Journal of
Mathematical
Inequalities
Volume 12, Number 3 (2018), 873–882 doi:10.7153/jmi-2018-12-65

ON A NEW FAMILY OF BIVARIATE MEANS II

EDWARD NEUMAN

(*Communicated by T. Burić*)

Abstract. Lower and upper bounds as well as inequalities satisfied by members of a family of bivariate means, introduced recently by this author in [18], are established. In particular, optimal convex combinations bounds are obtained. Also, the Wilker and Huygens-type inequalities involving means under discussion are obtained.

1. Introduction

In this paper we continue study of bivariate means introduced recently by this author in [18].

For the sake of presentation we recall first definition of the Schwab-Borchardt mean

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a<b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } b<a \end{cases} \tag{1.1}$$

(see, e.g., [5], [6]) which plays a crucial role in this paper. This mean has been studied extensively in [19], [20], and in [7]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables. Mean SB can also be represented in terms of the degenerated completely symmetric elliptic integral of the first kind (see, e.g., [13]). It has been pointed out in [19] that some well known bivariate means such as logarithmic mean and two Seiffert means (see [22, 23]) can be represented as the Schwab Borchardt mean of two simpler means such as geometric and arithmetic means or as the Schwab-Borchardt mean of arithmetic and the square root mean. This idea was employed lately by this author and other researchers as well. For more details see [7]–[15].

The mean under discussion, denoted by

$$R(a,b) \equiv R,$$

is defined as follows (see [18])

$$R(a,b) = be^{a/SB(a,b)-1}. \tag{1.2}$$

Mathematics subject classification (2010): 26E60, 26A09, 26D05.
Keywords and phrases: Schwab-Borchardt mean, Wilker and Huygens-type inequalities, lower and upper bounds.

Clearly function $R:\mathbb{R}_+^2 \to \mathbb{R}$ is nonsymmetric and homogeneous of degree one in its variables.

This paper can be regarded as continuation of investigations initiated in author's earlier papers [7, 10, 11, 12, 13, 14, 15, 16, 18] and is organized as follows. Some preliminary results are given in Section 2. Bounds for the means under discussion are established in Section 3. Therein the arithmetic and geometric bounds for the new means are derived. The Wilker and Huygens-type inequalities involving mean R are established in Section 4.

2. Preliminaries

First of all we will give new formulas for means SB. We have [17]

$$SB(a,b) \equiv SB = \begin{cases} b\dfrac{\sin r}{r} = a\dfrac{\tan r}{r} & \text{if } a<b, \\ b\dfrac{\sinh s}{s} = a\dfrac{\tanh s}{s} & \text{if } b<a, \end{cases} \tag{2.1}$$

where

$$\cos r = a/b \quad \text{if} \quad a<b \quad \text{and} \quad \cosh s = a/b \quad \text{if} \quad a>b. \tag{2.2}$$

Clearly

$$0<r<\pi/2 \tag{2.3}$$

and

$$0<s<\infty. \tag{2.4}$$

It has been proven in [18] that

$$\min(a,b) < \frac{ab}{SB(a,b)} < R(a,b) < \max(a,b). \tag{2.5}$$

It follows from (2.5) that $R(a,b)$ is the mean value of its arguments.

For later use let us record two formulas

$$R(a,b) = be^{r\cot r - 1} \tag{2.6}$$

if $a<b$ and

$$R(a,b) = be^{s\coth s - 1} \tag{2.7}$$

if $a>b$, where r and s are defined in (2.2). Formulas (2.6) and (2.7) follow immediately from (1.2) and (2.1).

In the next chapter we will utilize three lemmas which are included here for reader's convenience. The following one, often called L'Hôspital's-type rule for monotonicity, can be found, e.g., in [2]

LEMMA A. *Let the functions f and g be continuous on $[c,d]$, differentiable on (c,d) and such that $g'(t) \neq 0$ on (c,d). If $\frac{f'(t)}{g'(t)}$ is (strictly) increasing (decreasing) on (c,d), then the functions $\frac{f(t)-f(d)}{g(t)-g(d)}$ and $\frac{f(t)-f(c)}{g(t)-g(c)}$ are also (strictly) increasing (decreasing) on (c,d).*

Also, we will need the following monotonicity result [4]

LEMMA B. *Suppose that the power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ and $g(t) = \sum_{n=0}^{\infty} b_n t^n$ ($b_n > 0$ for all $n \geqslant 0$) both converge for $|t| < \infty$. Then the function $f(t)/g(t)$ is (strictly) increasing (decreasing) for $t > 0$ if the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing).*

The Wilker and Huygens-type inequalities involving mean R will be established with the aid of the following (see [8])

LEMMA C. *Let u, v, λ and μ be positive numbers. Assume that*

$$1 < u^{\gamma} v^{\delta} \tag{2.8}$$

holds for some nonnegative numbers γ and δ whose sume equals to 1. If u and v satisfy the separation condition $u < 1 < v$, then the inequality

$$1 < \frac{\lambda}{\lambda+\mu} u^{p} + \frac{\mu}{\lambda+\mu} v^{q} \tag{2.9}$$

holds if

$$q > 0 \qquad \text{and} \qquad p\lambda\delta \leqslant q\mu\gamma. \tag{2.10}$$

If $v < 1 < u$, then the inequality (2.9) holds true if

$$p > 0 \qquad \text{and} \qquad q\mu\gamma \leqslant p\lambda\delta. \tag{2.11}$$

3. Bounds for mean R

Our first result reads as follows:

THEOREM 3.1. *If $a < b$, then the following inequality*

$$\left(\frac{SB(a,b)}{b}\right)^{\alpha} < \frac{R(a,b)}{b} < \left(\frac{SB(a,b)}{b}\right)^{\beta} \tag{3.1}$$

holds true for all

$$\alpha \geqslant 1/\log(\pi/2) = 2.214\ldots \qquad \text{and} \qquad \beta \leqslant 2. \tag{3.2}$$

If $a > b$, then inequality (3.1) is satisfied for all

$$\alpha \leqslant 1 \qquad \text{and} \qquad \beta \geqslant 2. \tag{3.3}$$

Proof. Assume that $a < b$. It follows from (2.1) and (2.6), respectively that

$$\frac{SB(a,b)}{b} = \frac{\sin r}{r} \qquad \text{and} \qquad \frac{R(a,b)}{b} = e^{r\cot r - 1}. \tag{3.4}$$

Thus the two-sided inequality (3.1) can be written as follows:

$$\left(\frac{\sin r}{r}\right)^{\alpha} < e^{r\cot r - 1} < \left(\frac{\sin r}{r}\right)^{\beta} \tag{3.5}$$

$(0 < |r| < \pi/2)$. Let us introduce an auxiliary function

$$\Phi(r) = \frac{r\cot r - 1}{\ln\left(\frac{\sin r}{r}\right)} =: \frac{f(r)}{g(r)}.$$

It is clear that the inequality (3.5) is equivalent to

$$\beta < \Phi(r) < \alpha. \tag{3.6}$$

We shall now prove that the function $\Phi(r)$ is strictly increasing on $(0, \pi/2)$. To this aim we employ formulas (4.3.70) and (4.3.71) in [1] to obtain

$$f(r) = -\sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} r^{2n} =: -\sum_{n=1}^{\infty} a_n r^{2n}$$

and

$$g(r) = -\sum_{n=1}^{\infty} \frac{2^{2n-1}|B_{2n}|}{n(2n)!} r^{2n} =: -\sum_{n=1}^{\infty} b_n r^{2n},$$

respectively. Here B_{2n} stands for the Bernoulli number of order $2n$. Hence

$$\frac{a_n}{b_n} = 2n$$

for all $n \geqslant 1$. Use of Lemma B yields the monotonicity property of function Φ. Easy computations yield

$$\Phi(0^+) = 2 \qquad \text{and} \qquad \Phi(\pi/2^-) = 1/\ln(\pi/2).$$

This completes the proof of (3.1) when $a < b$. Let now $a > b$. It is easy to see that in this case inequality (3.1) can be written as follows

$$\left(\frac{\sinh s}{s}\right)^{\alpha} < e^{s\coth s - 1} < \left(\frac{\sinh s}{s}\right)^{\beta} \tag{3.7}$$

For the proof of the left inequality in (3.7) we employ the following one

$$\frac{2\cosh s + \lambda}{2 + \lambda} < e^{s\coth s - 1},$$

where $1 \leqslant \lambda \leqslant 4$, see [9, Theorem 3]. Letting $\lambda = 4$ we get

$$\frac{\cosh s + 2}{3} < e^{s\coth s - 1}.$$

Combining this with Huygens inequality (see [21])

$$\frac{\sinh s}{s} < \frac{\cosh s + 2}{3}$$

$(s > 0)$ we obtain the desired result. The right inequality in (3.7) has been established in [3]. The proof is complete. □

We will now deal with bounds for R in the form of either arithmetic or geometric convex combinations of mean's arguments.

We have the following:

THEOREM 3.2. *If $a > b$, then the two-sided inequality*

$$\alpha a + (1-\alpha)b < R(a,b) < \beta a + (1-\beta)b \tag{3.8}$$

holds for all

$$\alpha \leqslant 2/3 \qquad \text{and} \qquad \beta \geqslant 2/e = 0.735\ldots. \tag{3.9}$$

If $a < b$, then the inequality (3.8) is valid for all

$$\alpha \geqslant 2/3 \qquad \text{and} \qquad \beta \leqslant 1 - 1/e = 0.632\ldots. \tag{3.10}$$

Proof. We shall establish first the assertion when $a > b$. Let us note that the inequality (3.8) can be written as follows

$$\alpha < \frac{R/b - 1}{a/b - 1} < \beta.$$

Making use of (2.7) we get

$$R/b = e^{s\coth s - 1}$$

while (2.2) yields

$$a/b = \cosh s.$$

The last three statements can be combined in one

$$\alpha < \Psi(s) < \beta, \tag{3.11}$$

where

$$\Psi(s) = \frac{e^{s\coth s - 1} - 1}{\cosh s - 1} =: \frac{f_1(s)}{g_1(s)}$$

$(s > 0)$. Let $F(s) = f_1'(s)/g_1'(s)$. Differentiation yields

$$F(s) = e^{s\coth s - 1}(\coth s - s/\sinh^2(s))/\sinh s.$$

We shall prove now that the function $F(s)$ is strictly increasing on its domain. We differentiate again to obtain

$$F'(s) = \frac{e^{s\coth s-1}}{\sinh^3(s)}h(s),$$

where

$$h(s) = \frac{s^2}{\sinh^2(s)} + \frac{s\cosh s}{\sinh s} - 2.$$

We need to show that $h(s) > 0$ holds for all $s > 0$. The inequality $h(s) > 0$ is equivalent to

$$\frac{\sinh s}{s} < \frac{1}{2}\left(\frac{s}{\sinh s} + \cosh s\right)$$

which is established [21]. Thus the function $F(s)$ is strictly increasing. Making use of Lemma A we conclude that the function $\Psi(s)$ is strictly increasing. One can easily verify that

$$\Psi(0^+) = 2/3 \qquad \text{and} \qquad \Psi(\infty^-) = 2/e.$$

This in conjunction with (3.11) yields bounds (3.9). The proof of Theorem 3.2 in the case when $a > b$ is complete. Assume now that $a < b$. Using (2.6) we can write inequality (3.8) as follows

$$\beta < \Phi(r) < \alpha, \tag{3.12}$$

where

$$\Phi(r) = \frac{1 - e^{r\cot r-1}}{1-\cos r} =: \frac{f_2(r)}{g_2(r)},$$

$0 < |r| < \pi/2$. Clearly function $\Phi(r)$ is even on the stated domain. For the sake of simplicity we will assume, to the end of this proof, that $0 < r < \pi/2$. Let

$$G(r) := f_2'(r)/g_2'(r).$$

Differentiation yields

$$G'(r) = -\frac{e^{r\cot r-1}}{\sin^3 r}g(r),$$

where

$$g(r) = \frac{r^2}{\sin^2 r} + r\frac{\cos r}{\sin r} - 2.$$

Using inequality (2.11) of Theorem 2.3 in [21]:

$$\frac{\sin r}{r} < \frac{1}{2}\left(\frac{r}{\sin r} + \cos r\right)$$

we obtain $g(r) > 0$. This in turn implies that $G'(r) < 0$. Thus the function $\Phi(r)$ is strictly decreasing. One can easily verify that

$$\Phi(0^+) = 2/3 \qquad \text{and} \qquad \Phi(\pi/2^-) = 1 - 1/e. \tag{3.13}$$

This in conjunction with (3.12) gives (3.10). The proof is complete. □

It is worthy to mention that it follows from Theorem 3.2 that the inequality

$$\frac{2a+b}{3} < R(a,b)$$

holds true for all positive and unequal numbers a and b.

It has been proven in [18] that the inequality

$$a^{2/3}b^{1/3} < R(a,b)$$

is valid for all $a,b > 0$, $a \neq b$. We shall show now that this inequality is optimal in a certain sense.

The announced result reads as follows:

THEOREM 3.3. *If $a < b$, then the following inequality*

$$a^{\alpha}b^{1-\alpha} < R(a,b) < a^{\beta}b^{1-\beta} \tag{3.14}$$

holds true for all

$$\alpha \geqslant 2/3 \qquad \text{and} \qquad \beta \leqslant 0. \tag{3.15}$$

Proof. First we divide each member of (3.14) by b and next employ formulas (2.2) and (2.6) to obtain

$$(\cos r)^{\alpha} < \frac{R}{b} < (\cos r)^{\beta}$$

where $r \in (0, \pi/2)$. Taking logarithms of three members in the above inequality we get

$$\beta < \Psi(r) < \alpha, \tag{3.16}$$

where now

$$\Psi(r) = \frac{r\cot r - 1}{\ln(\cos r)}. \tag{3.17}$$

Our next task is to demonstrate that the function $\Psi(r)$ is strictly decreasing on its domain. Writing $\Psi(r) = f(r)/g(r)$ and next using formulas (4.3.70) and (4.3.72) in [1] we obtain

$$f(r) = r\cot r - 1 = -\sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} r^{2n} =: -\sum_{n=1}^{\infty} a_n r^{2n}$$

and

$$g(r) = -\sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{n(2n)!} r^{2n} =: -\sum_{n=1}^{\infty} b_n r^{2n},$$

respectively. This yields

$$\frac{a_n}{b_n} = \frac{2n}{2^{2n}-1}.$$

Since the sequence (a_n/b_n) is strictly decreasing we conclude that the function $|\Psi(r)|$ is strictly decreasing. This in conjunction with

$$\Psi(0^+) = 2/3 \qquad \text{and} \qquad \Psi(\pi/2^-) = 0$$

and (3.16) yields the assertion (3.15). $\square$

We close this section with the following:

THEOREM 3.4. *If*

$$a < b, \qquad c < d \qquad \textit{and} \qquad \frac{a}{b} < \frac{c}{d}, \tag{3.18}$$

then

$$\frac{R(a,b)}{b} < \frac{R(c,d)}{d}. \tag{3.19}$$

Inequality (3.19) is reversed if

$$a > b, \qquad c > d \qquad \textit{and} \qquad \frac{a}{b} > \frac{c}{d}.$$

Proof. We shall prove this theorem when $a < b$. Using (2.6) we have

$$\frac{R(a,b)}{b} = e^{r\cot r - 1} =: f(r).$$

Logarithmic differentiation gives

$$\frac{f'(r)}{f(r)} = \frac{1}{\sin r}\left(\cos r - \frac{r}{\sin r}\right).$$

Taking into account that $r/\sin r > 1$ holds for all $r \in (0, \pi/2)$ we conclude that $f'(r) < 0$ on the stated domain. Thus the function $f(r)$ is strictly decreasing for all $0 < r < \pi/2$. Using (2.2) we have $r = \cos^{-1}(a/b)$ and $p = \cos^{-1}(c/d)$. Assumption

$$\frac{a}{b} < \frac{c}{d}$$

gives $r > p$. Using monotonicity property of the function f we obtain the desired result (3.19) Because the proof of the second part of this theorem goes along the lines used above it is omitted. $\square$

4. Wilker and Huygens-type inequalities involving mean R

Inequalities similar to (2.9) are called in mathematical literature Wilker and Huygens-type inequalities. There is a wast literature about those inequalities for the trigonometric, hyperbolic and Jacobian elliptic functions and also for the Schwab-Borchardt means and other bivariate means. The interested reader is referred to [8] and the references therein.

For the sake of brevity we will write now R instead of $R(a,b)$. The main result of this section reads as follows:

THEOREM 4.1. *Let λ and μ be positive numbers. If $a<b$, then the inequality*

$$1<\frac{\lambda}{\lambda+\mu}\left(\frac{R}{b}\right)^{p}+\frac{\mu}{\lambda+\mu}\left(\frac{R}{a}\right)^{q} \tag{4.1}$$

is satisfied if

$$q>0 \qquad \text{and} \qquad 2p\lambda \leqslant q\mu . \tag{4.2}$$

If $a>b$, then the inequality (4.1) is valid if

$$p>0 \qquad \text{and} \qquad 2q\mu \leqslant p\lambda . \tag{4.3}$$

Proof. Let $a<b$. Then $a<R<b$. This implies that

$$\frac{R}{b}<1<\frac{R}{a}.$$

With

$$u=\frac{R}{b} \qquad \text{and} \qquad v=\frac{R}{a}$$

we see that the separation condition $u<1<v$ is satisfied. This together with $a^{2/3}b^{1/3}<R$ gives

$$1<u^{1/3}v^{2/3}.$$

Comparison with (2.8) yields $\gamma=1/3$ and $\delta=2/3$. To obtain conditions (4.2) it suffices to use (2.10). Proof of the second statement of this theorem goes along the above lines. We omit further details. The proof is complete. $\square$

We close this section with Wilker-type inequality involving the identric mean $I(a,b)\equiv I$ of two positive and unequal numbers a and b. Recall that

$$I(a,b)=\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}$$

Let A and G stand for the unweighted arithmetic and geometric means, respectively, of a and b. It is easy to show that $R(A,G)=I$. To prove the last identity we employ (1.2), with $a=A$ and $b=G$, to obtain

$$\ln R(A,G)=-1+\ln G+\frac{A}{L},$$

where L stands for the logarithmic mean of a and b. Here we have used a known fact that $SB(A,G)=L$ (see [19]). A straightforward algebra yields $\ln R(A,G)=\ln I(a,b)$.

To obtain an inequality involving identric mean I we use (4.3) to conclude that the following inequality

$$1<\frac{\lambda}{\lambda+\mu}\left(\frac{I}{G}\right)^{p}+\frac{\mu}{\lambda+\mu}\left(\frac{I}{A}\right)^{q}$$

is valid for all numbers λ,μ,p,q which satisfy conditions

$$\lambda,\mu,p>0 \qquad \text{and} \qquad 2q\mu \leqslant p\lambda .$$

Acknowledgements. The author is deeply indebted to an anonymous referee for several valuable remarks which helped him to improve the first version of this paper.

REFERENCES

[1] M. ABRAMOWITZ, A. I. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Ninth printing, Dover Publications Inc., New York, 1970.
[2] G. D. ANDERSON, M. K. VAMANAMURTHY, M. VUORINEN, *Monotonicity rules in calculus*, Amer. Math. Monthly **133** (2006), 805–816.
[3] B. A. BHAYO, J. SÁNDOR, *On certain old and new trigonometric and hyperbolic inequalities*, Analysis Math. **41** (2015), 3–15.
[4] M. BIERNACKI, J. KRZYŻ, *On the monotonicity of certain functionals in the theory of analytic functions*, Ann. Univ. Mariae Curie – Sklodowska **2** (1955), 134–145.
[5] J. M. BORWEIN, P. B. BORWEIN, *Pi and the AGM- A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1987.
[6] B. C. CARLSON, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly **78** (1971), 496–505.
[7] E. NEUMAN, *Inequalities for the Schwab-Borchardt mean and their applications*, J. Math. Inequal. **5**, 4 (2011), 601–609.
[8] E. NEUMAN, *Inequalities for weighted sums of powers and their applications*, Math. Inequal. Appl. **15**, 4 (2012), 995–1005.
[9] E. NEUMAN, *Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions*, Adv. Inequal. Appl. **1** (2012), 1–11.
[10] E. NEUMAN, *A note on a certain bivariate mean*, J. Math. Inequal. **6**, 4 (2012), 637–643.
[11] E. NEUMAN, *A one-parameter family of bivariate means*, J. Math. Inequal. **7**, 3 (2013), 399–412.
[12] E. NEUMAN, *Sharp inequalities involving Neuman-Sándor and logarithmic means*, J. Math. Inequal. **7**, 3 (2013), 413–419.
[13] E. NEUMAN, *On a new bivariate mean*, Aequat. Math. **88** (2014), 277–289.
[14] E. NEUMAN, *On some means derived from the Schwab-Borchardt mean*, J. Math. Inequal. **8**, 1 (2014), 171–183.
[15] E. NEUMAN, *On some means derived from the Schwab-Borchardt mean II*, J. Math. Inequal. **8**, 2 (2014), 359–368.
[16] E. NEUMAN, *Inequalities and bounds for a certain bivariate elliptic mean*, Math. Inequal. Appl. **19**, 4 (2016), 1375–1385.
[17] E. NEUMAN, *On two bivariate elliptic means*, J. Math. Inequal. **11**, 2 (2017), 345–354.
[18] E. NEUMAN, *On a new family of bivariate means*, J. Math. Inequal, **11**, 3 (2017), 673–681.
[19] E. NEUMAN, J. SÁNDOR, *On the Schwab-Borchardt mean*, Math. Pannon. **14**, 2 (2003), 253–266.
[20] E. NEUMAN, J. SÁNDOR, *On the Schwab-Borchardt mean II*, Math. Pannon. **17**, 1 (2006), 49–59.
[21] E. NEUMAN, J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl. **13**, 1 (2010), 715–723.
[22] H.-J. SEIFFERT, *Problem 887*, Nieuw. Arch. Wisk. **11** (1993), 176.
[23] H.-J. SEIFFERT, *Aufgabe 16*, Würzel **29** (1995), 87.

(Received February 26, 2017)

Edward Neuman
Mathematical Research Institute
144 Hawthorn Hollow, Carbondale IL 62903, USA
e-mail: edneuman76@gmail.com