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NECESSARY AND SUFFICIENT CONDITIONS FOR THE VALIDITY OF HILBERT TYPE INTEGRAL INEQUALITIES WITH A CLASS OF QUASI–HOMOGENEOUS KERNELS AND ITS APPLICATION IN OPERATOR THEORY

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Abstract. By using real analysis technique and the method of weight functions, the necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and the best constant factors are obtained, and its applications in operator theory are discussed.

1. Introduction and preliminary knowledge

Let $r>1$, and α be a constant. Set

$$L_{\alpha}^{r}(0,+\infty)=\left\{f(x)\geqslant 0:||f||_{r,\alpha}=\left(\int_{0}^{+\infty}x^{\alpha}f^{r}(x)\mathrm{d}x\right)^{1/r}<+\infty\right\}.$$

If $\frac{1}{p}+\frac{1}{q}=1(p>1)$, M is a nonnegative constant, $K(x,y)\geqslant 0$, $f(x)\in L_{\alpha}^{p}(0,+\infty)$, $g(y)\in L_{\beta}^{q}(0,+\infty)$, then the Hilbert type integral inequality is of the form

$$\int_{0}^{+\infty}\int_{0}^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant M||f||_{p,\alpha}||g||_{q,\beta}, \tag{1}$$

Define the singular integral operator T by

$$T(f)(y)=\int_{0}^{+\infty}K(x,y)f(x)\mathrm{d}x,\ f(x)\in L_{\alpha}^{p}(0,+\infty). \tag{2}$$

Then (1) can be equivalently written as

$$\int_{0}^{+\infty}g(y)T(f)(y)\mathrm{d}y\leqslant M||f||_{p,\alpha}||g||_{q,\beta}. \tag{3}$$

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An operator T is called the bounded operator from $L_\alpha^p(0,+\infty)$ to $L_\gamma^p(0,+\infty)$, if $||T(f)||_{p,\gamma} \leqslant M||f||_{p,\alpha}$. At this point, the operator norm of T is defined by

$$||T|| = \inf_{f\in L_\alpha^p(0,+\infty)} \frac{||T(f)||_{p,\gamma}}{||f||_{p,\alpha}}.$$

In particular, T is called a bounded operator in $L_\alpha^p(0,+\infty)$, if the bounded operator T is from $L_\alpha^p(0,+\infty)$ to itself. For the sake of convenience, denote by $||f||_{p,0} = ||f||_p$, $L_0^p(0,+\infty) = L^p(0,+\infty)$.

One can know from (3) that, the boundedness of the operator is closely related to the Hilbert type inequality. Although the study of Hilbert type inequality has made many achievements (cf. [1]–[16]), but most of them are discussed for a specific integral kernel. In this paper, we will study the necessary and sufficient conditions for the validity of Hilbert type inequality with a class of quasi-homogeneous kernels, and discuss its applications in operator theory.

DEFINITION 1. A function $K(x,y)$ is said to be quasi-homogeneous of order (λ_1,λ_2) if for $t>0$,

$$K(tx,y) = t^{\lambda_1}K(x,t^{-\frac{\lambda_1}{\lambda_2}}y), \quad K(x,ty) = t^{\lambda_2}K(t^{-\frac{\lambda_2}{\lambda_1}}x,y),$$

where λ_1 and λ_2 are nonzero constants. When $\lambda_1 = \lambda_2 = \lambda$, $K(x,y)$ becomes a homogeneous function of order λ.

LEMMA 1. *Let T be as in (2). Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, α, $\beta\in\mathbb{R}$, $\gamma=\beta(1-p)$. Then Hilbert type inequality (1) is equivalent to $||T(f)||_{p,\gamma}\leqslant M||f||_{p,\alpha}$.*

Proof. Necessity. If (1) holds, set

$$g(y) = y^{\beta(1-p)}(T(f)(y))^{p-1} = y^\gamma (T(f)(y))^{p-1},$$

then

$$\begin{aligned} ||T(f)||_{p,\gamma}^p &= \int_0^{+\infty} y^\gamma (T(f)(y))^p\,\mathrm{d}y = \int_0^{+\infty}\int_0^{+\infty} K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y \\ &\leqslant M||f||_{p,\alpha}||g||_{q,\beta} = M||f||_{p,\alpha}\left(\int_0^{+\infty} y^\beta g^q(y)\mathrm{d}y\right)^{1/q} \\ &= M||f||_{p,\alpha}\left(\int_0^{+\infty} y^\gamma (T(f)(y))^p\,\mathrm{d}y\right)^{1/q} = M||f||_{p,\alpha}||T(f)||_{p,\gamma}^{p/q}. \end{aligned}$$

If $||T(f)||_{p,\gamma}=0$, then inequality $||T(f)||_{p,\gamma}\leqslant M||f||_{p,\alpha}$ is naturally valid; if $||T(f)||_{p,\gamma} = \infty$, then inequality $||T(f)||_{p,\gamma}\leqslant M||f||_{p,\alpha}$ is impossible. Thus, in view of the above results, we have $||T(f)||_{p,\gamma}\leqslant M||f||_{p,\alpha}$.

Sufficiency. If $||T(f)||_{p,\gamma}\leqslant M||f||_{p,\alpha}$, then it is not difficult to derive (1), whence (1) and the inequality $||T(f)||_{p,\gamma}\leqslant M||f||_{p,\alpha}$ are equivalent. □

LEMMA 2. *Assume that* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda_1\lambda_2>0$, α, $\beta\in\mathbb{R}$, $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2$, $K(x,y)$ *is a quasi-homogeneous non-negative measurable function of order* (λ_1,λ_2), *set*

$$W_1=\int_0^{+\infty}u^{-\frac{\beta+1}{q}}K(1,u)\mathrm{d}u, W_2=\int_0^{+\infty}u^{-\frac{\alpha+1}{p}}K(u,1)\mathrm{d}u,$$

then $\lambda_1W_2=\lambda_2W_1$, *and*

$$\omega_1(x)=\int_0^{+\infty}y^{-\frac{\beta+1}{q}}K(x,y)\mathrm{d}y=x^{\lambda_1-\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+1}{q}-1\right)}W_1,$$

$$\omega_2(y)=\int_0^{+\infty}x^{-\frac{\alpha+1}{p}}K(x,y)\mathrm{d}x=y^{\lambda_2-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-1\right)}W_2.$$

Proof. Since $K(x,y)$ is a quasi-homogeneous function of order (λ_1,λ_2), and $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2>0$, we have

$$\begin{aligned}-\frac{\lambda_1}{\lambda_2}\left(\lambda_2-\frac{\beta+1}{q}\right)&=-\frac{1}{\lambda_2}\left(\lambda_1\lambda_2-\frac{\lambda_1\beta-\lambda_2}{q}-\frac{\lambda_1+\lambda_2}{q}\right)\\&=-\frac{1}{\lambda_2}\left(\lambda_1\lambda_2+\frac{\lambda_2\alpha-\lambda_1}{p}-\lambda_1\lambda_2-\frac{\lambda_1+\lambda_2}{q}\right)\\&=-\frac{\alpha}{p}+\frac{\lambda_1}{p\lambda_2}+\frac{\lambda_1+\lambda_2}{q\lambda_2}=-\frac{\alpha+1}{p}+1+\frac{\lambda_1}{\lambda_2},\end{aligned}$$

and

$$\begin{aligned}W_1&=\int_0^{+\infty}u^{\lambda_2-\frac{\beta+1}{q}}K(u^{-\frac{\lambda_2}{\lambda 1}},1)\mathrm{d}u\\&=\frac{\lambda_1}{\lambda_2}\int_0^{+\infty}t^{-\frac{\lambda_1}{\lambda_2}\left(\lambda_2-\frac{\beta+1}{q}\right)-\frac{\lambda_1}{\lambda_2}-1}K(t,1)\mathrm{d}t\\&=\frac{\lambda_1}{\lambda_2}\int_0^{+\infty}t^{-\frac{\alpha+1}{p}}K(t,1)\mathrm{d}t=\frac{\lambda_1}{\lambda_2}W_2,\end{aligned}$$

it follows that $\lambda_2W_1=\lambda_1W_2$. Substituting $x^{-\frac{\lambda_1}{\lambda_2}}y=u$, then

$$\begin{aligned}\omega_1(x)&=\int_0^{+\infty}x^{\lambda_1}y^{-\frac{\beta+1}{q}}K(1,x^{-\frac{\lambda_1}{\lambda_2}}y)\mathrm{d}y\\&=x^{\lambda_1-\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+1}{q}-1\right)}\int_0^{+\infty}u^{-\frac{\beta+1}{q}}K(1,u)\mathrm{d}u\\&=x^{\lambda_1-\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+1}{q}-1\right)}W_1.\end{aligned}$$

Similarly, we also have $\omega_2(y)=y^{\lambda_2-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-1\right)}W_2$. □

2. Main results

THEOREM 1. *Suppose that* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda_1\lambda_2>0$, α, $\beta\in\mathbb{R}$, $K(x,y)$ *is a quasi-homogeneous non-negative measurable function of order* (λ_1,λ_2), $K(1,u)$ $(K(u,1))>0$ *a.e. in* $(0,1)$ $((1,\infty))$, *and*

$$W_1=\int_0^{+\infty}u^{-\frac{\beta+1}{q}}K(1,u)\mathrm{d}u, W_2=\int_0^{+\infty}u^{-\frac{\alpha+1}{p}}K(u,1)\mathrm{d}u$$

are both convergent. Then

(i) there exists a constant M, *for any* $f\in L^p_\alpha(0,+\infty)$, $g\in L^q_\beta(0,+\infty)$, *the necessary and sufficient conditions for Hilbert type inequality*

$$\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant M||f||_{p,\alpha}||g||_{q,\beta} \tag{4}$$

to be true is that $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2$;

(ii) if (4) holds, then its best possible constant factor is $M=\frac{W}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}$, *where* $W=|\lambda_1|W_2=|\lambda_2|W_1$.

Proof. (i) *Necessity.* Assume that there exists a constant M, such that (4) holds. Set $c=\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}-\lambda_1\lambda_2$. First consider the case of $\lambda_1>0$, $\lambda_2>0$. If $c>0$, for sufficiently small $\varepsilon>0$ $(\varepsilon<\frac{c}{\lambda_1\lambda_2})$, take

$$f(x)=\begin{cases}x^{(-\alpha-1+\lambda_1\varepsilon)/p}, & 0<x\leqslant 1,\\ 0, & x>1,\end{cases}$$

$$g(y)=\begin{cases}y^{(-\beta-1+\lambda_2\varepsilon)/q}, & 0<y\leqslant 1,\\ 0, & y>1.\end{cases}$$

Then

$$||f||_{p,\alpha}||g||_{q,\beta}=\left(\int_0^1x^{-1+\lambda_1\varepsilon}\mathrm{d}x\right)^{1/p}\left(\int_0^1y^{-1+\lambda_2\varepsilon}\mathrm{d}y\right)^{1/q}=\frac{1}{\varepsilon\lambda_1{}^{1/p}\lambda_2{}^{1/q}}, \tag{5}$$

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\\ &=\int_0^1x^{(-\alpha-1+\lambda_1\varepsilon)/p}\left(\int_0^1K(x,y)y^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}y\right)\mathrm{d}x\\ &=\int_0^1x^{\lambda_1+(-\alpha-1+\lambda_1\varepsilon)/p}\left(\int_0^1K(1,x^{-\lambda_1/\lambda_2}y)y^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}y\right)\mathrm{d}x\\ &=\int_0^1x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\left(\int_0^{x^{-\lambda_1/\lambda_2}}K(1,t)t^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}t\right)\mathrm{d}x\\ &\geqslant\int_0^1x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\mathrm{d}x\int_0^1K(1,t)t^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}t.\end{aligned} \tag{6}$$

It follows from (4), (5) and (6) that

$$\int_0^1 x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\mathrm{d}x\int_0^1 K(1,t)t^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}t \leqslant \frac{M}{\varepsilon\lambda_1{}^{1/p}\lambda_2{}^{1/q}}. \tag{7}$$

Since $c>0,\ \lambda_2>0,\ \lambda_1\varepsilon-\frac{c}{\lambda_2}<0$ and $\int_0^1 x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\mathrm{d}x$ diverges to $+\infty$. Whence it is a contradiction of (7). In other words, it is not valid for $c>0$.

If $c<0$, for sufficiently small $\varepsilon>0$ $(\varepsilon<\frac{-c}{\lambda_1\lambda_2})$, take

$$f(x)=\begin{cases} x^{(-\alpha-1-\lambda_1\varepsilon)/p}, & x\geqslant 1,\\ 0, & 0<x<1,\end{cases}$$

$$g(y)=\begin{cases} y^{(-\beta-1-\lambda_2\varepsilon)/q}, & y\geqslant 1,\\ 0, & 0<y<1.\end{cases}$$

We get

$$||f||_{p,\alpha}||g||_{q,\beta}=\left(\int_1^{+\infty}x^{-1-\lambda_1\varepsilon}\mathrm{d}x\right)^{1/p}\left(\int_1^{+\infty}y^{-1-\lambda_2\varepsilon}\mathrm{d}y\right)^{1/q}=\frac{1}{\varepsilon\lambda_1{}^{1/p}\lambda_2{}^{1/q}},$$

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\\
&=\int_1^{+\infty}y^{(-\beta-1-\lambda_2\varepsilon)/q}\left(\int_1^{+\infty}K(x,y)x^{(-\alpha-1-\lambda_1\varepsilon)/p}\mathrm{d}x\right)\mathrm{d}y\\
&=\int_1^{+\infty}y^{\lambda_2+(-\beta-1-\lambda_2\varepsilon)/q}\left(\int_1^{+\infty}K(xy^{-\lambda_2/\lambda_1},1)x^{(-\alpha-1-\lambda_1\varepsilon)/p}\mathrm{d}x\right)\mathrm{d}y\\
&=\int_1^{+\infty}y^{-1-\lambda_2\varepsilon-\frac{c}{\lambda_1}}\left(\int_{y^{-\lambda_2/\lambda_1}}^{+\infty}K(t,1)t^{(-\alpha-1-\lambda_1\varepsilon)/p}\mathrm{d}t\right)\mathrm{d}y\\
&\geqslant\int_1^{+\infty}y^{-1-\lambda_2\varepsilon-\frac{c}{\lambda_1}}\mathrm{d}y\int_1^{+\infty}K(t,1)t^{(-\alpha-1-\lambda_1\varepsilon)/p}\mathrm{d}t.
\end{aligned}$$

Therefore

$$\int_1^{+\infty}y^{-1-\lambda_2\varepsilon-\frac{c}{\lambda_1}}\mathrm{d}y\int_1^{+\infty}K(t,1)t^{(-\alpha-1-\lambda_1\varepsilon)/p}\mathrm{d}t\leqslant\frac{M}{\varepsilon\lambda_1{}^{1/p}\lambda_2{}^{1/q}}.$$

Since $c<0,\ \lambda_1>0,\ -\lambda_2\varepsilon-\frac{c}{\lambda_1}>0$ and $\int_1^{+\infty}y^{-1-\lambda_2\varepsilon-\frac{c}{\lambda_1}}\mathrm{d}y$ diverges to $+\infty$, which contradicts the above inquality. Hence it does not hold for $c<0$.

To sum up, when $\lambda_1>0,\ \lambda_2>0$, we have $c=0$, that is $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2$.

Now let us consider the case of $\lambda_1<0,\ \lambda_2<0$. If $c>0$, for sufficiently small $\varepsilon>0$ $(\varepsilon<\frac{c}{\lambda_1\lambda_2})$, take

$$f(x)=\begin{cases} x^{(-\alpha-1+\lambda_1\varepsilon)/p}, & x\geqslant 1,\\ 0, & 0<x<1,\end{cases}$$

$$g(y)=\begin{cases} y^{(-\beta-1+\lambda_2\varepsilon)/q}, & y\geqslant 1,\\ 0, & 0<y<1.\end{cases}$$

Then

$$\begin{aligned}||f||_{p,\alpha}||g||_{q,\beta} &= \left(\int_1^{+\infty} x^{-1+\lambda_1\varepsilon}\mathrm{d}x\right)^{1/p}\left(\int_1^{+\infty} y^{-1+\lambda_2\varepsilon}\mathrm{d}y\right)^{1/q}\\ &= \frac{1}{\varepsilon\,(-\lambda_1)^{\,1/p}\,(-\lambda_2)^{\,1/q}},\end{aligned}$$

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\\ &=\int_1^{+\infty} x^{\lambda_1+(-\alpha-1+\lambda_1\varepsilon)/p}\left(\int_1^{+\infty} K(1,x^{-\lambda_1/\lambda_2}y)y^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}y\right)\mathrm{d}x\\ &=\int_1^{+\infty} x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\left(\int_{x^{-\lambda_1/\lambda_2}}^{+\infty} K(1,t)t^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}t\right)\mathrm{d}x\\ &\geqslant \int_1^{+\infty} x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\mathrm{d}x\int_1^{+\infty} K(1,t)t^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}t.\end{aligned}$$

Therefore

$$\int_1^{+\infty} x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\mathrm{d}x\int_1^{+\infty} K(1,t)t^{(-\beta-1+\lambda_2\varepsilon)/q}\mathrm{d}t\leqslant\frac{M}{\varepsilon\,(-\lambda_1)^{\,1/p}\,(-\lambda_2)^{\,1/q}}.$$

Since $c>0,\ \lambda_2<0,\ \lambda_1\varepsilon-\frac{c}{\lambda_2}>0$ and $\int_1^{+\infty} x^{-1+\lambda_1\varepsilon-\frac{c}{\lambda_2}}\mathrm{d}x$ diverges to $+\infty$. Thus it is also a contradiction of the above inequality. That is, it does not hold for $c>0$.

If $c<0$, for sufficiently small $\varepsilon>0$ $(\varepsilon<\frac{-c}{\lambda_1\lambda_2})$, take

$$f(x)=\begin{cases} x^{(-\alpha-1-\lambda_1\varepsilon)/p}, & 0<x\leqslant 1,\\ 0, & x>1,\end{cases}$$

$$g(y)=\begin{cases} y^{(-\beta-1-\lambda_2\varepsilon)/q}, & 0<y\leqslant 1,\\ 0, & y>1.\end{cases}$$

Similarly, one can get

$$\int_0^1 y^{-1-\lambda_2\varepsilon-\frac{c}{\lambda_1}}\mathrm{d}y\int_0^1 K(t,1)t^{(-\alpha-1-\lambda_1\varepsilon)/p}\mathrm{d}t\leqslant\frac{M}{\varepsilon\,(-\lambda_1)^{\,1/p}\,(-\lambda_2)^{\,1/q}}.$$

Since $c<0,\ \lambda_1<0,\ \lambda_2\varepsilon-\frac{c}{\lambda_1}<0$ and $\int_0^1 y^{-1-\lambda_2\varepsilon-\frac{c}{\lambda_1}}\mathrm{d}y$ diverges to $+\infty$, which contradicts the above inequality. It does not hold for $c<0$.

To sum up, when $\lambda_1<0,\ \lambda_2<0$, we also get $c=0$, that is $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2$.

Sufficiency. If $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2$, let $a=\frac{\alpha}{pq}+\frac{1}{pq}$, $b=\frac{\beta}{pq}+\frac{1}{pq}$. By Hölder's inequality and Lemma 2, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\\
&=\int_0^{+\infty}\int_0^{+\infty}\left[f(x)\frac{x^a}{y^b}\right]\left[g(y)\frac{y^b}{x^a}\right]K(x,y)\mathrm{d}x\mathrm{d}y\\
&\leqslant\left(\int_0^{+\infty}\int_0^{+\infty}f^p(x)\frac{x^{ap}}{y^{bp}}K(x,y)\mathrm{d}x\mathrm{d}y\right)^{1/p}\\
&\quad\times\left(\int_0^{+\infty}\int_0^{+\infty}g^q(y)\frac{y^{bq}}{x^{aq}}K(x,y)\mathrm{d}x\mathrm{d}y\right)^{1/q}\\
&=\left(\int_0^{+\infty}x^{\frac{\alpha+1}{q}}f^p(x)\omega_1(x)\mathrm{d}x\right)^{1/p}\left(\int_0^{+\infty}y^{\frac{\beta+1}{p}}g^q(y)\omega_2(y)\mathrm{d}y\right)^{1/q}\\
&=\left(\int_0^{+\infty}x^{\frac{\alpha+1}{q}+\lambda_1-\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+1}{q}-1\right)}f^p(x)W_1\mathrm{d}x\right)^{1/p}\\
&\quad\times\left(\int_0^{+\infty}y^{\frac{\beta+1}{p}+\lambda_2-\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-1\right)}g^q(y)W_2\mathrm{d}y\right)^{1/q}\\
&=W_1^{1/p}W_2^{1/q}\left(\int_0^{+\infty}x^{\frac{1}{\lambda_2}\left(\lambda_2\alpha-\frac{\lambda_2\alpha-\lambda_1}{p}-\frac{\lambda_1\beta-\lambda_2}{q}+\lambda_1\lambda_2\right)}f^p(x)\mathrm{d}x\right)^{1/p}\\
&\quad\times\left(\int_0^{+\infty}y^{\frac{1}{\lambda_1}\left(\lambda_1\beta-\frac{\lambda_2\alpha-\lambda_1}{p}-\frac{\lambda_1\beta-\lambda_2}{q}+\lambda_1\lambda_2\right)}g^q(y)\mathrm{d}y\right)^{1/q}\\
&=W_1^{1/p}W_2^{1/q}\left(\int_0^{+\infty}x^\alpha f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_0^{+\infty}y^\beta g^q(y)\mathrm{d}y\right)^{1/q}\\
&=W_1^{1/p}W_2^{1/q}||f||_{p,\alpha}||g||_{q,\beta},
\end{aligned}$$

thus (4) holds when taking any constant $M\geqslant W_1^{1/p}W_2^{1/q}$.

(ii) Suppose that (4) holds. If the constant factor $\frac{W}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}$ is not the best, then there exists a constant $M_0<\frac{W}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}$, for $f(x)\in L^p_\alpha(0,+\infty)$, $g(y)\in L^q_\beta(0,+\infty)$, such that

$$\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant M_0||f||_{p,\alpha}||g||_{q,\beta}. \tag{8}$$

For sufficiently small $\varepsilon>0$ and $\delta>0$, take

$$f(x)=\begin{cases}x^{(-\alpha-1-|\lambda_1|\varepsilon)/p}, & x\geqslant\delta,\\ 0, & 0<x<\delta,\end{cases}$$

$$g(y)=\begin{cases}y^{(-\beta-1-|\lambda_2|\varepsilon)/q}, & y\geqslant 1,\\ 0, & 0<y<1.\end{cases}$$

Then

$$||f||_{p,\alpha}||g||_{q,\beta} = \left(\int_{\delta}^{+\infty} x^{-1-|\lambda_1|\varepsilon}\mathrm{d}x\right)^{1/p}\left(\int_{1}^{+\infty} y^{-1-|\lambda_2|\varepsilon}\mathrm{d}y\right)^{1/q}$$
$$= \frac{1}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{1}{\delta^{|\lambda_1|\varepsilon}}\right)^{1/p}. \tag{9}$$

In view of $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2$, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} K(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y\\
&= \int_1^{+\infty} y^{-\frac{\beta+1+|\lambda_2|\varepsilon}{q}}\left(\int_{\delta}^{+\infty} K(x,y)x^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_1^{+\infty} y^{\lambda_2-\frac{\beta+1+|\lambda_2|\varepsilon}{q}}\left(\int_{\delta}^{+\infty} K(xy^{-\lambda_2/\lambda_1},1)x^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_1^{+\infty} y^{\lambda_2-\frac{\beta+1+|\lambda_2|\varepsilon}{q}-\frac{\lambda_2}{\lambda_1}\frac{\alpha+1+|\lambda_1|\varepsilon}{p}+\frac{\lambda_2}{\lambda_1}}\left(\int_{\delta y^{-\lambda_2/\lambda_1}}^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t\right)\mathrm{d}y\\
&\geqslant \int_1^{+\infty} y^{\frac{1}{\lambda_1}\left(\lambda_1\lambda_2-\frac{\lambda_2\alpha-\lambda_1}{p}-\frac{\lambda_1\beta-\lambda_2}{q}-\lambda_1-\frac{|\lambda_1|\lambda_2\varepsilon}{p}-\frac{|\lambda_2|\lambda_1\varepsilon}{q}\right)}\\
&\quad\times\left(\int_{\delta}^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t\right)\mathrm{d}y\\
&= \int_1^{+\infty} y^{-1-|\lambda_2|\varepsilon}\mathrm{d}y\int_{\delta}^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t\\
&= \frac{1}{|\lambda_2|\varepsilon}\int_{\delta}^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t,
\end{aligned} \tag{10}$$

it follows from (8), (9) and (10) that

$$\frac{1}{|\lambda_2|}\int_{\delta}^{+\infty} K(t,1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t \leqslant \frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}},$$

consequently,

$$\frac{1}{|\lambda_2|}\int_{\delta}^{+\infty} K(t,1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t \leqslant \frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}$$

as $\varepsilon \to 0^+$. Let $\delta \to 0^+$, we get

$$\frac{W_2}{|\lambda_2|} = \frac{1}{|\lambda_2|}\int_0^{+\infty} K(t,1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t \leqslant \frac{M_0}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}},$$

thus $\frac{W}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}} \leqslant M_0$, this is a contradiction. Hence the constant factor $\frac{W}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}$ is the best possible. □

3. Applications in operator theory

By using Lemma 1 and Theorem 1, we have

THEOREM 2. *Suppose that* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda_1\lambda_2>0,\ \alpha,\ \beta\in\mathbb{R},\ K(x,y)$ *is a quasi-homogeneous non-negative measurable function of order* (λ_1,λ_2), $K(1,u)$ $(K(u,1))>0$ *a.e. in* $(0,1)$ $((1,\infty))$. *Let the operator* T *be as in (2), and*

$$W_1=\int_0^{+\infty}u^{-\frac{\beta+1}{q}}K(1,u)\mathrm{d}u, W_2=\int_0^{+\infty}u^{-\frac{\alpha+1}{p}}K(u,1)\mathrm{d}u,$$

are both convergent. Then

(i) the necessary and sufficient conditions for the operator $T:L^p_\alpha(0,+\infty)\to L^p_{\beta(1-p)}(0,+\infty)$ *to be bounded is* $\frac{\lambda_2\alpha-\lambda_1}{p}+\frac{\lambda_1\beta-\lambda_2}{q}=\lambda_1\lambda_2$*;*

(ii) if T *is bounded from* $L^p_\alpha(0,+\infty)$ *to* $L^p_{\beta(1-p)}(0,+\infty)$, *then the operator norm of* T *is* $||T||=\frac{W}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}$, *where* $W=|\lambda_1|W_2=|\lambda_2|W_1$.

We get the following results by taking $\alpha=\beta=0$.

COROLLARY 1. *Assume that* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda_1<0,\ \lambda_2<0,\ K(x,y)$ *is a quasi-homogeneous non-negative measurable function of order* (λ_1,λ_2), $K(1,u)$ $(K(u,1))$ >0 *a.e. in* $(0,1)$ $((1,\infty))$. *Let the operator* T *be as in (2),* W_1 *and* W_2 *be as in Theorem 2. Then*

(i) the necessary and sufficient conditions for the operator T *in* $L^p(0,+\infty)$ *to be bounded is* $\frac{\lambda_1}{p}+\frac{\lambda_2}{q}+\lambda_1\lambda_2=0$.

(ii) if T *is bounded in* $L^p(0,+\infty)$, *then the operator norm of* T *is* $||T||=\frac{W}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}$, *where* $W=|\lambda_2|\int_0^{+\infty}u^{-\frac{1}{q}}K(1,u)\mathrm{d}u$.

COROLLARY 2. *Suppose that* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda_1>0,\ \lambda_2>0,\ \frac{1}{\lambda_2 p}<1$, *define the singular integral operator* T *by*

$$T(f)(y)=\int_0^{+\infty}\frac{1}{x^{\lambda_1}+y^{\lambda_2}}f(x)\mathrm{d}x.$$

Then

(i) T *is bounded in* $L^p(0,+\infty)$ *if and only if* $\frac{\lambda_1}{p}+\frac{\lambda_2}{q}=\lambda_1\lambda_2$.

(ii) if T *is bounded in* $L^p(0,+\infty)$, *then the operator norm of* T *is* $||T||=\frac{1}{\lambda_1{}^{1/q}\lambda_2{}^{1/p}}\frac{\pi}{\sin\frac{\pi}{\lambda_1 q}}$.

Proof. Assume that $K(x,y)=\frac{1}{x^{\lambda_1}+y^{\lambda_2}}$, then $K(x,y)$ is a quasi-homogeneous non-negative measurable function of order $(-\lambda_1,-\lambda_2)$. Since

$$\begin{aligned}\int_0^{+\infty}u^{-\frac{1}{q}}K(1,u)\mathrm{d}u&=\int_0^{+\infty}\frac{1}{1+u^{\lambda_2}}u^{-\frac{1}{q}}\mathrm{d}u\\&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{1}{1+t}t^{\frac{1}{\lambda_2 p}-1}\mathrm{d}t\\&=\frac{1}{\lambda_2}B\left(\frac{1}{\lambda_2 p},1-\frac{1}{\lambda_2 p}\right)\\&=\frac{1}{\lambda_2}\frac{\pi}{\sin\frac{\pi}{\lambda_2 p}},\end{aligned}$$

then according to Corollary 1, we know that Corollary 2 holds. □

COROLLARY 3. *Suppose that* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda_1>0,\ \lambda_2>0,\ \frac{1}{\lambda_2 p}<1$, *define the singular integral operator* T *by*

$$T(f)(y)=\int_0^{+\infty}\frac{1}{\max\{x^{\lambda_1},y^{\lambda_2}\}}f(x)\mathrm{d}x.$$

Then

(i) T *is bounded in* $L^p(0,+\infty)$ *if and only if* $\frac{\lambda_1}{p}+\frac{\lambda_2}{q}=\lambda_1\lambda_2$.

(ii) if T *is bounded in* $L^p(0,+\infty)$, *then the operator norm of* T *is* $||T||=\frac{\lambda_1 q+\lambda_2 p}{\lambda_1{}^{1/q}\lambda_2{}^{1/p}}$.

Proof. Set $K(x,y)=\frac{1}{\max\{x^{\lambda_1},y^{\lambda_2}\}}$, then $K(x,y)$ is a quasi-homogeneous non-negative measurable function of order $(-\lambda_1,-\lambda_2)$. Since $\lambda_2>0$ and $\frac{1}{\lambda_2 p}<1$, then

$$\begin{aligned}\int_0^{+\infty}u^{-\frac{1}{q}}K(1,u)\mathrm{d}u&=\int_0^{+\infty}\frac{1}{\max\{1,u^{\lambda_2}\}}u^{-\frac{1}{q}}\mathrm{d}u\\&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{1}{\max\{1,t\}}t^{\frac{1}{\lambda_2 p}-1}\mathrm{d}t\\&=\frac{1}{\lambda_2}\int_0^{1}t^{\frac{1}{\lambda_2 p}-1}\mathrm{d}t+\frac{1}{\lambda_2}\int_1^{+\infty}t^{\frac{1}{\lambda_2 p}-2}\mathrm{d}t\\&=p-\frac{1}{\lambda_2}\frac{1}{\frac{1}{\lambda_2 p}-1}=p-\frac{1}{\lambda_2}\frac{1}{-\frac{1}{\lambda_1 q}}\\&=\frac{1}{\lambda_2}(\lambda_2 p+\lambda_1 q).\end{aligned}$$

It follows from Corollary 1 that Corollary 3 holds. □

COROLLARY 4. *Suppose that* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda_1>0,\ \lambda_2>0,\ \frac{1}{\lambda_1 q}<1$, *define the singular integral operator* T *by*

$$T(f)(y)=\int_0^{+\infty}\frac{\ln x^{\lambda_1}-\ln y^{\lambda_1}}{x^{\lambda_1}-y^{\lambda_2}}f(x)\mathrm{d}x.$$

Then

(i) T is bounded in $L^p(0,+\infty)$ if and only if $\frac{\lambda_1}{p}+\frac{\lambda_2}{q}=\lambda_1\lambda_2$.

(ii) if T is bounded in $L^p(0,+\infty)$, then the operator norm of T is $||T||=\frac{1}{\lambda_1{}^{1/q}\lambda_2{}^{1/p}}\left(\frac{\pi}{\sin\frac{\pi}{\lambda_1 q}}\right)^2$.

Proof. Set $K(x,y)=\frac{\ln x^{\lambda_1}-\ln y^{\lambda_1}}{x^{\lambda_1}-y^{\lambda_2}}$, then $K(x,y)$ is a quasi-homogeneous non-negative measurable function of order $(-\lambda_1,-\lambda_2)$. Notice that $\int_0^{+\infty}\frac{\ln t}{t-1}t^{\frac{1}{r}}\mathrm{d}t=\left(\frac{\pi}{\sin\pi/r}\right)^2$ $(r>1)$ (cf. [17]), we find

$$\begin{aligned}\int_0^{+\infty}u^{-\frac{1}{q}}K(1,u)\mathrm{d}u&=\int_0^{+\infty}\frac{-\ln u^{\lambda_2}}{1-u^{\lambda_2}}u^{-\frac{1}{q}}\mathrm{d}u\\&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{\ln t}{t-1}t^{\frac{1}{\lambda_2 p}-1}\mathrm{d}t\\&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{\ln t}{t-1}t^{\frac{1}{\lambda_1 q}}\mathrm{d}t\\&=\frac{1}{\lambda 1}\left(\frac{\pi}{\sin\frac{\pi}{\lambda_1 q}}\right)^2.\end{aligned}$$

According to Corollary 1, we can see that Corollary 4 holds. □

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