inequality holds:*

$$J_n(\psi_1,\mathbf{x},\mathbf{p})\geqslant\varphi'(\overline{x})\sum_{i=1}^{n}p_i(x_i-\overline{x})^2=\varphi'(\overline{x})J_n\left(x^2,\mathbf{x},\mathbf{p}\right),$$

which is a refinement of Jensen Inequality if $\varphi'(\overline{x})>0$.

THEOREM 3. [1, Theorem 18] *Suppose that $\psi_N:[a,b)\to\mathbb{R}$, $0\leqslant a<b\leqslant\infty$, is N-quasiconvex function, that is $\psi_N=x^N\varphi(x)$, $N=1,2,\ldots$, when φ is convex on $[a,b)$. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, m, M, $\overline{x}_p$, $\overline{x}_q$ and x_i, $i=1,\ldots,n$ be as in Theorem 2. Then,*

$$\begin{aligned}&J_n(\psi_N,\mathbf{x}_1,\mathbf{p}_1)-mJ_n(\psi_N,\mathbf{x}_1,\mathbf{q})\\&\geqslant\sum_{i=1}^{n}(p_i-mq_i)(x_i-\overline{x}_p)^2\frac{\partial}{\partial\overline{x}_p}\left(\frac{x_i^N-\overline{x}_p^N}{x_i-\overline{x}_p}\varphi(\overline{x}_p)\right)\\&+m(\overline{x}_q-\overline{x}_p)^2\left(\frac{\overline{x}_q^N-\overline{x}_p^N}{\overline{x}_q-\overline{x}_p}\varphi(\overline{x}_p)\right),\end{aligned}$$

and

$$\begin{aligned} &J_n(\psi_N, \mathbf{x}_1, \mathbf{p}_1) - MJ_n(\psi_N, \mathbf{x}_1, \mathbf{q}) \\ &\leqslant \sum_{i=1}^{n} (p_i - Mq_i)(x_i - \overline{x}_q)^2 \frac{\partial}{\partial \overline{x}_q} \left(\frac{x_i^N - \overline{x}_q^N}{x_i - \overline{x}_q} \varphi(\overline{x}_q) \right) \\ &-M(\overline{x}_q - \overline{x}_p)^2 \frac{\partial}{\partial \overline{x}_q} \left(\frac{\overline{x}_q^N - \overline{x}_p^N}{\overline{x}_q - \overline{x}_p} \varphi(\overline{x}_q) \right). \end{aligned}$$

For $N = 1$ we get that

$$\begin{aligned} &J_n(\psi_1, \mathbf{x}, \mathbf{p}) - mJ_n(\psi_1, \mathbf{x}, \mathbf{q}) \\ &\geqslant \varphi'(\overline{x}_p)\left(J_n\left(x^2, \mathbf{x}, \mathbf{p}\right) - mJ_n\left(x^2, \mathbf{x}, \mathbf{q}\right)\right) \end{aligned}$$

and

$$\begin{aligned} &J_n(\psi_1, \mathbf{x}, \mathbf{p}) - MJ_n(\psi_1, \mathbf{x}, \mathbf{q}) \\ &\leqslant \varphi'(\overline{x}_q)\left(J_n\left(x^2, \mathbf{x}, \mathbf{p}\right) - MJ_n\left(x^2, \mathbf{x}, \mathbf{q}\right)\right). \end{aligned}$$

Let $0 \leqslant p_{i,1} \leqslant 1$, $0 < q_i \leqslant 1$, $\sum_{i=1}^{n} p_{i,1} = \sum_{i=1}^{n} q_i = 1$.

Denote $m_1 = \min\left(\frac{p_{i,1}}{q_i}\right)$, $i = 1, \ldots, n$ and s_1 the number of i-th for which m_1 occur.

Define

$$p_{i,k} = \begin{cases} p_{i,k-1} - m_{k-1} q_i, & m_{k-1} \neq \frac{p_{i,k-1}}{q_i} \\ \frac{1}{s_{k-1}} m_{k-1}, & m_{k-1} = \frac{p_{i,k-1}}{q_i} \end{cases}, \qquad k = 2, \ldots \tag{1.1}$$

$$m_{k-1} = \min_{1 \leqslant i \leqslant n} \left(\frac{p_{i,k-1}}{q_i}\right), \; k = 2, \ldots,$$

and denote s_{k-1} as the number of cases for which m_{k-1} occurs.

Let also $x_{i,1} \in (a,b)$, $i = 1, \ldots, n$ and denote

$$x_{i,k} = \begin{cases} x_{i,k-1}, & m_{k-1} \neq \frac{p_{i,k-1}}{q_i} \\ \sum_{i=1}^{n} q_i x_{i,k-1}, & m_{k-1} = \frac{p_{i,k-1}}{q_i} \end{cases}, \tag{1.2}$$

$i = 1, \ldots n$, $k = 2, \ldots,$.

In [8] the author uses the notations (1.1) and (1.2) for the special case $q_i = \frac{1}{n}$, $i = 1, \ldots, n$ and he proves:

THEOREM 4. [8, Theorem 1] *Let $f : I \to \mathbb{R}$, (I is an interval) be convex, and let $\mathbf{x}_1 = (x_{1,1}, \ldots, x_{n,1}) \subset I^n$, $\mathbf{p}_1 = (p_{1,1}, \ldots, p_{n,1}) \subset (0,1)^n$ be such that $\sum_{i=1}^{n} p_{i,1} = 1$. Then for every $N \in \mathbb{N}$ we have*

$$\begin{aligned} &\sum_{i=1}^{n} p_{i,1} f(x_{i,1}) - f\left(\sum_{i=1}^{n} p_{i,1} x_{i,1}\right) \\ &- \sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} \frac{1}{n} f(x_{i,k}) - f\left(\sum_{i=1}^{n} \frac{1}{n} x_{i,k}\right)\right) \geqslant 0, \end{aligned}$$

where $m_k = \min_{1 \leqslant i \leqslant n} \left(\frac{p_{i,k}}{q_i}\right)$ and $q_i = \frac{1}{n}$, $i = 1, \ldots, n$, $k = 1, \ldots, N$.

Using the theorems, notations and definitions stated above, and generalizing the technique used in [8], in the next sections we extend in several directions Theorem 1 proved in [5] by S. Dragomir and Theorem 4 proved by M. Sababheh in [8].

In Theorem 1 Dragomir compares a specific Jensen functional with another Jensen functional. In Theorem 5 and Theorem 6 in Section 2 we compare the specific Jensen functional with a sum of other functionals, see also Sababheh in [8], where a particular case is proved.

In Section 3, Theorem 7 we extend Theorem 1 and Theorem 5 for 1-quasiconvex functions.

A particular case of Theorem 7 is proved in [1].

In Theorem 8 we extend Theorem 1 and Theorem 5 for Superquadratic functions.

A particular case of Theorem 8 is proved in [2].

We show in the sequel that Theorem 4 is included in our Theorem 5 and therefore applications mentioned in [8] regarding inequalities of interest in Operator Theory – Matrix Inequalities (see for instance [6], [7], [9], [10], and [11]) can be seen as derived also from our theorems in this paper.

2. Convexity and extended normalized Jensen functional

We start with the following theorem:

THEOREM 5. *Suppose that* $f:[a,b)\to\mathbb{R},\ a<b\leqslant\infty$ *is a convex function. Then, for every integer* N

$$J_n(f,\mathbf{x}_1,\mathbf{p}_1)-\sum_{k=1}^{N}m_kJ_n(f,\mathbf{x}_k,\mathbf{q})\geqslant 0, \tag{2.1}$$

where $\mathbf{p}_1=(p_{1,1},\dots,p_{n,1})$, $\mathbf{q}=(q_1,\dots,q_n)$, $\mathbf{x}_k=\left(x_{1,k},\dots,x_{n,k}\right)$, $k=1,\dots,N$, *and* $p_{i,k}$, m_k, $x_{i,k}$, *are as denoted in* (1.1) *and* (1.2), $\sum_{i=1}^{n}p_{i,1}=\sum_{i=1}^{n}q_i=1$, *and* $p_{i,1}\geqslant 0$, $q_i>0,\quad i=1,\dots,n$, $m_1=\min_{1\leqslant i\leqslant n}\left(\frac{p_{i,1}}{q_i}\right)$.

Proof. According to (1.1) and (1.2)

$$\begin{aligned}
&\sum_{i=1}^{n}p_{i,1}f(x_{i,1})-m_1\left(\sum_{i=1}^{n}q_if(x_{i,1})-f\left(\sum_{i=1}^{n}q_ix_{i,1}\right)\right)\\
&=\sum_{i=1}^{n}(p_{i,1}-m_1q_i)f(x_{i,1})+\frac{s_1m_1}{s_1}f\left(\sum_{i=1}^{n}q_ix_{i,1}\right)\\
&=\sum_{i=1}^{n}p_{i,2}f(x_{i,2}).
\end{aligned}$$

Therefore, also

$$
\begin{aligned}
&\sum_{i=1}^{n} p_{i,1} f\left(x_{i,1}\right)-\sum_{k=1}^{N} m_k\left(\sum_{i=1}^{n} q_i f\left(x_{i,k}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,k}\right)\right) \qquad (2.2)\\
&=\sum_{i=1}^{n} p_{i,2} f\left(x_{i,2}\right)-\sum_{k=2}^{N} m_k\left(\sum_{i=1}^{n} q_i f\left(x_{i,k}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,k}\right)\right)\\
&=\sum_{i=1}^{n} p_{i,2} f\left(x_{i,2}\right)-m_2\left(\sum_{i=1}^{n} q_i f\left(x_{i,2}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,2}\right)\right)\\
&\quad-\sum_{k=3}^{N} m_k\left(\sum_{i=1}^{n} q_i f\left(x_{i,k}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,k}\right)\right)\\
&=\vdots\\
&=\sum_{i=1}^{n} p_{i,N-1} f\left(x_{i,N-1}\right)-m_{N-1}\left(\sum_{i=1}^{n} q_i f\left(x_{i,N-1}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,N-1}\right)\right)\\
&\quad-m_N\left(\sum_{i=1}^{n} q_i f\left(x_{i,N}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,N}\right)\right)\\
&=\sum_{i=1}^{n} p_{i,N} f\left(x_{i,N}\right)-m_N\left(\sum_{i=1}^{n} q_i f\left(x_{i,N}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,N}\right)\right)\\
&=\sum_{i=1}^{n}\left(p_{i,N}-m_N q_i\right) f\left(x_{i,N}\right)+s_N\left(\frac{m_N}{s_N}\right) f\left(\sum_{i=1}^{n} q_i x_{i,N}\right)\\
&=\sum_{i=1}^{n} p_{i,N+1} f\left(x_{i,N+1}\right),
\end{aligned}
$$

which means that

$$
\begin{aligned}
&\sum_{i=1}^{n} p_{i,1} f\left(x_{i,1}\right)-\sum_{k=1}^{N} m_k\left(\sum_{i=1}^{n} q_i f\left(x_{i,k}\right)-f\left(\sum_{i=1}^{n} q_i x_{i,k}\right)\right) \qquad (2.3)\\
&=\sum_{i=1}^{n}\left(p_{i,N}-m_N q_i\right) f\left(x_{i,N}\right)+s_N\left(\frac{m_N}{s_N}\right) f\left(\sum_{i=1}^{n} q_i x_{i,N}\right)=\sum_{i=1}^{n} p_{i,N+1} f\left(x_{i,N+1}\right).
\end{aligned}
$$

Also it is clear that

$$
\begin{aligned}
&\sum_{i=1}^{n} p_{i,k}=1, \quad p_{i,k} \geqslant 0, \quad i=1,\ldots,n, \quad k=1,\ldots,N+1, \qquad (2.4)\\
&\sum_{i=1}^{n} p_{i,1} x_{i,1}=\sum_{i=1}^{n} p_{i,k} x_{i,k}, \quad k=1,\ldots,N+1.
\end{aligned}
$$

Therefore as a result of (2.3) and (2.4), in order to prove (2.1) we have to show that

$$\begin{aligned}
&\sum_{i=1}^{n} p_{i,1} f(x_{i,1}) - \sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} q_i f(x_{i,k}) - f\left(\sum_{i=1}^{n} q_i x_{i,k} \right) \right) \\
&= \sum_{i=1}^{n} p_{i,N+1} f(x_{i,N+1}) \\
&\geqslant f\left(\sum_{i=1}^{n} p_{i,1} x_{i,1} \right) = f\left(\sum_{i=1}^{n} p_{i,k} x_{i,k} \right) = f\left(\sum_{i=1}^{n} p_{i,N+1} x_{i,N+1} \right).
\end{aligned}$$

That is, we have to show that

$$\sum_{i=1}^{n} p_{i,N+1} f(x_{i,N+1}) \geqslant f\left(\sum_{i=1}^{n} p_{i,N+1} x_{i,N+1} \right), \tag{2.5}$$

and (2.5) holds because it is given that the function f is a convex function.

The proof of the theorem is complete. □

COROLLARY 3. *Under the conditions of Theorem 5, if*

$$p_{i,N} = q_i, \quad i = 1, \ldots, n \tag{2.6}$$

we get an equality in (2.1).

Proof.

$$\begin{aligned}
&\sum_{i=1}^{n} p_{i,1} f(x_{i,1}) - \sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} q_i f(x_{i,k}) - f\left(\sum_{i=1}^{n} q_i x_{i,k} \right) \right) \qquad (2.7) \\
&= \sum_{i=1}^{n} p_{i,N} f(x_{i,N}) - m_N \left(\sum_{i=1}^{n} q_i f(x_{i,N}) - f\left(\sum_{i=1}^{n} q_i x_{i,N} \right) \right) \\
&= \sum_{i=1}^{n} p_{i,N} f(x_{i,N}) - 1 \left(\sum_{i=1}^{n} p_{i,N} f(x_{i,N}) - f\left(\sum_{i=1}^{n} p_{i,N} x_{i,N} \right) \right) \\
&= f\left(\sum_{i=1}^{n} p_{i,N} x_{i,N} \right).
\end{aligned}$$

Indeed, the first equality in (2.7) follows from the first equality in (2.3). The second and third equalities hold as $m_N = \frac{p_{i,N}}{q_i} = 1,\ i = 1, \ldots, n$. Therefore from (2.7)

$$\sum_{i=1}^{n} p_{i,1} f(x_{i,1}) - \sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} q_i f(x_{i,k}) - f\left(\sum_{i=1}^{n} q_i x_{i,k} \right) \right) - f\left(\sum_{i=1}^{n} p_{i,N} x_{i,N} \right) = 0,$$

holds, and as according to (2.4)

$$\sum_{i=1}^{n} p_{i,N} x_{i,N} = \sum_{i=1}^{n} p_{i,1} x_{i,1}$$

holds, (2.1) follows with equality. The proof is complete. □

Replacing $q_i,\ i=1,\dots,n$ by $\frac{1}{n}$ in Theorem 5 and Corollary 3 we get Theorem 4 (Theorem 1 in [8]) and Theorem 2 in [8].

We extend now the left hand-side inequality in Theorem 1.

We denote

$$\begin{aligned} \mathbf{p}_1 &= (p_{1,1},\dots,p_{1,n}), \quad \mathbf{q}=(q_1,\dots,q_n), \\ \mathbf{x}_k &= (x_{1,k},\dots,x_{n,k}), \quad k=1,\dots,N \end{aligned} \tag{2.8}$$

$$p_{i,1} \geqslant 0, \quad q_i > 0, \quad i=1,\dots,n, \quad \sum_{i=1}^{n} p_{i,1} = \sum_{i=1}^{n} q_i = 1,$$

$$M_1 = Max\left(\frac{p_{i,1}}{q_i}\right) = \frac{p_{j,1}}{q_j}, \quad i=1,\dots,n,$$

where j is a fixed specific integer for which M_1 holds.

We also denote

$$p_{i,1} = p^*_{i,1}, \ x^*_{i,1} = x_{i.1}, \ i=1,\dots,n, \tag{2.9}$$

$$p^*_{i,k} = p^*_{i,k-1} - M_{k-1}q_i, \ x^*_{i,k} = x^*_{i,k-1}, \text{ when } M_{k-1} \neq \frac{p^*_{i,k-1}}{q_i}, \ k=2,\dots,N$$

$$p^*_{i,k} = p^*_{i,k-1} - M_{k-1}q_i, \ x^*_{i,k} = x^*_{i,k-1}, \text{ when } M_{k-1} = \frac{p^*_{i,k-1}}{q_i}, \ i \neq j_k, \ k=2,\dots,N$$

$$p^*_{j_k,k} = M_{k-1}, \ x^*_{j,k} = \sum_{i=1}^{n} q_i x^*_{i,k-1}, \text{ when } M_{k-1} = \frac{p^*_{j_k,k-1}}{q_{j_{k-1}}},$$

$$M_k = \underset{1\leqslant i\leqslant n}{Max}\left(\frac{p^*_{i,k}}{q_i}\right) = \frac{p^*_{j_k,k}}{q_{j_k}}, \qquad k=1,\dots,N,$$

where j_k is a specific index for which M_k holds.

With the notations and conditions in (2.8) and (2.9) we get:

THEOREM 6. *Let $f:[a,b)\to\mathbb{R}$, $a\leqslant b\leqslant\infty$, be a convex function, and let (2.8) and (2.9) hold. Then, for every integer N*

$$J_n(f,\mathbf{x}_1,\mathbf{p}_1) - \sum_{k=1}^{N} M_k J_n(f,\mathbf{x}_k,\mathbf{q}) \leqslant 0, \tag{2.10}$$

and

$$M_k = \frac{p_{j_1,1}}{q^k_{j_1}}, \qquad k=1,\dots,N \tag{2.11}$$

hold, where j_1 is a fixed specific integer for which $M_1 = \frac{p_{j_1,1}}{q_{j_1}}$ is satisfied.

Proof. As j_1 is a specific integer for which $M_1 = \underset{1\leqslant i\leqslant n}{Max}\left(\frac{p_{i,1}}{q_i}\right) = \frac{p_{j_1,1}}{q_{j_1}}$, it is easy to see that

$$M_{k-1} = \underset{1\leqslant i\leqslant n}{Max}\frac{p^*_{i,k-1}}{q_i} = \frac{p_{j_1,1}}{q^{k-1}_{j_1}}, \quad k=2,\dots,N+1 \tag{2.12}$$

because the only positive $p^*_{i,k},\ k=2,\dots$ is $p^*_{j_1,k}$, as

$$\begin{aligned} &\text{when } \frac{p^*_{i,1}}{q_i} < \frac{p^*_{j_1,1}}{q_{j_1}}, \quad i \neq j_1, \quad \text{then } p^*_{i,2} < 0, \quad x^*_{i,2} = x^*_{i,1}, \\ &\text{when } \frac{p^*_{i,1}}{q_i} = \frac{p^*_{j_1,1}}{q_{j_1}}, \quad i \neq j_1, \quad \text{then } p^*_{i,2} = 0, \quad x^*_{i,2} = x^*_{i,1}, \\ &\qquad \text{when } i = j_1, \quad \text{then } p^*_{i,2} > 0, \quad x^*_{i,2} > 0, \end{aligned} \tag{2.13}$$

and

$$\sum_{i=1}^{n} p^*_{i,2} x^*_{i,2} = 1, \quad x^*_{j_1,2} = \sum_{i=1}^{n} q_i x_{i,1}.$$

Hence, also for $k=2,\dots,N$ the only positive $p^*_{i,k},\ i=2,\dots,n$ is $p^*_{j_1,k}$ where j_1 is the fixed integer that satisfies $M_1 = \frac{p_{j_1,1}}{q_{j_1}}$, and therefore we can replace in the last line of (2.9) j_k with j_1, which means that (2.11) holds.

In order to complete the proof of the theorem, we proceed now with proving (2.10): In a similar way as we get (2.2) we get

$$\begin{aligned} &\sum_{i=1}^{n} p_{i,1} f(x_{i,1}) - \sum_{k=1}^{N} M_k \left(\sum_{i=1}^{n} q_i f(x_{i,k}) - f\left(\sum_{i=1}^{n} q_i x_{i,k} \right) \right) \qquad (2.14) \\ &= \sum_{i=1}^{n} p^*_{i,2} f(x^*_{i,2}) - \sum_{k=2}^{N} M_k \left(\sum_{i=1}^{n} q_i f(x^*_{i,k}) - f\left(\sum_{i=1}^{n} q_i x^*_{i,k} \right) \right) \\ &= \sum_{i=1}^{n} p^*_{i,2} f(x^*_{i,2}) - M_2 \left(\sum_{i=1}^{n} q_i f(x^*_{i,2}) - f\left(\sum_{i=1}^{n} q_i x^*_{i,2} \right) \right) \\ &\quad - \sum_{k=3}^{N} M_k \left(\sum_{i=1}^{n} q_i f(x^*_{i,k}) - f\left(\sum_{i=1}^{n} q_i x^*_{i,k} \right) \right) \\ &= \vdots \\ &= \sum_{i=1}^{n} p^*_{i,N-1} f(x^*_{i,N-1}) - M_{N-1} \left(\sum_{i=1}^{n} q_i f(x^*_{i,N-1}) - f\left(\sum_{i=1}^{n} q_i x^*_{i,N-1} \right) \right) \\ &\quad - M_N \left(\sum_{i=1}^{n} q_i f(x^*_{i,N}) - f\left(\sum_{i=1}^{n} q_i x^*_{i,N} \right) \right) \\ &= \sum_{i=1}^{n} p^*_{i,N} f(x^*_{i,N}) - M_N \left(\sum_{i=1}^{n} q_i f(x^*_{i,N}) - f\left(\sum_{i=1}^{n} q_i x^*_{i,N} \right) \right) \\ &= \sum_{i=1}^{n} \left(p^*_{i,N} - M_N q_i \right) f(x^*_{i,N}) + M_N f\left(\sum_{i=1}^{n} q_i x^*_{i,N} \right) \\ &= \sum_{i=1}^{n} p^*_{i,N+1} f(x^*_{i,N+1}) \end{aligned}$$

which means that

$$\sum_{i=1}^{n} p_{i,1}^{*} f\left(x_{i,1}^{*}\right) - \sum_{k=1}^{N} M_k \left(\sum_{i=1}^{n} q_i f\left(x_{i,k}^{*}\right) - f\left(\sum_{i=1}^{n} q_i x_{i,k}^{*}\right)\right) \tag{2.15}$$
$$= \sum_{i=1}^{n} \left(p_{i,N}^{*} - M_N q_i\right) f\left(x_{i,N}^{*}\right) + M_N f\left(\sum_{i=1}^{n} q_i x_{i,N}^{*}\right)$$
$$= \sum_{i=1}^{n} p_{i,N+1}^{*} f\left(x_{i,N+1}^{*}\right).$$

Also it is clear that

$$p_{i,k}^{*} \leqslant 0, \quad i=1,\dots,n, \quad i \neq j_1, \quad p_{j_1,k} > 0, \quad k=2,\dots,N+1, \tag{2.16}$$
$$\sum_{i=1}^{n} p_{i,k}^{*} = 1, \quad \sum_{i=1}^{n} p_{i,1} x_{i,1} = \sum_{i=1}^{n} p_{i,k}^{*} x_{i,k}^{*}, \quad k=1,\dots,N+1.$$

From (2.15) it follows that we have to show that

$$\sum_{i=1}^{n} p_{i,1}^{*} f\left(x_{i,1}^{*}\right) - \sum_{k=1}^{N} M_k \left(\sum_{i=1}^{n} q_i f\left(x_{i,k}^{*}\right) - f\left(\sum_{i=1}^{n} q_i x_{i,k}^{*}\right)\right)$$
$$= \sum_{i=1}^{n} p_{i,N+1}^{*} f\left(x_{i,N+1}^{*}\right) \leqslant f\left(\sum_{i=1}^{n} p_{i,1} x_{i,1}\right),$$

again from (2.15) we have to show that

$$\sum_{i=1}^{n} \left(p_{i,N}^{*} - M_N q_i\right) f\left(x_{i,N}^{*}\right) + M_N f\left(\sum_{i=1}^{n} q_i x_{i,N}^{*}\right) \leqslant f\left(\sum_{i=1}^{n} p_{i,1} x_{i,1}\right).$$

Using (2.12) in other words, we have to show that

$$f\left(\sum_{i=1}^{n} p_{i,N}^{*} x_{i,N}^{*}\right) + \sum_{i=1,\quad i\neq j}^{n} \left(M_N q_i - p_{i,N}^{*}\right) f\left(x_{i,N}^{*}\right) \geqslant M_N f\left(\sum_{i=1}^{n} q_i x_{i,N.}^{*}\right),$$

holds, or that

$$\frac{1}{M_N} f\left(\sum_{i=1}^{n} p_{i,N}^{*} x_{i,N}^{*}\right) + \sum_{i=1,\ i\neq j}^{n} \left(q_i - \frac{p_{i,N}^{*}}{M_N}\right) f\left(x_{i,N}^{*}\right) \geqslant f\left(\sum_{i=1}^{n} q_i x_{i,N.}^{*}\right).$$

The last inequality follows from the convexity of f because $q_i - \frac{p_{i,N}^{*}}{M_N} \geqslant 0$, $i=1,\dots,n$, $i \neq j$ and $\frac{1}{M_N} > 0$, and the inequality (2.10) holds.

The proof of the theorem is complete. □

3. 1-quasiconvexity, superquadracity and normalized Jensen functional

In Theorem 7 we extend Theorem 3 and Theorem 5 for 1-quasiconvex functions, and in Theorem 8 we extend Theorem 2 for superquadratic functions.

THEOREM 7. *Let* $\psi_1 : [a,b) \to \mathbb{R},\ 0 \leqslant a < b \leqslant \infty$ *be a* 1*-quasiconvex function where* $\psi_1(x) = x\varphi(x)$, *and* φ *is a differentiable convex function. Let* $\overline{x}_{\mathbf{p}_k} = \sum_{i=1}^{n} p_{i,k}x_{i,k}$ *and* $\overline{x}_{\mathbf{q}_k} = \sum_{i=1}^{n} q_i x_{i,k}$, $k = 1,\dots,N$. *Then under the same notations and conditions as used in Theorem 5 for* $p_{i,k},\ x_{i,k},\ m_k,\ \mathbf{p}_1,\ \mathbf{q},\ k = 1,\dots,N,\ i = 1,\dots n$ *we get:*

$$\begin{aligned} & J_n(\psi_1, \mathbf{x}_1, \mathbf{p}_1) - \sum_{k=1}^{N} m_k J_n(\psi_1, \mathbf{x}_k, \mathbf{q}) \\ \geqslant\ & \varphi'(\overline{x}_{\mathbf{p}_1}) \left(\sum_{i=1}^{n} p_{i,N+1} x_{i,N+1}^2 - (\overline{x}_{\mathbf{p}_N})^2 \right) \\ =\ & \varphi'(\overline{x}_{\mathbf{p}_1}) \left(J_n\left(x^2, \mathbf{x}_1, \mathbf{p}_1\right) - \sum_{k=1}^{N} m_k J_n\left(x^2, \mathbf{x}_k, \mathbf{q}\right) \right). \end{aligned} \tag{3.1}$$

If φ *is also increasing then (3.1) refines Theorem 1 and Theorem 5.*

In particular, for $N = 1$ *we get that*

$$\begin{aligned} & J_n(\psi_1, \mathbf{x}_1, \mathbf{p}_1) - m_1 J_n(\psi_1, \mathbf{x}_1, \mathbf{q}) \\ \geqslant\ & \varphi'(\overline{x}_{\mathbf{p}_1}) \left(J_n\left(x^2, \mathbf{x}_1, \mathbf{p}_1\right) - m_1 J_n\left(x^2, \mathbf{x}_1, \mathbf{q}\right) \right). \end{aligned} \tag{3.2}$$

Proof. As ψ_1 is 1-quasiconvex, therefore from Corollary 2 we get that

$$\begin{aligned} & \sum_{i=1}^{n} p_{i,N+1} \psi_1(x_{i,N+1}) - \psi_1\left(\overline{x}_{\mathbf{p}_{N+1}}\right) \\ \geqslant\ & \varphi'\left(\overline{x}_{\mathbf{p}_{N+1}}\right) \left(\sum_{i=1}^{n} p_{i,N+1} x_{i,N+1}^2 - \left(\overline{x}_{\mathbf{p}_{N+1}}\right)^2 \right) \end{aligned} \tag{3.3}$$

and as (2.3) and (2.4) hold, we get that

$$\begin{aligned} & \sum_{i=1}^{n} p_{i,N+1} \psi_1(x_{i,N+1}) - \psi_1\left(\overline{x}_{\mathbf{p}_{N+1}}\right) \\ =\ & \sum_{i=1}^{n} p_{i,1} \psi_1(x_{i,1}) - \sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} q_i \psi_1\left(x_{i,k}\right) - \psi_1\left(\overline{x}_{\mathbf{q}_k}\right) \right) - \psi_1\left(\overline{x}_{\mathbf{p}_{N+1}}\right) \\ =\ & \sum_{i=1}^{n} p_{i,1} \psi_1(x_{i,1}) - \sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} q_i \psi_1\left(x_{i,k}\right) - \psi_1\left(\overline{x}_{\mathbf{q}_k}\right) \right) - \psi_1\left(\overline{x}_{\mathbf{p}_1}\right) \end{aligned} \tag{3.4}$$

holds.

(3.3) and (3.4) lead to

$$\sum_{i=1}^{n} p_{i,1}\psi_1\left(x_{i,1}\right)-\psi_1\left(\overline{x}_{\mathbf{p}_1}\right)-\sum_{k=1}^{N} m_k\left(\sum_{i=1}^{n} q_i\psi_1\left(x_{i,k}\right)-\psi_1\left(\overline{x}_{\mathbf{q}_k}\right)\right)$$
$$\geqslant \varphi'\left(\overline{x}_{\mathbf{p}_{N+1}}\right)\left(\sum_{i=1}^{n} p_{i,N+1}x_{i,N+1}^2-\left(\overline{x}_{\mathbf{p}_{N+1}}\right)^2\right).$$

From this inequality and by using again (2.3) and (2.4) for the convex function $f(x)=x^2$ we get that (3.1) holds.

In the case that φ is also non-decreasing, ψ_1 is convex too, and as $f(x)=x^2$ is convex, (3.1) refines (2.1).

Inequality (3.2) follows by inserting in (3.1) $N=1$.

The proof is complete. □

Inequality (3.2) appears also in Theorem 3 ([1, Theorem 17]).

Similarly, we get for superquadratic functions (see Definition 1) the following theorem which extends Theorem 2:

THEOREM 8. *Let $f:[0,b)\to\mathbb{R}$, $0<b\leqslant\infty$ be a superquadratic function. Let $p_{i,k}$, $x_{i,k}$, m_k and s_k, $k=1,\dots,N$, $i=1,\dots,n$ satisfy (1.1) and (1.2). Let $\overline{x}_{\mathbf{p}_j}=\sum_{i=1}^{n} p_{i,j}x_{i,j}$ and $\overline{x}_{\mathbf{q}_j}=\sum_{i=1}^{n} q_ix_{i,j}$, $j=1,\dots,N$, $p_{i,1}\geqslant 0$, $q_i>0$, $i=1,\dots,n$, $\mathbf{x}=(x_i,\dots,x_n)\in[0,b)^n$. Then*

$$J_n\left(f,\mathbf{x}_1,\mathbf{p}_1\right)-\sum_{k=1}^{N} m_kJ_n\left(f,\mathbf{x}_k,\mathbf{q}\right)\geqslant\sum_{k=1}^{n} p_{i,N+1}f\left(\left|x_{i,N+1}-\overline{x}_{\mathbf{p}_1}\right|\right) \tag{3.5}$$

If f is also non-negative then f is convex and (3.5) refines Theorem 4.

In particular for $N=1$ we get that

$$J_n\left(f,\mathbf{x}_1,\mathbf{p}_1\right)-mJ_n\left(f,\mathbf{x}_1,\mathbf{q}\right)\geqslant mf\left(\left|\overline{x}_q-\overline{x}_{p_1}\right|\right)+\sum_{i=1}^{n}\left(p_i-mq_i\right)f\left(\left|x_i-\overline{x}_{p_1}\right|\right), \tag{3.6}$$

Proof. From (2.3) we get the identity

$$\sum_{i=1}^{n} p_{i,1}f\left(x_{i,1}\right)-\sum_{k=1}^{N} m_k\left(\sum_{i=1}^{n} q_if\left(x_{i,k}\right)-f\left(\overline{x}_{\mathbf{q}_k}\right)\right)=\sum_{i=1}^{n} p_{i,N+1}f\left(x_{i,N+1}\right), \tag{3.7}$$

and because

$$p_{i,N+1}\geqslant 0,\quad i=1,\dots,n,\quad \sum_{i=1}^{n} p_{i,N+1}=1$$

we get from the superquadracity of f that Corollary 1 holds, that is

$$\sum_{i=1}^{n} p_{i,N+1}f\left(x_{i,N+1}\right)-f\left(\overline{x}_{\mathbf{p}_{N+1}}\right)\geqslant\sum_{i=1}^{n} p_{i,N+1}f\left(\left|x_{i,N+1}-\overline{x}_{\mathbf{p}_{N+1}}\right|\right). \tag{3.8}$$

Using (2.4) we can rewrite (3.8) as

$$\sum_{i=1}^{n} p_{i,N+1}f\left(x_{i,N+1}\right)-f\left(\overline{x}_{\mathbf{p}_1}\right)\geqslant\sum_{i=1}^{n} p_{i,N+1}f\left(\left|x_{i,N+1}-\overline{x}_{\mathbf{p}_1}\right|\right). \tag{3.9}$$

(3.7) and (3.9) lead to (3.5).

By inserting $N = 1$ in (3.5) and making simple calculations using (2.3) (for $N = 1$) we get (3.6).

The proof of the theorem is complete. □

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