ON GENERALIZATION OF D'AURIZIO–SÁNDOR
INEQUALITIES INVOLVING A PARAMETER

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Abstract. In this work, we generalize the D'Aurizio-Sándor inequalities ([2, 4]) using an elementary approach. In particular, our approach provides an alternative proof of the D'Aurizio-Sándor inequalities. Moreover, as an immediate consequence of the generalized D'Aurizio-Sándor inequalities, we establish the D'Aurizio-Sándor-type inequalities for hyperbolic functions.

1. Introduction

Based on infinite product expansions and inequalities on series and the Riemann's zeta function, D'Aurizio ([2]) proved the following inequality:

$$\frac{1 - \dfrac{\cos x}{\cos \frac{x}{2}}}{x^2} < \frac{4}{\pi^2}, \tag{1}$$

where $x \in (0, \pi/2)$. Using an elementary approach, Sándor ([4]) offered an alternative proof of (1) by employing trigonometric inequalities and an auxiliary function. In the same paper, Sándor also provided the converse to (1):

$$\frac{1 - \dfrac{\cos x}{\cos \frac{x}{2}}}{x^2} > \frac{3}{8}, \tag{2}$$

where $x \in (0, \pi/2)$. In addition, Sándor found the following analogous inequality (4) holds true for the case of sine functions:

THEOREM 1. (D'Aurizio-Sándor inequalities ([2, 4])) *The two double inequalities*

$$\frac{3}{8} < \frac{1 - \dfrac{\cos x}{\cos \frac{x}{2}}}{x^2} < \frac{4}{\pi^2} \tag{3}$$

and

$$\frac{4}{\pi^2}(2 - \sqrt{2}) < \frac{2 - \dfrac{\sin x}{\sin \frac{x}{2}}}{x^2} < \frac{1}{4} \tag{4}$$

hold for any $x \in (0, \pi/2)$.

Throughout this paper, we denote $\dfrac{1-\frac{\cos x}{\cos\frac{x}{p}}}{x^2}$ and $\dfrac{p-\frac{\sin x}{\sin\frac{x}{p}}}{x^2}$ respectively by

$$f_p^c(x) = \frac{1 - \dfrac{\cos x}{\cos\frac{x}{p}}}{x^2}, \tag{5}$$

$$f_p^s(x) = \frac{p - \dfrac{\sin x}{\sin\frac{x}{p}}}{x^2}. \tag{6}$$

Our aim is to generalize the D'Aurizio-Sándor inequalities for the case of $f_p^c(x)$ and $f_p^s(x)$ as follows:

THEOREM 2. (Generalized D'Aurizio-Sándor inequalities) *Let $0 < x < \pi/2$. Then the two double inequalities*

$$\frac{4}{\pi^2} < \frac{1 - \dfrac{\cos x}{\cos\frac{x}{p}}}{x^2} < \frac{p^2 - 1}{2p^2} \tag{7}$$

and

$$\frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right) < \frac{p - \dfrac{\sin x}{\sin\frac{x}{p}}}{x^2} < \frac{p^2 - 1}{6p} \tag{8}$$

hold for $p = 3, 4, 5, \cdots$. In particular, (8) remains true when $p = 2$ while (7) is reversed when $p = 2$.

We remark that the inequality (7) for the cosine function has been established in [5]. In this paper, we prove (7) by using a different approach.

The remainder of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 2 and an alternative proof of Theorem 1. In Section 3, we establish an analogue of Theorem 2 for hyperbolic functions. As an application of Theorem 2, we apply in Section 4 the inequality (8) to the Chebyshev polynomials of the second kind and establish a trigonometric inequality.

2. Proof of the main results

At first we will prove the following lemma. The lemma provides some formulas for the higher-order derivative $\frac{d^2}{dx^2}(x^3 \frac{d}{dx} f_p^\triangle(x))$ involving $f_p^\triangle(x)$ $(\triangle = c, s)$, which are the key ingredients of the proof of Theorem 2. It will turn out that the sign of $\frac{d^2}{dx^2}(x^3 \frac{d}{dx} f_p^\triangle(x))$ plays a crucial role in proving Theorem 2.

LEMMA 1. *Let $0 < x < \pi/2$ and $k = 1, 2, 3, \cdots$. Then when $p \in \mathbb{R}$ and $p \neq 0$, we have*

(i)

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^c(x)\right) = -\frac{x\csc^4(\frac{x}{p})}{8\,p^3}\left((p+1)^3\sin\left(x-\frac{3x}{p}\right)+(p-1)^3\sin\left(x+\frac{3x}{p}\right)\right.$$
$$+(3p^3+3p^2-15p-23)\sin\left(x-\frac{x}{p}\right)$$
$$\left.+(3p^3-3p^2-15p+23)\sin\left(x+\frac{x}{p}\right)\right); \tag{9}$$

(ii)

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right) = \frac{x\csc^4(\frac{x}{p})}{8\,p^3}\left((p+1)^3\sin\left(x-\frac{3x}{p}\right)-(p-1)^3\sin\left(x+\frac{3x}{p}\right)\right.$$
$$+(-3p^3-3p^2+15p+23)\sin\left(x-\frac{x}{p}\right)$$
$$\left.+(3p^3-3p^2-15p+23)\sin\left(x+\frac{x}{p}\right)\right). \tag{10}$$

In particular,

(iii) *when* $p=2k$,

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right) = -\frac{x}{4k^3}\sum_{j=0}^{k-1}(2j+1)^3\sin\left(\frac{2j+1}{2k}x\right); \tag{11}$$

(iv) *when* $p=2k+1$,

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^c(x)\right) = -\frac{16x}{(2k+1)^3}\sum_{j=1}^{k}j^3\sin\left(\frac{2j}{2k+1}x\right)(-1)^{j-1}, \tag{12}$$

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right) = -\frac{16x}{(2k+1)^3}\sum_{j=1}^{k}j^3\sin\left(\frac{2j}{2k+1}x\right). \tag{13}$$

(v) *For* $p\in\mathbb{R}$ *and* $p\neq 0$,

$$\lim_{x\to0}\frac{d}{dx}\left(x^3\frac{d}{dx}f_p^\triangle(x)\right) = \lim_{x\to0}x^3\frac{d}{dx}f_p^\triangle(x) = 0, \tag{14}$$

where $\triangle = c,s$.

Proof. (i), (ii) and (v) follow directly from calculations using elementary Calculus. In particular, trigonometric addition formulas are used in proving (i) and (ii). To prove (11) in (iii), we claim

$$-\frac{x}{4k^3}\sum_{j=0}^{k-1}(2j+1)^3\sin\left(\frac{2j+1}{2k}x\right) = -x\frac{d^3}{dx^3}\left(\frac{\sin x}{\sin\left(\frac{x}{2k}\right)}\right). \tag{15}$$

We rewrite

$$\frac{1}{4k^3} \sum_{j=0}^{k-1} (2j+1)^3 \sin\left(\frac{2j+1}{2k}x\right) = 2\frac{d^3}{dx^3}\left(\sum_{j=0}^{k-1} \cos\left(\frac{2j+1}{2k}x\right)\right). \tag{16}$$

On the other hand, making use of Euler's formula $e^{iz} = \cos z + i \sin z$ leads to an alternative expression of the right-hand side of (16):

$$\sum_{j=0}^{k-1} \cos\left(\frac{2j+1}{2k}x\right) = \sum_{j=0}^{k-1} \Re\left\{e^{i\left(\frac{x}{2k}+\frac{x}{k}j\right)}\right\} = \Re\left\{e^{i\frac{x}{2k}} \sum_{j=0}^{k-1}\left(e^{i\frac{x}{k}}\right)^j\right\} \tag{17}$$

$$= \Re\left\{e^{i\frac{x}{2k}}\frac{1-e^{ix}}{1-e^{i\frac{x}{k}}}\right\} = \Re\left\{e^{i\frac{x}{2k}}\frac{e^{\frac{ix}{2}}(e^{-\frac{ix}{2}}-e^{\frac{ix}{2}})}{e^{\frac{ix}{2k}}(e^{-\frac{ix}{2k}}-e^{\frac{ix}{2k}})}\right\} \tag{18}$$

$$= \Re\left\{e^{i\frac{x}{2}}\frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2k}\right)}\right\} = \cos\left(\frac{x}{2}\right)\frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2k}\right)} = \frac{\sin x}{2\sin\left(\frac{x}{2k}\right)}, \tag{19}$$

where $\Re\{z\}$ is the real part of z and $i = \sqrt{-1}$. We complete the proof of the claim. Now it suffices to show

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_p^s(x)\right) = -x\frac{d^3}{dx^3}\left(\frac{\sin x}{\sin\left(\frac{x}{2k}\right)}\right). \tag{20}$$

Using (10) in (ii), this can be achieved by straightforward calculations of the right-hand side of the above equation. Thus (iii) is proved. The proof of (iv) is similar, and we omit the details. We complete the proof of Lemma 1. ☐

We provide here an alternative proof of the two double inequalities in Theorem 1.

Proof of Theorem 1. To this end, we show that for $x \in (0, \pi/2)$, $f_2^c(x) = \frac{1-\frac{\cos x}{\cos\frac{x}{2}}}{x^2}$ is strictly increasing while $f_2^s(x) = \frac{2-\frac{\sin x}{\sin\frac{x}{2}}}{x^2}$ is strictly decreasing. These lead to the desired inequalities since it is easy to see that

$$\lim_{x\to 0} f_2^c(x) = \frac{3}{8}, \quad \lim_{x\to\pi/2} f_2^c(x) = \frac{4}{\pi^2}, \tag{21}$$

$$\lim_{x\to 0} f_2^s(x) = \frac{1}{4}, \quad \lim_{x\to\pi/2} f_2^s(x) = \frac{4}{\pi^2}(2-\sqrt{2}). \tag{22}$$

To see if $f_2^c(x)$ is strictly increasing, we employ (9) in Lemma 1 to obtain

$$\frac{d^2}{dx^2}\left(x^3\frac{d}{dx}f_2^c(x)\right) = -\frac{x}{64}\sec^4\left(\frac{x}{2}\right)\left(-44\sin\left(\frac{x}{2}\right)+5\sin\left(\frac{3x}{2}\right)+\sin\left(\frac{5x}{2}\right)\right)$$

$$= -\frac{x}{16}\sec^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)(\cos x-2)(\cos x+5) > 0. \tag{23}$$

As $\lim_{x\to 0} \frac{d}{dx}\left(x^3 \frac{d}{dx} f_2^c(x)\right) = 0$, it follows that $\frac{d}{dx}\left(x^3 \frac{d}{dx} f_2^c(x)\right) > 0$. We are led to $x^3 \frac{d}{dx} f_2^c(x) > 0$ or $\frac{d}{dx} f_2^c(x) > 0$ since $\lim_{x\to 0}\left(x^3 \frac{d}{dx} f_2^c(x)\right) = 0$. This shows that $f_2^c(x)$ is strictly increasing.

By means of (11) in Lemma 1, we have

$$\frac{d^2}{dx^2}\left(x^3 \frac{d}{dx} f_2^s(x)\right) = -\frac{x}{4}\sin\left(\frac{x}{2}\right) < 0, \tag{24}$$

from which we infer that $\frac{d}{dx}\left(x^3 \frac{d}{dx} f_2^s(x)\right) < 0$ because of $\lim_{x\to 0}\frac{d}{dx}\left(x^3 \frac{d}{dx} f_2^s(x)\right) = 0$. Then

$$\frac{d}{dx}\left(x^3 \frac{d}{dx} f_2^s(x)\right) < 0, \tag{25}$$

together with the fact that $\lim_{x\to 0}\left(x^3 \frac{d}{dx} f_2^s(x)\right) = 0$ yields $x^3 \frac{d}{dx} f_2^s(x) < 0$ or $\frac{d}{dx} f_2^s(x) < 0$. Thus we have shown that $f_2^s(x)$ is strictly decreasing. We note that when $p = 2$, (v) of Lemma 1

$$\lim_{x\to 0}\frac{d}{dx}\left(x^3 \frac{d}{dx} f_2^\triangle(x)\right) = \lim_{x\to 0} x^3 \frac{d}{dx} f_2^\triangle(x) = 0, \tag{26}$$

where $\triangle = c, s$ has been employed in the above arguments. This completes the proof of the theorem. $\square$

We are now in the position to give the proof of Theorem 2.

Proof of Theorem 2. The proof of the case when $p = 2$ has been given in Theorem 2. For $p \geqslant 3$, we prove the desired inequalities by showing that $\frac{d}{dx} f_p^\triangle(x) < 0$ for $\triangle = c, s$. Due to (i) of Lemma 1, we see that $\frac{d^2}{dx^2}\left(x^3 \frac{d}{dx} f_p^c(x)\right) < 0$. On the other hand, we use (11) and (13) in Lemma 1 to conclude that $\frac{d^2}{dx^2}\left(x^3 \frac{d}{dx} f_p^s(x)\right) < 0$. Thus we have shown that for $\triangle = c, s$,

$$\frac{d^2}{dx^2}\left(x^3 \frac{d}{dx} f_p^\triangle(x)\right) < 0. \tag{27}$$

Because of the first vanishing limit in (v) of Lemma 1, it follows that

$$\frac{d}{dx}\left(x^3 \frac{d}{dx} f_p^\triangle(x)\right) < 0, \tag{28}$$

which, together with the fact that the second limit in (v) of Lemma 1 vanishes, implies that $x^3 \frac{d}{dx} f_p^\triangle(x) < 0$ or $\frac{d}{dx} f_p^\triangle(x) < 0$ for $\triangle = c, s$. It remains to find the following limits:

$$\lim_{x\to 0} f_p^c(x) = \frac{p^2 - 1}{2 p^2}, \quad \lim_{x\to \pi/2} f_p^c(x) = \frac{4}{\pi^2}, \tag{29}$$

$$\lim_{x\to 0} f_p^s(x) = \frac{p^2 - 1}{6 p}, \quad \lim_{x\to \pi/2} f_p^s(x) = \frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right). \tag{30}$$

We immediately have

$$\frac{4}{\pi^2} = \lim_{x\to\pi/2} f_p^c(x) < \frac{1 - \dfrac{\cos x}{\cos \frac{x}{p}}}{x^2} < \lim_{x\to 0} f_p^c(x) = \frac{p^2 - 1}{2p^2} \tag{31}$$

and

$$\frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right) = \lim_{x\to\pi/2} f_p^s(x) < \frac{p - \dfrac{\sin x}{\sin \frac{x}{p}}}{x^2} < \lim_{x\to 0} f_p^s(x) = \frac{p^2 - 1}{6p}. \tag{32}$$

The proof is completed. $\square$

3. Generalized D'Aurizio-Sándor inequalities for hyperbolic functions

In this section, we show an analogue of Theorem 2 for the case of hyperbolic functions holds true. Let

$$h_p^c(x) = \frac{1 - \dfrac{\cosh x}{\cosh \frac{x}{p}}}{x^2}, \tag{33}$$

$$h_p^s(x) = \frac{p - \dfrac{\sinh x}{\sinh \frac{x}{p}}}{x^2}. \tag{34}$$

Following the same arguments for proving Lemma 1, it can be shown that Lemma 1 with $\cos x$, $\sin x$ and $f_p^{\triangle}(x)$ $(\triangle = c, s)$ replaced by $\cosh x$, $\sinh x$ and $h_p^{\triangle}(x)$ $(\triangle = c, s)$ respectively, remains valid. It follows that we can prove $\frac{d}{dx}h_p^{\triangle}(x) < 0$ for $\triangle = c, s$ as in the proof of Theorem 2. It remains to calculate the following limits:

$$\lim_{x\to 0} f_p^c(x) = \frac{1 - p^2}{2p^2}, \quad \lim_{x\to\pi/2} f_p^c(x) = \frac{4}{\pi^2}\left(1 - \cosh\left(\frac{\pi}{2}\right)\operatorname{sech}\left(\frac{\pi}{2p}\right)\right), \tag{35}$$

$$\lim_{x\to 0} f_p^s(x) = \frac{1 - p^2}{6p}, \quad \lim_{x\to\pi/2} f_p^s(x) = \frac{4}{\pi^2}\left(p - \sinh\left(\frac{\pi}{2}\right)\operatorname{csch}\left(\frac{\pi}{2p}\right)\right). \tag{36}$$

Thus, we have the following analogue of Theorem 2 for $\cosh x$ and $\sinh x$.

THEOREM 3. *Let $0 < x < \pi/2$. Then the two double inequalities*

$$\frac{4}{\pi^2}\left(1 - \cosh\left(\frac{\pi}{2}\right)\operatorname{sech}\left(\frac{\pi}{2p}\right)\right) < \frac{1 - \dfrac{\cosh x}{\cosh \frac{x}{p}}}{x^2} < \frac{1 - p^2}{2p^2} \tag{37}$$

and

$$\frac{4}{\pi^2}\left(p - \sinh\left(\frac{\pi}{2}\right)\operatorname{csch}\left(\frac{\pi}{2p}\right)\right) < \frac{p - \dfrac{\sinh x}{\sinh \frac{x}{p}}}{x^2} < \frac{1 - p^2}{6p} \tag{38}$$

hold for $p = 3,4,5,\cdots$. In particular, (37) is reversed when $p = 2$ while (38) remains true when $p = 2$.

4. Application of the generalized D'Aurizio-Sándor inequalities to the Chebyshev polynomials of the second kinds

The first few Chebyshev polynomials of the second kind $U_n(x)$ $(n = 0,1,2,\cdots)$ are ([1, 3])

$$U_0(x) = 1, \tag{39}$$
$$U_1(x) = 2x, \tag{40}$$
$$U_2(x) = 4x^2 - 1, \tag{41}$$
$$U_3(x) = 8x^3 - 4x, \tag{42}$$
$$U_4(x) = 16x^4 - 12x^2 + 1, \tag{43}$$
$$U_5(x) = 32x^5 - 32x^3 + 6x, \tag{44}$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1. \tag{45}$$

In this section, we apply Theorem 2 to $U_n(x)$ with $x = \cos\theta$. By means of the formula $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$, we obtain the following corollary.

COROLLARY 1. *Let* $y \in (0, \frac{\pi}{2p})$. *The double inequality*

$$\frac{p}{6}\left((1-p^2)y^2+6\right) < U_{p-1}(\cos y) < p - \frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right)py^2 \tag{46}$$

holds for $p = 2,3,4,5,\cdots$.

Proof. The double inequality (8) in Theorem 2 can be written as

$$p - \frac{p^2-1}{6p}x^2 < \frac{\sin x}{\sin\frac{x}{p}} < p - \frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right)x^2, \quad x \in (0, \pi/2). \tag{47}$$

Letting $x/p = y$, we have

$$\frac{p}{6}\left(6 - (p^2-1)y^2\right) < \frac{\sin(py)}{\sin y} < p - \frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right)p^2y^2, \quad y \in (0, \frac{\pi}{2p}). \tag{48}$$

Due to $\frac{\sin(py)}{\sin y} = U_{p-1}(\cos y)$, the proof is completed. □

EXAMPLE 1. Letting $p = 7$ in Corollary 1 results in the following inequality

$$7 - 56y^2 < 64\cos^6 y - 80\cos^4 y + 24\cos^2 y - 1 < 7 - \frac{196\left(7 - \csc\left(\frac{\pi}{14}\right)\right)}{\pi^2}y^2, \tag{49}$$

where $y \in (0, \frac{\pi}{14}) \approx (0, 0.2244)$ and $\frac{196\left(7 - \csc\left(\frac{\pi}{14}\right)\right)}{\pi^2} \approx 49.7673$.

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