FRACTIONAL INTEGRAL ASSOCIATED WITH SCHRÖDINGER OPERATOR ON VANISHING GENERALIZED MORREY SPACES

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Abstract. Let \( L = -\Delta + V \) be a Schrödinger operator, where the non-negative potential \( V \) belongs to the reverse Hölder class \( RH_{n/2} \), let \( b \) belong to a new \( BMO_0(\rho) \) space, and let \( \mathcal{I}_L^\beta \) be the fractional integral operator associated with \( L \). In this paper, we study the boundedness of the operator \( \mathcal{I}_L^\beta \) and its commutators \([b, \mathcal{I}_L^\beta]\) with \( b \in BMO_0(\rho) \) on generalized Morrey spaces associated with Schrödinger operator \( M_{p,\varphi}^V \) and vanishing generalized Morrey spaces associated with Schrödinger operator \( VM_{p,\varphi_1}^V \). We find the sufficient conditions on the pair \((\varphi_1, \varphi_2)\) which ensures the boundedness of the operator \( \mathcal{I}_L^\beta \) from \( M_{p,\varphi_1}^V \) to \( M_{q,\varphi_2}^V \) and from \( VM_{p,\varphi_1}^V \) to \( VM_{q,\varphi_2}^V \), \( 1/p - 1/q = \beta/n \). When \( b \) belongs to \( BMO_0(\rho) \) and \((\varphi_1, \varphi_2)\) satisfies some conditions, we also show that the commutator operator \([b, \mathcal{I}_L^\beta]\) is bounded from \( M_{p,\varphi_1}^V \) to \( M_{q,\varphi_2}^V \) and from \( VM_{p,\varphi_1}^V \) to \( VM_{q,\varphi_2}^V \), \( 1/p - 1/q = \beta/n \).

1. Introduction and results

Let us consider the Schrödinger operator

\[ L = -\Delta + V \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 3, \]

where \( V \) is a non-negative, \( V \neq 0 \), and belongs to the reverse Hölder class \( RH_q \) for some \( q \geq n/2 \), i.e., there exists a constant \( C > 0 \) such that the reverse Hölder inequality

\[ \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y)dy \right)^{1/q} \leq \frac{C}{|B(x,r)|} \int_{B(x,r)} V(y)dy \]

holds for every \( x \in \mathbb{R}^n \) and \( 0 < r < \infty \), where \( B(x,r) \) denotes the ball centered at \( x \) with radius \( r \). In particular, if \( V \) is a nonnegative polynomial, then \( V \in RH_{\infty} \).

Obviously, \( RH_{q_2} \subset RH_{q_1} \), if \( q_2 > q_1 \). The most important property of the class \( RH_q \) is its self-improvement, that is, if \( V \in RH_q \), then \( V \in RH_{q+\varepsilon} \) for some \( \varepsilon > 0 \).


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As in [18], for a given potential \( V \in RH_q \) with \( q \geq n/2 \), we define the auxiliary function
\[
\rho(x) := \frac{1}{m_V(x)} = \sup_{r > 0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}.
\]

It is well-known that that \( 0 < \rho(x) < \infty \) for any \( x \in \mathbb{R}^n \).

According to [4], the new BMO space \( BMO_\theta(\rho) \) with \( \theta \geq 0 \) is defined as a set of all locally integrable functions \( b \) such that
\[
\frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_B| dy \leq C(1 + \frac{r}{\rho(x)})^\theta
\]
for all \( x \in \mathbb{R}^n \) and \( r > 0 \), where \( b_B = \frac{1}{|B|} \int_B b(y) dy \). A norm for \( b \in BMO_\theta(\rho) \), denoted by \([b]_\theta\), is given by the infimum of the constants in the inequalities above. Clearly, \( BMO \subset BMO_\theta(\rho) \).

We now present the definition of generalized Morrey spaces (including weak version) related to potential, which introduced by Guliyev in [12].

**Definition 1.** Let \( \varphi(x, r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \), \( 1 \leq p < \infty \), \( \alpha \geq 0 \), and \( V \in RH_q \), \( q \geq 1 \). We denote by \( M^{\alpha, V}_{p, \varphi} = M^{\alpha, V}_{p, \varphi}(\mathbb{R}^n) \) the generalized Morrey space associated with Schrödinger operator, the space of all functions \( f \in L^P_{loc}(\mathbb{R}^n) \) with finite quasinorm
\[
\|f\|_{M^{\alpha, V}_{p, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} r^{-n/p} \|f\|_{L^p(B(x, r))}.
\]

Also \( WM^{\alpha, V}_{p, \varphi} = WM^{\alpha, V}_{p, \varphi}(\mathbb{R}^n) \) we denote the weak generalized Morrey space associated with Schrödinger operator, the space of all functions \( f \in WL^P_{loc}(\mathbb{R}^n) \) with
\[
\|f\|_{WM^{\alpha, V}_{p, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} r^{-n/p} \|f\|_{WL^p(B(x, r))} < \infty.
\]

**Remark 1.** (i) When \( \alpha = 0 \), and \( \varphi(x, r) = r^{(\lambda - n)/p} \), \( M^{\alpha, V}_{p, \varphi}(\mathbb{R}^n) \) is the classical Morrey space \( L^P_{p, \lambda}(\mathbb{R}^n) \) introduced by Morrey in [13];

(ii) When \( \varphi(x, r) = r^{(\lambda - n)/p} \), \( M^{\alpha, V}_{p, \varphi}(\mathbb{R}^n) \) is the Morrey space associated with Schrödinger operator \( L^P_{\alpha, \lambda}(\mathbb{R}^n) \) studied by Tang and Dong in [21];

(iii) When \( \alpha = 0 \), \( M^{\alpha, V}_{p, \varphi}(\mathbb{R}^n) \) is the generalized Morrey space \( M^{\alpha, V}_{p, \varphi}(\mathbb{R}^n) \) introduced by Mizuhara and Nakai in [14, 15].

(iv) The generalized Morrey space associated with Schrödinger operator \( M^{\alpha, V}_{p, \varphi}(\mathbb{R}^n) \) was introduced by Guliyev in [12].

The classical Morrey spaces \( L^P_{p, \lambda}(\mathbb{R}^n) \) was introduced by Morrey in [13] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers
to [7, 8, 9, 13]. The generalized Morrey spaces are defined with $r^\lambda$ replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [10, 14, 15, 19] and etc).

For brevity, in the sequel we use the notations

$$\mathfrak{A}^\alpha_{p,\varphi}(f;x,r) := \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/p} \varphi(x,r)^{-1} \|f\|_{L^p(B(x,r))}$$

and

$$\mathfrak{A}^\beta_{\varphi,\psi}(f;x,r) := \left(1 + \frac{r}{\rho(x)}\right)^\beta r^{-n/p} \varphi(x,r)^{-1} \|f\|_{W^{1,p}(B(x,r))}.$$  

**Definition 2.** The vanishing generalized Morrey space associated with Schrödinger operator $V M^\alpha_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M^\alpha_{p,\varphi}(\mathbb{R}^n)$ such that

$$\limsup_{r \to 0} \mathfrak{A}^\alpha_{p,\varphi}(f;x,r) = 0. \quad (1)$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $V W M^\alpha_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in W M^\alpha_{p,\varphi}(\mathbb{R}^n)$ such that

$$\limsup_{r \to 0} \mathfrak{A}^\alpha_{p,\varphi}(f;x,r) = 0.$$ 

The vanishing spaces $V M^\alpha_{p,\varphi}(\mathbb{R}^n)$ and $V W M^\alpha_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{V M^\alpha_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \mathfrak{A}^\alpha_{p,\varphi}(f;x,r),$$

$$\|f\|_{V W M^\alpha_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \mathfrak{A}^\alpha_{p,\varphi}(f;x,r),$$

respectively.

In the case $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$ $V M^\lambda_{\varphi,\psi}(\mathbb{R}^n)$ is the vanishing Morrey space $V M_{\varphi,\psi}$ introduced in [22], where applications to PDE were considered.

We refer to [1, 6, 16, 17] for some properties of vanishing generalized Morrey spaces.

**Definition 3.** Let $L = -\triangle + V$ with $V \in RH_{n/2}$. The fractional integral associated with $L$ is defined by

$$\mathcal{I}_\beta f(x) = L^{-\beta/2} f(x) = \int_0^\infty e^{-tL}(f)(x) t^{\beta/2-1} dt$$

for $0 < \beta < n$. The commutator of $\mathcal{I}_\beta$ is defined by

$$[b, \mathcal{I}_\beta] f(x) = b(x) \mathcal{I}_\beta f(x) - \mathcal{I}_\beta (b f)(x).$$
In this paper, we consider the boundedness of the fractional integral operator $\mathcal{I}_\beta^L$ on the generalized Morrey spaces $M^\alpha_{p,\varphi}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM^\alpha_{p,\varphi}(\mathbb{R}^n)$. When $b$ belongs to the new BMO space $BMO_\theta(\rho)$, we also show that $[b, \mathcal{I}_\beta^L]$ is bounded from $M^\alpha_{p,\varphi}(\mathbb{R}^n)$ to $M^\alpha_{q,\varphi}(\mathbb{R}^n)$ and from $VM^\alpha_{p,\varphi}(\mathbb{R}^n)$ to $VM^\alpha_{q,\varphi}(\mathbb{R}^n)$.

Our main results are as follows.

**Theorem 1.** Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^\alpha$, $\varphi_2 \in \Omega_q^\alpha$ satisfies the condition

$$\int_r^\infty \text{ess inf}_{t \leq s < \infty} \varphi_1(x,s) \frac{s^n}{t^q} dt \leq c_0 \varphi_2(x,r),$$

where $c_0$ does not depend on $x$ and $r$. Then the operator $\mathcal{I}_\beta^L$ is bounded on $M^\alpha_{p,\varphi_1}$ to $M^\alpha_{q,\varphi_2}$ for $p > 1$ and from $M^\alpha_{1,\varphi_1}$ to $WM^\alpha_{n,\frac{n-\beta}{n}} \varphi_2$. Moreover, for $p > 1$

$$\|\mathcal{I}_\beta^L f\|_{M^\alpha_{q,\varphi_2}} \leq C \|f\|_{M^\alpha_{p,\varphi_1}},$$

and for $p = 1$

$$\|\mathcal{I}_\beta^L f\|_{WM^\alpha_{1/\beta,\varphi_2}} \leq C \|f\|_{M^\alpha_{1,\varphi_1}},$$

where $C$ does not depend on $f$.

**Theorem 2.** Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^\alpha$, $\varphi_2 \in \Omega_q^\alpha$ satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{r}{t}\right) \frac{\text{ess inf}_{t \leq s < \infty} \varphi_1(x,s) s^n}{t^q} dt \leq c_0 \varphi_2(x,r),$$

where $c_0$ does not depend on $x$ and $r$. If $b \in BMO_\theta(\rho)$, then the operator $[b, \mathcal{I}_\beta^L]$ is bounded from $M^\alpha_{p,\varphi_1}$ to $M^\alpha_{q,\varphi_2}$ and

$$\|[b, \mathcal{I}_\beta^L] f\|_{M^\alpha_{q,\varphi_2}} \leq C[b]_\theta \|f\|_{M^\alpha_{p,\varphi_1}},$$

where $C$ does not depend on $f$. 
THEOREM 3. Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 \leq p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_{p,1}^{\alpha,V}$, $\varphi_2 \in \Omega_{q,1}^{\alpha,V}$ satisfies the conditions

$$c_\delta := \int_\delta^\infty \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t} < \infty$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x,t) \frac{dt}{t^{1-\beta}} \leq C_0 \varphi_2(x,r),$$

where $C_0$ does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $\mathcal{I}_\beta$ is bounded from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$ for $p > 1$ and from $VM_{1,\varphi_1}^{\alpha,V}$ to $VM_{n-\beta,\varphi_2}^{\alpha,V}$.

THEOREM 4. Let $V \in RH_{n/2}$, $b \in BMO(\rho)$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, and $\varphi_1 \in \Omega_{p,1}^{\alpha,V}$, $\varphi_2 \in \Omega_{q,1}^{\alpha,V}$ satisfies the conditions

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1(x,t) \frac{dt}{t^{1-\beta}} \leq c_0 \varphi_2(x,r),$$

where $c_0$ does not depend on $x$ and $r$,

$$\lim_{r \to 0} \inf_{x \in \mathbb{R}^n} \frac{\ln \frac{r}{\varphi_2(x,r)}}{x} = 0$$

and

$$c_\delta := \int_\delta^\infty \left(1 + |\ln t|\right) \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t^{1-\beta}} < \infty$$

for every $\delta > 0$. Then the operator $[b, \mathcal{I}_\beta]$ is bounded from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$.

REMARK 2. Note that, Theorems 1 and 2 in the case of $V \equiv 0$ was proved in [11, Corollary 5.5 and 7.5] and in the case of $\varphi(x,r) = r^{(\lambda-n)/p}$ in [21, Theorems 1.3 and 1.4].

REMARK 3. Note that, in [2] the Nikolskii-Morrey type spaces were introduced and the authors studied some embedding theorems. In the next paper, we shall introduce the generalized Nikolskii-Morrey spaces associated with Schrödinger operator and will study some embedding theorems. We will also investigate the boundedness of fractional integral associated with Schrödinger operator on the generalized Nikolskii-Morrey spaces associated with Schrödinger operator.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant $C$, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. 

2. Some preliminaries

We would like to recall the important properties concerning the critical function.

**Lemma 1.** [18] Let $V \in RH_{n/2}$. For the associated function $\rho$ there exist $C$ and $k_0 \geq 1$ such that

$$C^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{k_0}$$  \hspace{1cm} (8)

for all $x, y \in \mathbb{R}^n$.

**Lemma 2.** [3] Suppose $x \in B(x_0, r)$. Then for $k \in \mathbb{N}$ we have

$$\frac{1}{\left(1 + \frac{2kr \rho(x)}{\rho(x_0)}\right)^N} \lesssim \frac{1}{\left(1 + \frac{2kr \rho(x_0)}{\rho(x_0)}\right)^{(k_0+1)N/k_0}}.$$  

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

**Lemma 3.** [4] Let $1 \leq s < \infty$. If $b \in BMO_\theta(\rho)$, then

$$\left(\frac{1}{|B|} \int_B |b(y) - b_B|^s dy\right)^{1/s} \lesssim [b]_\theta \left(1 + \frac{r}{\rho(x)}\right)^{\theta'}$$

for all $B = B(x, r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0 + 1)\theta$ and $k_0$ is the constant appearing in (8).

**Lemma 4.** [4] Let $1 \leq s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x, r)$. Then

$$\left(\frac{1}{|2^k B|} \int_{2^k B} |b(y) - b_B|^s dy\right)^{1/s} \lesssim [b]_\theta k \left(1 + \frac{2kr}{\rho(x)}\right)^{\theta'}$$

for all $k \in \mathbb{N}$, with $\theta'$ as in Lemma 3.

Let $K_\beta$ be the kernel of $\mathcal{I}_\beta^L$. The following result give the estimate on the kernel $K_\beta(x, y)$.

**Lemma 5.** [5] If $V \in RH_{n/2}$, then for every $N$, there exists a constant $C$ such that

$$|K_\beta(x, y)| \lesssim \frac{C}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N \frac{1}{|x-y|^{n-\beta}}}.$$  \hspace{1cm} (9)

Finally, we recall a relationship between essential supremum and essential infimum.
Lemma 6. [23] Let $f$ be a real-valued nonnegative function and measurable on $E$. Then

$$\left(\text{ess inf}_{x \in E} f(x)\right)^{-1} = \text{ess sup}_{x \in E} \frac{1}{f(x)}.$$

Lemma 7. [3] Let $\phi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q \geq 1$.

(i) If $\sup_{t < r < \infty} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \frac{r^{-\frac{\alpha}{p}}}{\phi(x, r)} = \infty$ for some $t > 0$ and for all $x \in \mathbb{R}^n$, then $M_{p, \phi}^{\alpha, V}(\mathbb{R}^n) = \Theta$.

(ii) If $\sup_{0 < r < \tau} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \phi(x, r)^{-1} = \infty$ for some $\tau > 0$ and for all $x \in \mathbb{R}^n$, then $M_{p, \phi}^{\alpha, V}(\mathbb{R}^n) = \Theta$.

Remark 4. We denote by $\Omega_{p, 1}^{\alpha, V}$ the sets of all positive measurable functions $\phi$ on $\mathbb{R}^n \times (0, \infty)$ such that for all $t > 0$,

$$\sup_{x \in \mathbb{R}^n} \left\| \left(1 + \frac{r}{\rho(x)}\right)^\alpha \frac{r^{-\frac{\alpha}{p}}}{\phi(x, r)} \right\|_{L_\infty(t, \infty)} < \infty,$$

and

$$\sup_{x \in \mathbb{R}^n} \left\| \left(1 + \frac{r}{\rho(x)}\right)^\alpha \phi(x, r)^{-1} \right\|_{L_\infty(0, t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 7, we always assume that $\phi \in \Omega_{p, 1}^{\alpha, V}$.

Remark 5. We denote by $\Omega_{p, 1}^{\alpha, V}$ the sets of all positive measurable functions $\phi$ on $\mathbb{R}^n \times (0, \infty)$ such that

$$\inf_{x \in \mathbb{R}^n} \inf_{r > \delta} \left(1 + \frac{r}{\rho(x)}\right)^{-\alpha} \phi(x, r) > 0,$$

for some $\delta > 0$, (10)

and

$$\lim_{r \to 0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \frac{r^{n/p}}{\phi(x, r)} = 0.$$

For the non-triviality of the space $VM_{p, \phi}^{\alpha, V}(\mathbb{R}^n)$ we always assume that $\phi \in \Omega_{p, 1}^{\alpha, V}$.

3. Proof of Theorem 1

We first prove the following conclusions

Theorem 5. Let $V \in RH_{n/2}$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality

$$\|J_{p, \beta}^L(f)\|_{L_q(B(x_0, r))} \lesssim \frac{1}{r^{\frac{n}{q}}} \int_{2r}^\infty \left\|L_p(B(x_0, t)) \right\|_{t^q} dt.$$
holds for any $f \in L^p_{\text{loc}}(\mathbb{R}^n)$. Moreover, for $p = 1$ the inequality
\[
\| \mathcal{J}^L_\beta(f) \|_{W L_{\frac{n}{n-\beta}}(B(x_0, r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\| f \|_{L^1(B(x_0, t))}}{t^{n-\beta}} \, dt
\]
holds for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

**Proof.** For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ and $\lambda B = B(x_0, \lambda r)$ for any $\lambda > 0$. We write $f$ as $f = f_1 + f_2$, where $f_1(y) = f(y) \chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then
\[
\| \mathcal{J}^L_\beta(f) \|_{L^q(B(x_0, r))} \leq \| \mathcal{J}^L_\beta(f_1) \|_{L^q(B(x_0, r))} + \| \mathcal{J}^L_\beta(f_2) \|_{L^q(B(x_0, r))}.
\]
Since $f_1 \in L^p(\mathbb{R}^n)$ and from the boundedness of $\mathcal{J}^L_\beta$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ (see [20]) it follows that
\[
\| \mathcal{J}^L_\beta(f_1) \|_{L^q(B(x_0, r))} \lesssim \| f \|_{L^p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \lesssim r^n \| f \|_{L^p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}}}.
\]
(11)
To estimate $\| \mathcal{J}^L_\beta(f_2) \|_{L^p(B(x_0, r))}$, observe that $x \in B$, $y \in (2B)^c$ implies $|x-y| \approx |x_0-y|$. Then by (9) we have
\[
\sup_{x \in B} | \mathcal{J}^L_\beta(f_2)(x) | \lesssim \sup_{x \in B} \int_{(2B)^c} |\mathcal{K}_\beta(x,y)f(y)| \, dy \
\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0-y|^{n-\beta}} \, dy \
\lesssim \sum_{k=1}^{\infty} (2^{k+1}r)^{-n+\beta} \int_{2^{k+1}B} |f(y)| \, dy.
\]
By Hölder’s inequality we get
\[
\sup_{x \in B} | \mathcal{J}^L_\beta(f_2)(x) | \lesssim \sum_{k=1}^{\infty} \| f \|_{L^p(2^{k+1}B)} (2^{k+1}r)^{-1-\frac{n}{p}+\beta} \int_{2^{k}r}^{2^{k+1}r} \, dt \\
\lesssim \sum_{k=1}^{\infty} \int_{2^{k}r}^{2^{k+1}r} \frac{\| f \|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \, dt \\
\lesssim \int_{2r}^{\infty} \frac{\| f \|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \, dt.
\]
(12)
Then
\[ \|J_\beta^L(f_2)\|_{L^q(B(x_0,r))} \lesssim r^{\frac{\alpha}{q}} \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \] (13)
holds for \( 1 \leq p < n/\beta \). Therefore, by (11) and (13) we get
\[ \|J_\beta^L(f)\|_{L^q(B(x_0,r))} \lesssim r^{\frac{\alpha}{q}} \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \] (14)
holds for \( 1 \leq p < n/\beta \).

When \( p = 1 \), by the boundedness of \( J_\beta^L \) from \( L^1(\mathbb{R}^n) \) to \( WL_{n^{-\beta}}(\mathbb{R}^n) \), we get
\[ \|J_\beta^L(f_1)\|_{WL_{n^{-\beta}}(B(x_0,r))} \lesssim \|f\|_{L^1(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^\infty \frac{\|f\|_{L^1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}. \]

By (13) we have
\[ \|J_\beta^L(f_2)\|_{WL_{n^{-\beta}}(B(x_0,r))} \lesssim \|J_\beta^L(f_2)\|_{L^\infty(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^\infty \frac{\|f\|_{L^1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}. \]

Then
\[ \|J_\beta^L(f)\|_{WL_{n^{-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^\infty \frac{\|f\|_{L^1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}. \quad \square \]

**Proof of Theorem 1.** From Lemma 6, we have
\[ \frac{1}{\text{ess inf}_{t<s<\infty} \varphi_1(x,s)^{\frac{n}{\beta}}} = \frac{1}{\text{ess sup}_{t<s<\infty} \varphi_1(x,s)^{\frac{n}{\beta}}}. \]

Note the fact that \( \|f\|_{L^p(B(x_0,t))} \) is a nondecreasing function of \( t \), and \( f \in M_{p,\varphi_1}^\alpha \), then
\[ \left( 1 + \frac{t}{\rho(x_0)} \right)^\alpha \|f\|_{L^p(B(x_0,t))} \lesssim \text{ess sup}_{t<s<\infty} \left( 1 + \frac{t}{\rho(x_0)} \right)^\alpha \|f\|_{L^p(B(x_0,t))} \]
\[ \lesssim \sup_{0<s<\infty} \left( 1 + \frac{s}{\rho(x_0)} \right)^\alpha \|f\|_{L^p(B(x_0,s))} \]
\[ \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,\psi}}. \]

Since \( \alpha \geq 0 \), and \( (\varphi_1, \varphi_2) \) satisfies the condition (2), then
\[ \int_{2r}^\infty \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} = \int_{2r}^\infty \frac{\left( 1 + \frac{t}{\rho(x_0)} \right)^\alpha \|f\|_{L^p(B(x_0,t))}}{\text{ess inf}_{t<s<\infty} \varphi_1(x_0,s)^{\frac{n}{\beta}}} \left( 1 + \frac{t}{\rho(x_0)} \right)^{\alpha \frac{n}{q}} \frac{dt}{t}. \]
\[ \|f\|_{M_p, \varphi_1} \int_{2r}^{\infty} \frac{\text{ess inf}_{t < s < \infty} \varphi_1(t, s)^{\frac{n}{p}}}{(1 + \frac{t}{\rho(x_0)})^{\frac{n}{q}}} \frac{dt}{t^\alpha r^{\frac{n}{q}}} \]

Then by Theorem 5 we get

\[ \|f\|_{M_p, \varphi_1} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{2r}^{\infty} \frac{\text{ess inf}_{t < s < \infty} \varphi_1(t, s)^{\frac{n}{p}}}{t^\alpha r^{\frac{n}{q}}} dt \]

\[ \|f\|_{M_p, \varphi_1} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0, r). \]

(15)

Let \( q = \frac{n}{n - \beta} \), similar to the estimates of (15) we have

\[ \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0, t))} dt}{t^{n - \beta}} \lesssim \|f\|_{M_p, \varphi_1} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0, r). \]

Thus by Theorem 5 we get

\[ \|f\|_{W,M_p, \varphi_1} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0, r). \]

\[ \|f\|_{M_p, \varphi_1}. \]

4. Proof of Theorem 2

As the proof of Theorem 1, it suffices to prove the following result.

THEOREM 6. Let \( V \in RH_{n/2}, b \in BMO_\theta(\rho) \). If \( 1 < p < n/\beta, 1/q = 1/p - \beta/n \) then the inequality

\[ \| [b, \mathcal{J}_\beta^L(f)] \|_{L_q(B(x_0, r))} \lesssim [b] \rho r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))} dt}{t^{\frac{n}{q}}} \]

(16)

holds for any \( f \in L^p_{loc}(\mathbb{R}^n) \).
Proof. We write \( f \) as \( f = f_1 + f_2 \), where \( f_1(y) = f(y) \chi_{B(x_0,2r)}(y) \). Then

\[
\||b, \mathcal{J}_B^L(f)||_{L_q(B(x_0,r))} \leq ||b, \mathcal{J}_B^L(f_1)||_{L_q(B(x_0,r))} + ||b, \mathcal{J}_B^L(f_2)||_{L_q(B(x_0,r))}.
\]

By the boundedness of \([b, \mathcal{J}_B^L]\) on \( L_p(\mathbb{R}^n) \) to \( L_q(\mathbb{R}^n) \) (see [21]) and (11) we get

\[
\begin{align*}
\||b, \mathcal{J}_B^L(f_1)||_{L_q(B(x_0,r))} &\lesssim [b]_\theta ||f||_{L_p(B(x_0,2r))} \\
&\lesssim [b]_\theta r^\frac{n}{\theta} \int_{2r}^\infty \frac{||f||_{L_p(B(x_0,t))}}{t^{\frac{n}{\theta}}} \frac{dt}{t} \\
&\lesssim [b]_\theta r^\frac{n}{\theta} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \frac{||f||_{L_p(B(x_0,t))}}{t^{\frac{n}{\theta}}} \frac{dt}{t}.
\end{align*}
\]

We now turn to deal with the term \( ||b, \mathcal{J}_B^L(f_2)||_{L_q(B(x_0,r))} \). For any given \( x \in B(x_0, r) \) we have

\[
||b, \mathcal{J}_B^L(f_2)(x)|| \leq |b(x) - b_{2B}||f_2(x)|| + |\mathcal{J}_B^L((b - b_{2B})f_2)(x)|.
\]

Then by (12), Lemma 3, and taking \( N \geq (k_0 + 1)\theta \) we get

\[
\begin{align*}
\|(b(x) - b_{2B})\mathcal{J}_B^L(f_2)||_{L_q(B(x_0,r))} &\lesssim [b]_\theta r^\frac{n}{\theta} \left(1 + \frac{2r}{\rho(x_0)}\right)^{\theta - N/(k_0 + 1)} \int_{2r}^\infty \frac{||f||_{L_p(B(x_0,t))}}{t^{\frac{n}{\theta}}} \frac{dt}{t} \\
&\lesssim [b]_\theta r^\frac{n}{\theta} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \frac{||f||_{L_p(B(x_0,t))}}{t^{\frac{n}{\theta}}} \frac{dt}{t}.
\end{align*}
\]

Finally, let us estimate \( ||\mathcal{J}_B^L((b - b_{2B})f_2)||_{L_q(B(x_0,r))} \). By (9), Lemma 2 and (12) we have

\[
\begin{align*}
\sup_{x \in B} |\mathcal{J}_B^L((b - b_{2B})f_2)(x)| &\leq \sup_{x \in B} \int_{(2B)^c} \frac{1}{\left(1 + \frac{|x - y|}{\rho(x)}\right)^N} \frac{|b(y) - b_{2B}| ||f(y)||}{|x_0 - y|^{n - \beta}} \, dy \\
&\leq \sup_{x \in B} \sum_{k=1}^\infty \frac{1}{(2kr)^{n - \beta} \left(1 + \frac{2kr}{\rho(x)}\right)^N} \int_{2^{k+1}B} |b(y) - b_{2B}| ||f(y)|| \, dy \\
&\leq \sum_{k=1}^\infty \frac{1}{(2kr)^{n - \beta} \left(1 + \frac{2kr}{\rho(x_0)}\right)^{N/(k_0 + 1)}} \int_{2^{k+1}B} |b(y) - b_{2B}| ||f(y)|| \, dy.
\end{align*}
\]
Note that
\[
\int_{2^{k+1}B} |b(y) - b_{2B}| |f(y)| dy \lesssim \left( \int_{2^{k+1}B} |b(y) - b_{2B}|^p \right)^{1/p'} \|f\|_{L_p(B(x_0, 2^{k+1}r))}
\lesssim \|b\|_\theta k \left( 1 + \frac{2^kr}{p(x_0)} \right)^{\theta'} (2^kr)^{\frac{n}{p'}} \|f\|_{L_p(B(x_0, 2^{k+1}r))}.
\]

Then
\[
sup_{x \in B} |\mathcal{I}_{\beta}^L ((b - b_{2B})f_2)(x)| \lesssim \|b\|_\theta \sum_{k=1}^{\infty} k \left( \frac{(2^kr)^{\beta}}{1 + \frac{2^kr}{p(x_0)}} \right)^{N/(k_0+1)} \|f\|_{L_p(B(x_0, 2^{k+1}r))}
\lesssim \|b\|_\theta \sum_{k=1}^{\infty} k (2^kr)^{-\frac{n}{p} + \beta} \|f\|_{L_p(B(x_0, 2^{k+1}r))}
\lesssim \|b\|_\theta \sum_{k=1}^{\infty} k \int_{2^{k}r}^{2^{k+1}r} \left( \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \right) dt.
\]

Since $2^k r \leq t \leq 2^{k+1} r$, then $k \approx \ln \frac{t}{r}$. Thus
\[
sup_{x \in B} |\mathcal{I}_{\beta}^L ((b - b_{2B})f_2)(x)| \lesssim \|b\|_\theta \sum_{k=1}^{\infty} k \int_{2^k r}^{2^{k+1}r} \left( \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \right) dt.
\]

Then
\[
\|\mathcal{I}_{\beta}^L ((b - b_{2B})f_2)\|_{L_q(B(x_0,r))} \lesssim \|b\|_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \left( \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \right) dt.
\]

Combining (17), (18) and (19), the proof of Theorem 6 is completed. \(\square\)

**Proof of Theorem 2.** Since $f \in M_{p,q}^{r,\alpha}$ and $(\varphi_1, \varphi_2)$ satisfies the condition (3), by (15) we have
\[
\int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \left( \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \right) dt
= \int_{2r}^{\infty} \left( 1 + \frac{t}{p(x_0)} \right)^{\alpha} \left( \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \right) \left( 1 + \ln \frac{t}{r} \right) \left( \frac{\ess inf \varphi_1(x_0,s)}{s^{\frac{n}{p}}} \right) dt
\lesssim \|f\|_{M_{p,q}^{r,\alpha}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \left( 1 + \frac{t}{p(x_0)} \right)^{\alpha} \left( \frac{\ess inf \varphi_1(x_0,s)}{s^{\frac{n}{p}}} \right) dt
\lesssim \|f\|_{M_{p,q}^{r,\alpha}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) \left( 1 + \frac{t}{p(x_0)} \right)^{\alpha} \left( \frac{\ess inf \varphi_1(x_0,s)}{s^{\frac{n}{p}}} \right) dt.
\]
\[ \lesssim \| f \|_{M^{\alpha,V}_{p,q_1}} \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\alpha} \int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \frac{\text{ess inf} \, \varphi_1(x_0,s) s^{-\frac{\alpha}{p}}}{t^{-\frac{\alpha}{q}}} \, dt \]
\[ \lesssim \| f \|_{M^{\alpha,V}_{p,q_1}} \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\alpha} \varphi_2(x_0,r). \]  

Then from Theorem 6 and by (20) we get
\[ \| [b, \mathcal{F}_\beta^L](f) \|_{M^{\alpha,V}_{q,q_2}} \lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \| [b, \mathcal{F}_\beta^L](f) \|_{L_q(B(x_0,r))} \]
\[ \lesssim [b]_\theta \sup_{x_0 \in \mathbb{R}^n, r > 0} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \frac{\| f \|_{L_p(B(x_0,t))}}{t^{-\frac{\alpha}{q}}} \, dt \]
\[ \lesssim [b]_\theta \| f \|_{M^{\alpha,V}_{p,q_1}}. \square \]

5. Proof of Theorem 3

The statement is derived from the estimate (14). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1. So we only have to prove that
\[ \limsup_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{A}^\alpha_{p,q_1}(f;x,r) = 0 \Rightarrow \limsup_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{A}^\alpha_{w,q_2}(\mathcal{F}_\beta^L(f);x,r) = 0 \quad (21) \]
and
\[ \limsup_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{A}^\alpha_{1,q_1}(f;x,r) = 0 \Rightarrow \limsup_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{A}^{w,\alpha}_{n/(\alpha-\beta),q_2}(\mathcal{F}_\beta^L(f);x,r) = 0. \quad (22) \]

To show that \( \sup_{x \in \mathbb{R}^n} \left( 1 + \frac{r}{\rho(x)} \right)^\alpha \varphi_2(x,r)^{-1} r^{-n/p} \| \mathcal{F}_\beta^L(f) \|_{L_q(B(x,r))} < \varepsilon \) for small \( r \), we split the right-hand side of (14):
\[ \left( 1 + \frac{r}{\rho(x)} \right)^\alpha \varphi_2(x,r)^{-1} r^{-n/p} \| \mathcal{F}_\beta^L(f) \|_{L_q(B(x,r))} \leq C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \quad (23) \]
where \( \delta_0 > 0 \) (we may take \( \delta_0 > 1 \)), and
\[ I_{\delta_0}(x,r) := \left( \frac{1 + \frac{r}{\rho(x)}}{\varphi_2(x,r)} \right)^\alpha \int_r^{\delta_0} t^{-\frac{\alpha}{q} - 1} \| f \|_{L_p(B(x,t))} \, dt \]
and
\[ J_{\delta_0}(x,r) := \left( \frac{1 + \frac{r}{\rho(x)}}{\varphi_2(x,r)} \right)^\alpha \int_{\delta_0}^{\infty} t^{-\frac{\alpha}{q} - 1} \| f \|_{L_p(B(x,t))} \, dt \]
and it is supposed that \( r < \delta_0 \). We use the fact that \( f \in VM_{p,\phi_1}^{\alpha,V}(\mathbb{R}^n) \) and choose any fixed \( \delta_0 > 0 \) such that

\[
\sup_{x \in \mathbb{R}^n} \left( 1 + \frac{t}{\rho(x)} \right)^{\alpha} \phi_1(x,t)^{-1} t^{-n/p} \| f \|_{L_p(B(x,t))} < \frac{\epsilon}{2CC_0},
\]

where \( C \) and \( C_0 \) are constants from (4) and (23). This allows to estimate the first term uniformly in \( r \in (0, \delta_0) \):

\[
\sup_{x \in \mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\epsilon}{2}, \quad 0 < r < \delta_0.
\]

The estimation of the second term now may be made already by the choice of \( r \) sufficiently small. Indeed, thanks to the condition (10) we have

\[
J_{\delta_0}(x,r) \leq c_{\sigma_0} \left( 1 + \frac{r}{\rho(x)} \right)^{\alpha} \| f \|_{VM_{p,\phi_1}^{\alpha,V}},
\]

where \( c_{\sigma_0} \) is the constant from (1). Then, by (10) it suffices to choose \( r \) small enough such that

\[
\sup_{x \in \mathbb{R}^n} \left( 1 + \frac{r}{\rho(x)} \right)^{\alpha} \phi_2(x,r)^{-1} r^{-n/p} \| \| [b, \mathcal{G}_\beta(f)] \|_{L_q(B(x,r))} < \frac{\epsilon}{2C_0}.
\]

which completes the proof of (21).

The proof of (22) is similar to the proof of (21).

6. Proof of Theorem 4

The norm inequality having already been provided by Theorem 2, we only have to prove the implication

\[
\lim sup_{r \to 0, x \in \mathbb{R}^n} \left( 1 + \frac{r}{\rho(x)} \right)^{\alpha} \phi_2(x,r)^{-1} r^{-n/p} \| [b, \mathcal{G}_\beta(f)] \|_{L_q(B(x,r))} = 0.
\]

To check that

\[
\sup_{x \in \mathbb{R}^n} \left( 1 + \frac{r}{\rho(x)} \right)^{\alpha} \phi_2(x,r)^{-1} r^{-n/p} \| [b, \mathcal{G}_\beta(f)] \|_{L_q(B(x,r))} < \epsilon \quad \text{for small } r,
\]
we use the estimate (16):
\[
\varphi_2(x,r)^{-1}r^{-n/p}\| [b, S^L_\beta(f)] \|_{L^q(B(x,r))} \lesssim \frac{[b]_\theta}{\varphi_2(x,r)} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{\beta}{n}}} \frac{dt}{t}.
\]
We take \( r < \delta_0 \), where \( \delta_0 \) will be chosen small enough and split the integration:
\[
\left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1}r^{-n/p}\| [b, S^L_\beta(f)] \|_{L^q(B(x,r))} \leq C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)],
\]
where
\[
I_{\delta_0}(x,r) := \frac{\left(1 + \frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)} \int_r^{\delta_0} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{\beta}{n}}} \frac{dt}{t}
\]
and
\[
J_{\delta_0}(x,r) := \frac{\left(1 + \frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L^p(B(x_0,t))}}{t^{\frac{\beta}{n}}} \frac{dt}{t}.
\]
We choose a fixed \( \delta_0 > 0 \) such that
\[
\sup_{x \in \mathbb{R}^n} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L^p(B(x,r))} < \frac{\epsilon}{2C_0}, \quad r < \delta_0,
\]
where \( C \) and \( C_0 \) are constants from (24) and (5), which yields the estimate of the first term uniform in \( r \in (0, \delta_0) : \sup_{x \in \mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\epsilon}{2}, \quad 0 < r < \delta_0 \).

For the second term, writing \( 1 + \ln \frac{t}{r} \leq 1 + |\ln t| + \frac{1}{r} \), we obtain
\[
J_{\delta_0}(x,r) \leq \frac{c_{\delta_0} + \widetilde{c}_{\delta_0} \ln \frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M^\alpha_{p,q}},
\]
where \( c_{\delta_0} \) is the constant from (7) with \( \delta = \delta_0 \) and \( \widetilde{c}_{\delta_0} \) is a similar constant with omitted logarithmic factor in the integrand. Then, by (6) we can choose small \( r \) such that \( \sup_{x \in \mathbb{R}^n} J_{\delta_0}(x,r) < \frac{\epsilon}{2} \), which completes the proof.

7. Conclusions

In this paper, we study the boundedness of the of the fractional integral operator \( S^L_\beta \) associated with Schrödinger operator and its commutators \([b, S^L_\beta]\) with \( b \in BMO_\theta(\rho) \) on generalized Morrey spaces \( M^\alpha_{p,q} \) associated with Schrödinger operator and vanishing generalized Morrey spaces \( VM^\alpha_{p,q} \) associated with Schrödinger operator. We find the sufficient conditions on the pair \((\varphi_1, \varphi_2)\) which ensures the boundedness of the operator \( S^L_\beta \) from \( M^\alpha_{p,\varphi_1} \) to \( M^\alpha_{q,\varphi_2} \) and from \( VM^\alpha_{p,\varphi_1} \) to \( VM^\alpha_{q,\varphi_2} \), \( 1/p - 1/q = \beta/n \). When \( b \) belongs to \( BMO_\theta(\rho) \) and \((\varphi_1, \varphi_2)\) satisfies some conditions, we also
show that the commutator operator $[b, \mathcal{L}_\beta]$ is bounded from $M^{\alpha,V}_{p,\phi_1}$ to $M^{\alpha,V}_{q,\phi_2}$ and from $VM^{\alpha,V}_{p,\phi_1}$ to $VM^{\alpha,V}_{q,\phi_2}$, $1/p - 1/q = \beta/n$.

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