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NEW EXPLICIT BOUNDS ON GAMIDOV TYPE INTEGRAL INEQUALITIES ON TIME SCALES AND APPLICATIONS

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Abstract. In this paper, we derive some generalizations of certain Gamidov type integral inequalities on time scales, which provide explicit bounds on unknown functions. Also, some examples are presented to show the feasibility of these results.

1. Introduction

Integral inequalities that give explicit bounds on unknown functions provide a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of differential and integral equations. During the past few years, many such new inequalities have been discovered, which are motivated by certain applications. For example, see [2–3, 6–16] and the references therein. In [10], Sh. G. Gamidov, while studying the boundary value problem for higher order differential equations, initiated the study of obtaining explicit upper bounds on the integral inequalities of the forms

$$u(t) \leqslant c + \int_a^t f(s)u(s)ds + \int_a^b n(s)u(s)ds. \tag{1.1}$$

for $t \in [a,b]$, under some suitable conditions on the functions involved in (1.1).

In [16], Pachpatte established more general Gamidov inequalities as follows:

$$u(t) \leqslant a(t) + \int_a^t f(t,s)u(s)ds + \int_a^b n(s)u(s)ds. \tag{1.2}$$

In [8], the authors considered the followig Gronwall-Bellman-Gamidov integral inequality with power nonlinearity

$$u^p(t) \leqslant a(t) + b(t)\int_0^t f(s)u^q(s)ds + c(t)\int_0^T n(s)u^r(s)ds. \tag{1.3}$$

In the present paper we shall consider the problem of obtaining new explicit upper bounds on the general versions of (1.3) on time scales which can be used as tools in the study of qualitative behavior of solutions of certain classes of integral equations on time scales.

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2. Some preliminaries

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{T}$ is an arbitrary time scale. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) =: \sup\{s \in \mathbb{T} : s < t\}$. C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^k$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0, \infty[$ is defined by $\mu(t) := \sigma(t) - t$. $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions.

We define the set of all positively regressive functions by

$$\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Also we define the time scales interval by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}, \tag{2.1}$$

note that $[a,b]_{\mathbb{T}}^k = \begin{cases} [a,b]_{\mathbb{T}} \text{ if } b \text{ is left- dense,} \\ [a,b[_{\mathbb{T}} = [a,\rho(b)]_{\mathbb{T}} = [a,b[\text{ if } b \text{ is left- scattered.} \end{cases}$

The following lemmas are very useful in our main results.

LEMMA 1. (Theorem 6.1 in [4]) *Suppose $u, b \in C_{rd}$, $a \in \mathfrak{R}^+$. Then*

$$u^{\Delta}(t) \leqslant a(t)u(t) + b(t), \quad t \geqslant t_0, \quad t \in \mathbb{T}, \tag{2.2}$$

implies

$$u(t) \leqslant u(t_0)e_a(t,t_0) + \int_{t_0}^{t} b(\tau)e_a(t,\sigma(\tau))\Delta\tau, \quad t \geqslant t_0, \quad t \in \mathbb{T}. \tag{2.3}$$

LEMMA 2. (Theorem 1.117 in [4]) *Let $t_0 \in \mathbb{T}^k$ and assume $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > t_0$. Also assume that $f^{\Delta}(t,.)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left| f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t) - s) \right| \leqslant \varepsilon \left| \sigma(t) - s \right| \text{ for all } s \in U, \tag{2.4}$$

where f^{Δ} denotes the derivative of f with respect to the first variable. Then

$$g(t) := \int_a^t f(t,\tau)\Delta\tau \text{ implies } g^{\Delta}(t) = \int_a^t f^{\Delta}(t,\tau)\Delta\tau + f(\sigma(t),t). \tag{2.5}$$

For more discussion on time scales, we refer the reader to [4, 5].

LEMMA 3. [11] *Assume that $a \geqslant 0$, $p \geqslant q \geqslant 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{2.6}$$

for any $K > 0$.

Now we state the main results of this work.

3. Main result

In this section, some time scale Gamidov type integral inequalities are investigated. For convenience, it is always assumed that p,q,r are real constants such that $0 < q,\ r \leqslant p,\ p \geqslant 1$ and $a,b \in \mathbb{T}$.

LEMMA 4. *Suppose* $u,m,l,n \in C_{rd}\left([a,b]_{\mathbb{T}},\mathbb{R}_+\right)$*, and* $g:\mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0,+\infty[$ *with continuous nonincreasing first derivative* g' *on* $]0,+\infty[$*. If*

$$u(t) \leqslant m(t) + l(t)\int_a^b n(s)g(u(s))\Delta s. \tag{3.1}$$

Then

$$u(t) \leqslant m(t) + \frac{l(t)\int_a^b n(s)g(m(s))\Delta s}{1-\int_a^b g'(m(s))n(s)l(s)\Delta s}, \tag{3.2}$$

for all $t \in [a,b]_{\mathbb{T}}^k$ *provided that*

$$\int_a^b g'(m(s))n(s)l(s)\Delta s < 1. \tag{3.3}$$

Proof. Let

$$\Sigma = \int_a^b n(s)g(u(s))\Delta s, \tag{3.4}$$

it's clear that Σ is a constant, substituting (3.4) in (3.1), we get

$$u(t) \leqslant m(t) + l(t)\Sigma. \tag{3.5}$$

Applying the mean value Theorem for the function g, then for every $x \geqslant y > 0$, there exists $c \in]y,x[$ such that

$$g(x) - g(y) = g'(c)(x-y) \leqslant g'(y)(x-y) \tag{3.6}$$

which gives

$$g(u(t)) \leqslant g(m(t)+l(t)\Sigma) \leqslant g'(m(t))l(t)\Sigma + g(m(t)). \tag{3.7}$$

Multiplying both sides of (3.7) by $n(t)$, then integrating the result from a to b, it yields

$$\int_a^b n(s)g(u(s))\Delta s \leqslant \int_a^b n(s)g(m(s))\Delta s + \Sigma\int_a^b g'(m(s))n(s)l(s)\Delta s. \tag{3.8}$$

(3.8) can be rewritten as

$$\Sigma \leqslant \int_a^b n(s)g(m(s))\Delta s + \Sigma \int_a^b g'(m(s))n(s)l(s)\Delta s. \tag{3.9}$$

The inequality (3.9) implies the estimate

$$\Sigma \left(1 - \int_a^b g'(m(s))n(s)l(s)\Delta s\right) \leqslant \int_a^b n(s)g(m(s))\Delta s. \tag{3.10}$$

From (3.3), we observe that

$$\Sigma \leqslant \frac{\int_a^b n(s)g(m(s))\Delta s}{1 - \int_a^b g'(m(s))n(s)l(s)\Delta s}. \tag{3.11}$$

Therefore, from (3.11) and (3.5), one can deduce inequality (3.2). □

REMARK 1. If we take $\mathbb{T} = \mathbb{R}$ and $g(x) = x$, $a = 0$, $b = T$, Lemma 4 reduces to Lemma 3 in [8].

COROLLARY 1. *Assume that the hypotheses of Lemma 4 hold. Then*

$$u(t) \leqslant m(t) + l(t)\int_a^b n(s)\arctan(u(s))\Delta s$$

implies

$$u(t) \leqslant m(t) + \frac{l(t)\int_a^b n(s)\arctan(m(s))\Delta s}{1 - \int_a^b \frac{n(s)l(s)}{1+m^2(s)}\Delta s},$$

for all $t \in [a,b]_{\mathbb{T}}^{k}$ *provided that*

$$\int_a^b \frac{n(s)l(s)}{1+m^2(s)}\Delta s < 1,$$

and if

$$u(t) \leqslant m(t) + l(t)\int_a^b n(s)\ln(u(s)+1)\Delta s,$$

then

$$u(t) \leqslant m(t) + \frac{l(t) \int_a^b n(s) \ln(m(s)+1) \Delta s}{1 - \int_a^b \frac{n(s)l(s)}{1+m(s)} \Delta s},$$

for all $t \in [a,b]_{\mathbb{T}}^{k}$ *provided that*

$$\int_a^b \frac{n(s)l(s)}{1+m(s)} \Delta s < 1.$$

Now by using Lemma 4, we give a new versions of the inequality (1.3) on time scale.

THEOREM 1. *Let* $u, c, n \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_+)$ *and* $c^{\Delta} \geqslant 0$, *and* f *is defined as in Lemma 2 such that* $f(t,s) \geqslant 0$ *and* $f^{\Delta}(t,s) \geqslant 0$ *for* $t,s \in [a,b]_{\mathbb{T}}$ *with* $s \leqslant t$. *If* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0,+\infty[$ *with continuous nonincreasing first derivative* g' *on* $]0,+\infty[$. *Then*

$$u(t) \leqslant c(t) + \int_a^t f(t,s)u(s)\Delta s + \int_a^b n(s)g(u(s))\Delta s, \tag{3.12}$$

implies

$$u(t) \leqslant \left(m(t) + \frac{l(t) \int_a^b n(s)g(m(s))\Delta s}{1 - \int_a^b g'(m(s))n(s)l(s)\Delta s} \right), \tag{3.13}$$

for all $t \in [a,b]_{\mathbb{T}}^{k}$, *where*

$$m(t) = c(a)e_P(t,a) + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s, \tag{3.14}$$

$$l(t) = e_P(t,a),$$

and

$$P(t) = \left[f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s \right], \tag{3.15}$$

$$Q(t) = c^{\Delta}(t),$$

with

$$\int_a^b g'(m(s))n(s)l(s)\Delta s < 1. \tag{3.16}$$

Proof. Define a function $z(t)$ by

$$z(t) = c(t) + \int_a^t f(t,s)u(s)\Delta s + \int_a^b n(s)g(u(s))\Delta s, \tag{3.17}$$

then

$$u(t) \leqslant z(t), \tag{3.18}$$

$$z(a) = c(a) + \int_a^b n(s)g(u(s))\Delta s,$$

and from (3.17)–(3.18), we have

$$z(a) \leqslant c(a) + \int_a^b n(s)g(z(s))\Delta s, \tag{3.19}$$

$$z^{\Delta}(t) = c^{\Delta}(t) + f(\sigma(t),t)u(t) + \int_a^t f^{\Delta}(t,s)u(s)\Delta s, \tag{3.20}$$

$$z^{\Delta}(t) \leqslant c^{\Delta}(t) + \left(f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s \right) z(t). \tag{3.21}$$

The inequality (3.21) can be reformulated as

$$z^{\Delta}(t) \leqslant P(t)z(t) + Q(t), \tag{3.22}$$

where P and Q are defined as in (3.15).

Applying Lemma 1 to (3.22), we obtain

$$z(t) \leqslant z(a)e_P(t,a) + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s. \tag{3.23}$$

The substituting of (3.19) in (3.23), gives

$$z(t) \leqslant \int_a^t Q(s)e_P(t,\sigma(s))\Delta s + c(a)e_P(t,a) + e_P(t,a)\int_a^b n(s)g(z(s))\Delta s. \tag{3.24}$$

The inequality (3.24) can be rewritten as

$$z(t) \leqslant m(t) + l(t) \int_a^b n(s) g(z(s)) \Delta s, \tag{3.25}$$

where m and l are defined as in (3.14).

Applying Lemma 4 to (3.25), we get the desired inequality. □

REMARK 2. If we take $\mathbb{T} = \mathbb{R}$ and $g(x) = x$, the inequality given in Theorem 1 reduces to the inequality given in [16, Theorem 1, (a_3)].

THEOREM 2. *Let* $u, a, b, n \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_+)$, *and* f *is defined as in Lemma 2 such that* $f(t,s) \geqslant 0$ *and* $f^{\Delta}(t,s) \geqslant 0$ *for* $t, s \in [a,b]_{\mathbb{T}}$ *with* $s \leqslant t$. *If* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0, +\infty[$ *with continuous nonincreasing first derivative* g' *on* $]0, +\infty[$, *then*

$$u^p(t) \leqslant a(t) + b(t) \left(\int_a^t f(t,s) u^q(s) \Delta s + \int_a^b n(s) g(u(s)) \Delta s \right), \tag{3.26}$$

implies

$$u(t) \leqslant m^*(t) + \frac{l^*(t) \int_a^b n(s) g(m^*(s)) \Delta s}{1 - \int_a^b g'(m^*(s)) n(s) l^*(s) \Delta s},$$

for all $t \in [a,b]^k_{\mathbb{T}}$, *where* $m^*(t)$ *and* $l^*(t)$ *are given by*

$$m^*(t) = \frac{1}{p} K^{\frac{1-p}{p}} b(t) \int_a^t Q^*(s) e_{P^*}(t, \sigma(s)) \Delta s + \frac{1}{p} K^{\frac{1-p}{p}} a(t) + \frac{p-1}{p} K^{\frac{1}{p}}, \tag{3.27}$$

$$l^*(t) = \frac{1}{p} K^{\frac{1-p}{p}} b(t) e_{P^*}(t,a),$$

and

$$P^*(t) = \frac{q}{p} K^{\frac{q-p}{p}} b(t) f(\sigma(t), t) + \int_a^t \frac{q}{p} K^{\frac{q-p}{p}} b(s) f^{\Delta}(t,s) \Delta s, \tag{3.28}$$

$$Q^*(t) = f(\sigma(t), t) \Big(\frac{q}{p} K^{\frac{q-p}{p}} a(t) + \frac{p-q}{p} K^{\frac{q}{p}} \Big) + \int_a^t f^{\Delta}(t,s) \Big(\frac{q}{p} K^{\frac{q-p}{p}} a(s) + \frac{p-q}{p} K^{\frac{q}{p}} \Big) \Delta s$$

with

$$\int_a^b g'(m^*(s)) n(s) l^*(s) \Delta s < 1.$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_a^t f(t,s)u^q(s)\Delta s + \int_a^b n(s)g(u(s))\Delta s, \tag{3.29}$$

then

$$u(t) \leqslant (a(t) + b(t)z(t))^{\frac{1}{p}}. \tag{3.30}$$

Using Lemma 3, we obtain

$$u(t) \leqslant \frac{1}{p}K^{\frac{1-p}{p}}(a(t) + b(t)z(t)) + \frac{p-1}{p}K^{\frac{1}{p}} = w(t), \tag{3.31}$$

$$z(a) = \int_a^b n(s)g(u(s))\Delta s \leqslant \int_a^b n(s)g(w(s))\Delta s.$$

Using Lemma 2, we get

$$\begin{aligned} z^{\Delta}(t) &= f(\sigma(t),t)u^q(t) + \int_a^t f^{\Delta}(t,s)u^q(s)\Delta s \\ &\leqslant f(\sigma(t),t)(a(t) + b(t)z(t))^{\frac{q}{p}} + \int_a^t f^{\Delta}(t,s)(a(s) + b(t)z(s))^{\frac{q}{p}}\Delta s. \end{aligned} \tag{3.32}$$

Applying Lemma 3 to (3.32), we obtain

$$\begin{aligned} z^{\Delta}(t) \leqslant{} & f(\sigma(t),t)\Big(\frac{q}{p}K^{\frac{q-p}{p}}(a(t) + b(t)z(t)) + \frac{p-q}{p}K^{\frac{q}{p}}\Big) \\ & + \int_a^t f^{\Delta}(t,s)\Big(\frac{q}{p}K^{\frac{q-p}{p}}(a(s) + b(s)z(s)) + \frac{p-q}{p}K^{\frac{q}{p}}\Big)\Delta s. \end{aligned} \tag{3.33}$$

Then, (3.33) can be rewritten as

$$\begin{aligned} z^{\Delta}(t) \leqslant{} & \Big(\frac{q}{p}K^{\frac{q-p}{p}}b(t)f(\sigma(t),t) + \int_a^t \frac{q}{p}K^{\frac{q-p}{p}}f^{\Delta}(t,s)b(s)\Delta s\Big)z(t) \\ & + f(\sigma(t),t)\Big(\frac{q}{p}K^{\frac{q-p}{p}}a(t) + \frac{p-q}{p}K^{\frac{q}{p}}\Big) \\ & + \int_a^t f^{\Delta}(t,s)\Big(\frac{q}{p}K^{\frac{q-p}{p}}a(s) + \frac{p-q}{p}K^{\frac{q}{p}}\Big)\Delta s. \end{aligned} \tag{3.34}$$

Now (3.34) can be restated as

$$z^{\Delta}(t) \leqslant P^*(t)z(t) + Q^*(t), \tag{3.35}$$

where $P^*(t)$ and $Q^*(t)$ are given by (3.28).

Using Lemma 1, from (3.35), we easily obtain

$$z(t) \leqslant z(a)e_{P^*}(t,a) + \int_a^t Q^*(s)e_{P^*}(t,\sigma(s))\Delta s. \tag{3.36}$$

From (3.31) and (3.36), we have

$$z(t) \leqslant e_{P^*}(t,a)\int_a^b n(s)g(w(s))\Delta s + \int_a^t Q^*(s)e_{P^*}(t,\sigma(s))\Delta s. \tag{3.37}$$

Multiplying both sides of the inequality (3.37) by $\frac{1}{p}K^{\frac{1-p}{p}}b(t)$ and adding $\frac{1}{p}K^{\frac{1-p}{p}}a(t)+\frac{p-1}{p}K^{\frac{1}{p}}$ to both sides of the resultant inequality, we obain

$$\begin{aligned} w(t) \leqslant{}& \frac{1}{p}K^{\frac{1-p}{p}}b(t)e_{P^*}(t,a)\int_a^b n(s)g(w(s))\Delta s \\ &+ \frac{1}{p}K^{\frac{1-p}{p}}b(t)\int_a^t Q^*(s)e_{P^*}(t,\sigma(s))\Delta s + \frac{1}{p}K^{\frac{1-p}{p}}a(t) + \frac{p-1}{p}K^{\frac{1}{p}}, \end{aligned} \tag{3.38}$$

then (3.38) can be reformulated as

$$w(t) \leqslant m^*(t) + l^*(t)\int_a^b n(s)g(w(s))\Delta s, \tag{3.39}$$

where m^* and l^* are defined as in (3.27).

Using Lemma 4, from (3.39) we have

$$u(t) \leqslant w(t) \leqslant m^*(t) + \frac{l^*(t)\int_a^b n(s)g(m^*(s))\Delta s}{1-\int_a^b g'(m^*(s))n(s)l^*(s)\Delta s}. \tag{3.40}$$

The proof of Theorem 2 is complete. □

COROLLARY 2. *Let $u,a,b,c,n \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_+)$ and f is defined as in Lemma 2 such that $f(t,s) \geqslant 0$ and $f^{\Delta}(t,s) \geqslant 0$ for $t,s \in [a,b]_{\mathbb{T}}$, with $s \leqslant t$. If $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $]0,+\infty[$ with continuous nonincreasing first derivative g' on $]0,+\infty[$. Then*

$$u^p(t) \leqslant a(t) + b(t)\int_a^t f(t,s)u^q(s)\Delta s + c(t)\int_a^b n(s)g(u(s))\Delta s, \tag{3.41}$$

implies

$$u(t) \leqslant m^*(t) + \frac{l^*(t) \int_a^b n(s) g(m^*(s)) \Delta s}{1 - \int_a^b g'(m^*(s)) n(s) l^*(s) \Delta s} \tag{3.42}$$

for $t \in [a,b]_{\mathbb{T}}^k$ *where* $m^*(t)$, $l^*(t), P^*(t)$ *and* $Q^*(t)$ *are defined as in (3.27)–(3.28) by replacing* $b(t)$ *by* $(b(t)+c(t))$, *with*

$$\int_a^b g'(m^*(s)) n(s) l^*(s) \Delta s < 1.$$

Proof. The inequality (3.41) can be estimated as

$$u^p(t) \leqslant a(t) + (b(t)+c(t)) \left(\int_a^t f(t,s) u^q(s) \Delta s + \int_a^b n(s) g(u(s)) \Delta s \right). \tag{3.43}$$

Applying Theorem 2 to (3.43), we obtain the desired inequality (3.42). □

THEOREM 3. *Let* $u, c, n \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_+)$, *with* $c^{\Delta} \geqslant 0$ *and* f *is defined as in Lemma 2 such that* $f(t,s) \geqslant 0$ *and* $f^{\Delta}(t,s) \geqslant 0$ *for* $t, s \in [a,b]_{\mathbb{T}}$ *with* $s \leqslant t$. *Assume that* $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *are a differentiable increasing functions on* $]0,+\infty[$ *with continuous nonincreasing first derivatives on* $]0,+\infty[$.

Then

$$u^p(t) \leqslant c(t) + \int_a^t f(t,s) g_1(u(s)) \Delta s + \int_a^b n(s) g_2(u(s)) \Delta s, \tag{3.44}$$

implies

$$u(t) \leqslant m(t) + \frac{l(t) \int_a^b n(s) g_2(m(s)) \Delta s}{1 - \int_a^b g_2'(m(s)) n(s) l(s) \Delta s}, \tag{3.45}$$

for all $t \in [a,b]_{\mathbb{T}}^k$, *provided that*

$$\int_a^b g_2'(m(s)) n(s) l(s) \Delta s < 1,$$

where

$$m(t) = \frac{1}{p} K^{\frac{1-p}{p}} (c(a)e_P(t,a) + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s) + \frac{p-1}{p} K^{\frac{1}{p}}, \tag{3.46}$$
$$l(t) = \frac{1}{p} K^{\frac{1-p}{p}} e_P(t,a),$$

and

$$P(t) = \frac{1}{p} K^{\frac{1-p}{p}} g_1'\Big(\frac{p-1}{p} K^{\frac{1}{p}}\Big)\Big(f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s\Big), \tag{3.47}$$
$$Q(t) = c^{\Delta}(t) + g_1\Big(\frac{p-1}{p} K^{\frac{1}{p}}\Big)\Big(f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s\Big).$$

Proof. Define a function $z(t)$ by

$$z(t) = c(t) + \int_a^t f(t,s)g_1(u(s))\Delta s + \int_a^b n(s)g_2(u(s))\Delta s, \tag{3.48}$$

then

$$u(t) \leqslant z^{\frac{1}{p}}(t) \leqslant \frac{1}{p} K^{\frac{1-p}{p}} z(t) + \frac{p-1}{p} K^{\frac{1}{p}} = w(t), \tag{3.49}$$
$$z^{\Delta}(t) = c^{\Delta}(t) + f(\sigma(t),t)g_1(u(t)) + \int_a^t f^{\Delta}(t,s)g_1(u(s))\Delta s,$$
$$z^{\Delta}(t) \leqslant c^{\Delta}(t) + \Big(f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s\Big) g_1(w(t)),$$
$$z(a) = c(a) + \int_a^b n(s)g_2(u(s))\Delta s \leqslant c(a) + \int_a^b n(s)g_2(w(s))\Delta s,$$

from the properties of g_1 and using (3.49), it follows that

$$z^{\Delta}(t) \leqslant c^{\Delta}(t) + (f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s)\Big(g_1'\Big(\frac{p-1}{p} K^{\frac{1}{p}}\Big)\frac{1}{p} K^{\frac{1-p}{p}} z(t) + g_1\Big(\frac{p-1}{p} K^{\frac{1}{p}}\Big)\Big), \tag{3.50}$$

the inequality (3.50) can be reformulated as

$$z^{\Delta}(t) \leqslant P(t)z(t) + Q(t), \tag{3.51}$$

where P and Q are defined as in (3.47).

Using Lemma 1 from (3.51), we have

$$z(t) \leqslant z(a)e_P(t,a) + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s. \tag{3.52}$$

Substituting the last inequality of (3.49) in (3.52), we get

$$z(t) \leqslant c(a)e_P(t,a) + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s + e_P(t,a)\int_a^b n(s)g_2(w(s))\Delta s, \tag{3.53}$$

multiplying both sides of (3.53) by $\frac{1}{p}K^{\frac{1-p}{p}}$ and adding $\frac{p-1}{p}K^{\frac{1}{p}}$ to both sides of the resultant inequality, we obtain

$$\begin{aligned} w(t) \leqslant{}& \frac{1}{p}K^{\frac{1-p}{p}}\left(c(a)e_P(t,a) + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s\right) + \frac{p-1}{p}K^{\frac{1}{p}} \\ &+ \frac{1}{p}K^{\frac{1-p}{p}}e_P(t,a)\int_a^b n(s)g_2(w(s))\Delta s, \end{aligned} \tag{3.54}$$

the inequality (3.54) can be expressed as

$$w(t) \leqslant m(t) + l(t)\int_a^b n(s)g_2(w(s))\Delta s, \tag{3.55}$$

where $m(t)$ and $l(t)$ are defined as in (3.46). Using lemma 4, from (3.55), we have

$$u(t) \leqslant w(t) \leqslant m(t) + \frac{l(t)\int_a^b n(s)g_2(m(s))\Delta s}{1 - \int_a^b g_2'(m(s))n(s)l(s)\Delta s}.$$

This completes the proof of Theorem 3. □

THEOREM 4. *Let* $u,h,n \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}^+)$, $c \geqslant 0$, *and* f *is defined as in Lemma 2 such that* $f(t,s) \geqslant 0$ *and* $f^{\Delta}(t,s) \geqslant 0$ *for* $t,s \in [a,b]_{\mathbb{T}}$ *with* $s \leqslant t$. *If* g *is differentiable increasing functions on* $]0,+\infty[$ *with continuous nonincreasing first derivative* g' *on* $]0,+\infty[$. *Then*

$$u^p(t) \leqslant c + \int_a^t h(s)\left[u^q(s) + \int_a^s f(s,\tau)u^q(\tau)\Delta\tau + \int_a^b n(\tau)g(u(\tau))\Delta\tau\right]\Delta s, \tag{3.56}$$

implies

$$u(t) \leqslant \left[c + qK^{\frac{q-1}{p}} \int_a^t h(\tau)(m(\tau) + \frac{l(\tau)\int_a^b n(t)g(m(t))\Delta t}{1 - \int_a^b g'(m(t))n(t)l(t)\Delta t})\Delta\tau \right]^{\frac{1}{p}}. \tag{3.57}$$

for all $t \in [a,b]_{\mathbb{T}}^k$, *provided that*

$$\int_a^b g'(m(t))n(t)l(t)\Delta t < 1, \tag{3.58}$$

where

$$m(t) = \Big(\frac{1}{p}K^{\frac{1-p}{p}}c + \frac{p-1}{qp}K^{\frac{1}{p}}\Big)e_P(t,a) + \frac{p-1}{p}K^{\frac{1}{p}} \tag{3.59}$$

$$+ \frac{1}{q}K^{\frac{1-q}{p}} \int_a^t Q(s)e_P(t,\sigma(s))\Delta s,$$

$$l(t) = \frac{1}{q}K^{\frac{1-q}{p}} e_P(t,a),$$

$$P(t) = \frac{q}{p}K^{\frac{q-p}{p}} h(t) + f(\sigma(t),t) + \int_a^t f^{\Delta}(t,\tau)\Delta\tau, \tag{3.60}$$

$$Q(t) = \frac{p-q}{p}K^{\frac{q}{p}}(f(\sigma(t),t) + \int_a^t f^{\Delta}(t,\tau)\Delta\tau).$$

Proof. Define function $z(t)$ as follows

$$z(t) = c + \int_a^t h(s)\left[u^q(s) + \int_a^s f(s,\tau)u^q(\tau)\Delta\tau + \int_a^b n(\tau)g(u(\tau))\Delta\tau\right]\Delta s, \tag{3.61}$$

then

$$z(a) = c, \tag{3.62}$$

$$u(t) \leqslant z^{\frac{1}{p}}(t) \leqslant \frac{1}{p}K^{\frac{1-p}{p}}z(t) + \frac{p-1}{p}K^{\frac{1}{p}}, \tag{3.63}$$

and

$$z^{\Delta}(t) = h(t)\left[u^q(t) + \int_a^t f(t,\tau)u^q(\tau)\Delta\tau + \int_a^b n(\tau)g(u(\tau))\Delta\tau\right]. \tag{3.64}$$

Using (3.63) and taking into account that $z(t)$ is nondecreasing, (3.64) becomes

$$z^{\Delta}(t) \leqslant h(t)\left[\left(z^{\frac{q}{p}}(t)+\int_{a}^{t} f(t,\tau) z^{\frac{q}{p}}(\tau)\Delta\tau\right)+\left(\int_{a}^{b} n(\tau) g(z^{\frac{1}{p}}(\tau))\Delta\tau\right)\right]. \tag{3.65}$$

Using Lemma 3 from (3.65), we obtain

$$z^{\Delta}(t) \leqslant h(t)\left[\left(\frac{q}{p}K^{\frac{q-p}{p}} z(t)+\frac{p-q}{p}K^{\frac{q}{p}}\right)+\int_{a}^{t} f(t,\tau)\left(\frac{q}{p}K^{\frac{q-p}{p}} z(\tau)+\frac{p-q}{p}K^{\frac{q}{p}}\right)\Delta\tau\right.$$
$$\left.+\int_{a}^{b} n(\tau) g\left(\frac{1}{p}K^{\frac{1-p}{p}} z(\tau)+\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta\tau\right]. \tag{3.66}$$

Define a function $v(t)$ by

$$v(t)=\left[\left(\frac{q}{p}K^{\frac{q-p}{p}} z(t)+\frac{p-q}{p}K^{\frac{q}{p}}\right)+\int_{a}^{t} f(t,\tau)\left(\frac{q}{p}K^{\frac{q-p}{p}} z(\tau)+\frac{p-q}{p}K^{\frac{q}{p}}\right)\Delta\tau\right.$$
$$\left.+\int_{a}^{b} n(\tau) g\left(\frac{1}{p}K^{\frac{1-p}{p}} z(\tau)+\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta\tau\right]. \tag{3.67}$$

Then

$$v(a)=\frac{q}{p}K^{\frac{q-p}{p}} c+\frac{p-q}{p}K^{\frac{q}{p}}+\int_{a}^{b} n(\tau) g\left(\frac{1}{p}K^{\frac{1-p}{p}} z(\tau)+\frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta\tau. \tag{3.68}$$

Remarking that

$$\frac{q}{p}K^{\frac{q-p}{p}} z(t) \leqslant v(t), \quad z^{\Delta}(t) \leqslant h(t)v(t), \tag{3.69}$$

and $v(t)$ is nondecreasing for $t \in [a,b]_{\mathbb{T}}^{k}$. Using Lemma 2, we obtain

$$v^{\Delta}(t)=\frac{q}{p}K^{\frac{q-p}{p}} z^{\Delta}(t)+f(\sigma(t),t)\left(\frac{q}{p}K^{\frac{q-p}{p}} z(t)+\frac{p-q}{p}K^{\frac{q}{p}}\right) \tag{3.70}$$
$$+\int_{a}^{t} f^{\Delta}(t,\tau)\left(\frac{q}{p}K^{\frac{q-p}{p}} z(\tau)+\frac{p-q}{p}K^{\frac{q}{p}}\right)\Delta\tau.$$

It follows from (3.69) and (3.70) that

$$v^{\Delta}(t) \leqslant \left(\frac{q}{p}K^{\frac{q-p}{p}} h(t)+f(\sigma(t),t)+\int_{a}^{t} f^{\Delta}(t,\tau)\Delta\tau\right) v(t) \tag{3.71}$$
$$+\frac{p-q}{p}K^{\frac{q}{p}}(f(\sigma(t),t)+\int_{a}^{t} f^{\Delta}(t,\tau)\Delta\tau),$$

then the inequality (3.71), can be reformulated as

$$v^{\Delta}(t) \leqslant P(t)v(t) + Q(t). \tag{3.72}$$

where $P(t)$ and $Q(t)$ are defined as in (3.60).

Using lemma 1 from (3.72), we have

$$v(t) \leqslant v(a)e_P(t,a) + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s, \tag{3.73}$$

from (3.73) and using (3.68), it is easy to observe that

$$\begin{aligned} v(t) \leqslant{} & \left(\frac{q}{p}K^{\frac{q-p}{p}}c + \frac{p-q}{p}K^{\frac{q}{p}}\right)e_P(t,a) \\ & + e_P(t,a)\int_a^b n(\tau)g\left(\frac{1}{p}K^{\frac{1-p}{p}}z(\tau) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta\tau \\ & + \int_a^t Q(s)e_P(t,\sigma(s))\Delta s. \end{aligned} \tag{3.74}$$

Multiplying both sides of (3.74) by $\frac{1}{q}K^{\frac{1-q}{p}}$ and adding $\frac{p-1}{p}K^{\frac{1}{p}}$ to both sides of the resultant inequality and using (3.69), we obtain

$$\frac{1}{q}K^{\frac{1-q}{p}}v(t) + \frac{p-1}{p}K^{\frac{1}{p}} \leqslant m(t) + l(t)\int_a^b n(\tau)g\left(\frac{1}{q}K^{\frac{1-q}{p}}v(\tau) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\Delta\tau, \tag{3.75}$$

where $m(t)$ and $l(t)$ are defined as in (3.59).

The inequality (3.75) can be restated as

$$w(t) \leqslant m(t) + l(t)\int_a^b n(\tau)g(w(\tau))\Delta\tau, \tag{3.76}$$

where

$$w(t) = \frac{1}{q}K^{\frac{1-q}{p}}v(t) + \frac{p-1}{p}K^{\frac{1}{p}}. \tag{3.77}$$

Applying Lemma 4 to (3.77), we get

$$w(t) \leqslant m(t) + \frac{l(t)\int_a^b n(s)g(m(s))\Delta s}{1 - \int_a^b g'(m(s))n(s)l(s)\Delta s}. \tag{3.78}$$

It follows from (3.77) that

$$\frac{1}{q}K^{\frac{1-q}{p}}v(t)\leqslant w(t), \tag{3.79}$$

using (3.78) and (3.79), we obtain

$$v(t)\leqslant qK^{\frac{q-1}{p}}\left(m(t)+\frac{l(t)\int_a^b n(s)g(m(s))\Delta s}{1-\int_a^b g'(m(s))n(s)l(s)\Delta s}\right). \tag{3.80}$$

It follows from (3.69) that

$$z^{\Delta}(t)\leqslant qK^{\frac{q-1}{p}}h(t)\left(m(t)+\frac{l(t)\int_a^b n(s)g(m(s))\Delta s}{1-\int_a^b g'(m(s))n(s)l(s)\Delta s}\right). \tag{3.81}$$

Integrating the inequality (3.81) from a to t, we obtain

$$z(t)\leqslant c+qK^{\frac{q-1}{p}}\int_a^t h(\tau)\left(m(\tau)+\frac{l(\tau)\int_a^b n(s)g(m(s))\Delta s}{1-\int_a^b g'(m(s))n(s)l(s)\Delta s}\right)\Delta\tau. \tag{3.82}$$

Therefore, from (3.82) and (3.63), one can deduce the inequality (3.57). □

REMARK 3. If we take $\mathbb{T}=\mathbb{R}$, $g(x)=x$, $f(t,s)=f(t)$, $p=q=1$ the inequality given in Theorem 4 reduces to the inequality given in Theorem 1.5.3 (c_2) in [15, page 47].

4. Applications

In this section we present some examples for our main results to investigate certain properties of solutions of dynamic equation on time scales.

EXAMPLE 1. Consider the following general mixed nonlinear integral equation

$$y^{p}(t)=x(t)+\int_a^t F(s,y^{q}(s))\Delta s+\int_a^b G(s,y(s))\Delta s, \tag{4.1}$$

for $t\in[a,b]_{\mathbb{T}}$, where $p\geqslant q\geqslant 1$, $p\geqslant r\geqslant 1$, $y(t)$ is unknown function, $x\in C_{rd}\left([a,b]_{\mathbb{T}},\mathbb{R}\right)$, $F,G\in C_{rd}\left([a,b]_{\mathbb{T}}\times\mathbb{R},\mathbb{R}\right)$.

Suppose that the functions x,y,F,G in equation (4.1) satisfy the following conditions:

$$\begin{aligned} |x(t)| &\leqslant a(t), \\ |F(s,y^q(s))| &\leqslant f(s)\,|y|^q\,, \\ |G(s,y(s))| &\leqslant n(s)g(|y|). \end{aligned} \tag{4.2}$$

where $a,n,f \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_+)$ and g defined as in Theorem 2.

PROPOSITION 1. *Assume that* $y(t)$ *is the unique solution of equation* (4.1) *and* $\int_a^b g'(m^*(s))n(s)l^*(s)\Delta s < 1$*, then*

$$|y(t)| \leqslant \left\{ m^*(t) + \frac{l^*(t)\int_a^b n(s)g(m^*(s))\Delta s}{1-\int_a^b g'(m^*(s)n(s)l^*(s)\Delta s} \right\}, \tag{4.3}$$

holds for all $t \in [a,b]_{\mathbb{T}}^k$*, where* $m^*(t)$ *and* $l^*(t)$ *are defined as in Theorem 2.*

Proof. From (4.1)–(4.2), we obtain

$$|y(t)|^p \leqslant a(t) + \int_a^t f(s)\,|y(s)|^q\,\Delta s + \int_a^b n(s)g(|y|)\Delta s. \tag{4.4}$$

Applying Theorem 2 to (4.4), we get (4.3). □

REMARK 4. If we take $g(x) = \arctan(x)$ in (4.4), then the solution of (4.1)–(4.2) can be estimated as

$$|y(t)| \leqslant \left\{ m^*(t) + \frac{l^*(t)\int_a^b n(s)\arctan(m^*(s))\Delta s}{1-\int_a^b \dfrac{n(s)l^*(s)}{1+m^{*2}(s)}\Delta s} \right\}.$$

where m^* and l^* are as same defined in Theorem 2.

EXAMPLE 2. Consider the following initial value problem

$$y^\Delta(t) = h(t)\left[y^q(t) + \int_a^t f(t,s)y^q(s)\Delta s + \int_a^b n(s)\ln(|y(s)|+1)\Delta s\right], \quad y(a)=c, \tag{4.5}$$

where $h(t)$, $f(t,s)$ and $n(t)$ are as same defined in Theorem 4, and c is a constant.

PROPOSITION 2. *Assume that* $y(t)$ *is the unique solution of* (4.5)*, then*

$$y(t) \leqslant \left[c + qK^{\frac{q-1}{p}} \int_a^t h(\tau) \left(m(\tau) + \frac{l(\tau)\displaystyle\int_a^b n(s)\ln(m(s)+1)\Delta s}{1-\displaystyle\int_a^b \frac{n(s)l(s)}{1+m(s)}\Delta s} \right) \Delta\tau \right]^{\frac{1}{p}}, \tag{4.6}$$

holds for all $t \in [a,b]_{\mathbb{T}}^{k}$, *provided that*

$$\int_a^b \frac{n(s)l(s)}{1+m(s)}\Delta s < 1. \tag{4.7}$$

where m *and* l *as same defined in Theorem 4.*

Proof. If $y(t)$ is the unique solution of (4.5), then $y(t)$ can be expressed as

$$y(t) = c + \int_a^t h(s)\left[y^q(s) + \int_a^s f(s,\tau)y^q(\tau)\Delta\tau + \int_a^b n(\tau)\ln(|y(\tau)|+1)\Delta\tau\right]\Delta s. \tag{4.8}$$

Then

$$|y(t)| \leqslant |c| + \int_a^t h(s)\left[|y^q(s)| + \int_a^s f(s,\tau)\,|y^q(\tau)|\,\Delta\tau + \int_a^b n(\tau)\ln(|y(\tau)|+1)\Delta\tau\right]\Delta s. \tag{4.9}$$

Applying Theorem 4 to (4.9), we obtain the desired inequality (4.6). □

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