Journal of
Mathematical
Inequalities
Volume 12, Number 3 (2018), 889–899
doi:10.7153/jmi-2018-12-67

A FAMILY OF WINDSCHITL TYPE APPROXIMATIONS FOR GAMMA FUNCTION

ZHEN-HANG YANG AND JING-FENG TIAN*

(*Communicated by G. Nemes*)

Abstract. In this paper, we present a family of high accurate approximation formulas

$$\mathscr{W}_p(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35}\right)$$

for gamma function $\Gamma(x+1)$ with parameter $p \geqslant -33/35$, and prove the function

$$x \mapsto \ln\Gamma(x+1) - \ln\mathscr{W}_p(x)$$

is strictly increasing and concave on $(0,\infty)$ if and only if $p \geqslant 158/315$. This yields some new sharp approximations for gamma function.

1. Introduction

It is known that the Stirling's formula

$$n! \sim \sqrt{2\pi n}n^n e^{-n} \tag{1.1}$$

for $n \in \mathbb{N}$ has important applications in probability theory, statistical physics, number theory, combinatorics and other related fields. There are many improvements for the Stirling's formula, see for example, Burnside's [1], Gosper [2], Batir [3], Mortici [4].

Because the gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$ is related to the factorial function, many scholars were devoted to seeking various better approximations for the gamma function, for instance, Ramanujan [5, p. 339], Windschitl [6, Eq. (42)], [7], Nemes [8, Corollary 4.1], Mortici [9, 10], Yang and Chu [11, Propositions 4 and 5], Chen [12, 13], Lu et al. [14, 15].

More results involving the approximation formulas for the factorial or gamma function can be found in [16, 17, 18, 19, 20, 21, 22, 23] and the references cited therein. Several nice inequalities between gamma function and the truncations of its asymptotic series can be found in [24], [25].

Now we focus on the Windschitl's approximation formulas (see [6, Eq. (42)], [7]) given by

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} := W_0(x), \tag{1.2}$$

Mathematics subject classification (2010): Primary 33B15, 26D15, Secondary 26A48, 26A51.
Keywords and phrases: Gamma function, monotonicity, convexity, Windschitl type approximation.
* Corresponding author.

Paper JMI-12-67

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}+\frac{1}{810x^6}\right)^{x/2}:=W_1(x), \tag{1.3}$$

as $x\to\infty$. It was showed in [13] that the rates of Windschitl's approximation formulas $W_0(x)$ and $W_1(x)$ converging to $\Gamma(x+1)$ is like x^{-5} and x^{-7} as $x\to\infty$, respectively. These show that $W_0(x)$ and $W_1(x)$ are excellent approximations for gamma function.

Very recently, by using a little known power series Yang and Tian [26] developed Windschitl's approximation formula $W_0(x)$ for the gamma function to asymptotic expansion

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\exp\left(\sum_{n=3}^{\infty}\frac{2n(2n-2)!-2^{2n-1}}{2n(2n)!}\frac{B_{2n}}{x^{2n-1}}\right) \tag{1.4}$$

as $x\to\infty$, where B_{2n} is the Bernoulli number. While the approximation formula $W_1(x)$ was developed to another asymptotic expansion by Chen and Paris in [27, Theorem 2]:

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}+\frac{1}{810x^6}-\frac{163}{170100x^8}+\frac{1019}{680400x^{10}}+\cdots\right)^{x/2}$$

as $x\to\infty$. Lu, Song and Ma [28] extended Windschitl's formula $W_0(n)$ to another asymptotic expansion in the form of

$$\Gamma(n+1)\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left[n\sinh\left(\frac{1}{n}+\frac{a_7}{n^7}+\frac{a_9}{n^9}+\frac{a_{11}}{n^{11}}+\cdots\right)\right]^{n/2}$$

with $a_7=1/810, a_9=-67/42525, a_{11}=19/8505,\ldots$ without a general formula for the coefficients a_k for $k\geqslant 7$. This general formula was given by Chen in [12, Theorem 2], and in the same paper, Chen presented a new asymptotic expansion

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2+\sum_{j=0}^{\infty}r_jx^{-j}},\quad x\to\infty, \tag{1.5}$$

which was improved in [26] by Yang and Tian.

On the other hand, Alzer [29] showed that the double inequality

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\alpha}{x^5}\right)<\Gamma(x+1)<\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right) \tag{1.6}$$

holds for $x>0$ with the best possible constants $\alpha=0$ and $\beta=1/1620$. Yang and Tian [30] presented a very accurate Windschitl type approximation:

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\exp\left(\frac{7}{324}\frac{1}{x^3(35x^2+33)}\right)=W_2(x), \tag{1.7}$$

as $x\to\infty$, and prove that the function $x\mapsto\ln\Gamma(x+1)-\ln W_2(x)$ is decreasing and concave on $(1,\infty)$. This yields some new inequalities for gamma function in terms of $W_0(x)$.

Inspired by the Windschitl type approximation (1.7), we consider a family of Windschitl type approximations for gamma function defined by

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\exp\left(\frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35}\right)=\mathscr{W}_p(x) \tag{1.8}$$

as $x \to \infty$, where $p \geqslant -33/35$. It is easy to see that

$$\mathscr{W}_\infty(x) = \lim_{p\to\infty} \mathscr{W}_p(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\right)$$

$$\mathscr{W}_{-33/35}(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5} - \frac{11}{18\,900x^7}\right),$$

$$\mathscr{W}_0(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\frac{x^2}{x^2+33/35}\right) = W_2(x).$$

The aim of this paper is to determine the parameter $p \geqslant -33/35$ such that the function $x \mapsto \ln\Gamma(x+1) - \ln\mathscr{W}_p(x)$ is increasing and concave on $(0,\infty)$. Our main result is the following theorem.

THEOREM 1. *Let $p \geqslant -33/35$. Then the function*

$$F_p(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x+\frac{1}{2}\right)\ln x + x$$
$$-\frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right) - \frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35} \tag{1.9}$$

is strictly increasing and concave on $(0,\infty)$ if and only if $p \geqslant p_0 = 158/315$.

2. Proof of Theorem 1

As is well known, analytic inequality [31, 32, 33] is playing a very important role in different branches of modern mathematics. In order to prove Theorem 1, we need the following inequality.

LEMMA 1. *The inequality*

$$\psi'\left(x+\frac{1}{2}\right) < \frac{1}{x}\frac{x^6+\frac{445}{78}x^4+\frac{199\,801}{34\,320}x^2+\frac{3072}{5005}}{x^6+\frac{301}{52}x^4+\frac{14\,357}{2288}x^2+\frac{9069}{9152}} \tag{2.1}$$

holds for $x > 0$.

Proof. Let

$$g(x) = \psi'\left(x+\frac{1}{2}\right) - \frac{1}{x}\frac{q_6(x)}{p_6(x)},$$

where

$$p_6(x) = x^6 + \frac{301}{52}x^4 + \frac{14\,357}{2288}x^2 + \frac{9069}{9152},$$
$$q_6(x) = x^6 + \frac{445}{78}x^4 + \frac{199\,801}{34\,320}x^2 + \frac{3072}{5005}.$$

In view of

$$\psi'\left(x+\frac{3}{2}\right) - \psi'\left(x+\frac{1}{2}\right) = -\frac{1}{(x+1/2)^2}$$

(see [34, p. 260, (6.4.6)]), we have

$$g(x+1)-g(x) = \psi'\left(x+\frac{3}{2}\right)-\frac{1}{x+1}\frac{q_6(x+1)}{p_6(x+1)}-\psi'\left(x+\frac{1}{2}\right)+\frac{1}{x}\frac{q_6(x)}{p_6(x)}$$
$$= -\frac{1}{(x+1/2)^2}-\frac{1}{x+1}\frac{q_6(x+1)}{p_6(x+1)}+\frac{1}{x}\frac{q_6(x)}{p_6(x)}$$
$$= \frac{176400}{20449}\frac{1}{x(x+1)(2x+1)^2 p_6(x)p_6(x+1)} > 0.$$

This yields

$$g(x) < g(x+1) < \ldots < \lim_{n\to\infty} g(x+n) = 0,$$

which completes the proof. □

Proof of Theorem 1. Using the asymptotic expansion (1.4), we have that as $x\to\infty$,

$$F_p(x) = \left[\ln\Gamma(x+1)-\ln\sqrt{2\pi}-\left(x+\frac{1}{2}\right)\ln x+x-\frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right)\right]$$
$$-\frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35}$$
$$\sim \sum_{n=3}^{5}\frac{2n(2n-2)!-2^{2n-1}}{2n(2n)!}\frac{B_{2n}}{x^{2n-1}}-\frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35}$$
$$= -\frac{11}{170100}\frac{(315p-158)x^2-(455p+429)}{x^9(35x^2+35p+33)},$$

which implies that

$$\lim_{x\to\infty}\frac{F_p(x)}{x^{-9}} = -\frac{11}{18900}\left(p-\frac{158}{315}\right). \tag{2.2}$$

In view of $\lim_{x\to\infty}F_p(x)=0$, making use of L'Hospital rule gives

$$\lim_{x\to\infty}\frac{F_p'(x)}{x^{-10}} = \frac{11}{2100}\left(p-\frac{158}{315}\right),$$
$$\lim_{x\to\infty}\frac{F_p''(x)}{x^{-11}} = -\frac{11}{210}\left(p-\frac{158}{315}\right).$$

(i) We now prove the necessity. If F_p is increasing and concave on $(0,\infty)$, then there must be

$$\lim_{x\to\infty}\frac{F_p'(x)}{x^{-10}} = \frac{11}{2100}\left(p-\frac{158}{315}\right)\geqslant 0 \text{ and } \lim_{x\to\infty}\frac{F_p''(x)}{x^{-11}} = -\frac{11}{210}\left(p-\frac{158}{315}\right)\leqslant 0,$$

which yields $p\geqslant p_0=158/315$.

(ii) We next prove the sufficiency. Differentiation yields

$$F_p'(x) = \psi(x+1)-\frac{1}{2}\ln\left(x\sinh\frac{1}{x}\right)+\frac{1}{2x}\coth\frac{1}{x}-\ln x-\frac{1}{2x}-\frac{1}{2}$$
$$+\frac{7}{324}\frac{175x^4+(350p+99)x^2+5p(35p+33)}{x^6(35x^2+35p+33)^2},$$

$$F_p''(x) = \psi'(x+1) + \frac{1}{2x^3}\frac{1}{\sinh^2(1/x)} - \frac{3}{2x} + \frac{1}{2x^2}$$

$$-\frac{7}{54}\frac{6125x^6 + 35(525p+187)x^4 + 3(35p+33)(175p+22)x^2 + 5p(35p+33)^2}{x^7(35x^2+35p+33)^3}.$$

Since

$$\frac{\partial F_p'(x)}{\partial p} = \frac{385}{108}\frac{63x^2+35p+33}{x^6(35x^2+35p+33)^3} > 0$$

$$\frac{\partial F_p''(x)}{\partial p} = -\frac{385}{18}\frac{3675x^4 + 112(35p+33)x^2 + (35p+33)^2}{x^7(35x^2+35p+33)^4} < 0$$

for $p \geqslant -33/35$ and $x > 0$, that are, the functions $p \mapsto F_p'(x)$ and $p \mapsto F_p''(x)$ are increasing and decreasing on $[-33/35, \infty)$. Thus, if we prove $F_{p_0}''(x) < 0$, then $F_{p_0}'(x) > \lim_{x\to\infty} F_{p_0}'(x) = 0$ for $x > 0$. And, for $p \geqslant p_0$, it is easy to see that $F_p''(x) \leqslant F_{p_0}''(x) < 0$ and $F_p'(x) \geqslant F_{p_0}'(x) > 0$, then the sufficiency follows.

Replacing x by $(x+1/2)$ in the inequity of (2.1) leads to

$$\psi'(x+1) < \frac{1}{210}\frac{(60060x^6+180180x^5+567875x^4+835450x^3+919933x^2+532238x+146631)}{(2x+1)(143x^6+429x^5+1364x^4+2013x^3+2273x^2+1338x+420)},$$

which indicates that

$$F_{p_0}''(x) < \frac{1}{210}\frac{60060x^6+180180x^5+567875x^4+835450x^3+919933x^2+532238x+146631}{(2x+1)(143x^6+429x^5+1364x^4+2013x^3+2273x^2+1338x+420)}$$

$$-\frac{3}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3}\frac{1}{\sinh^2(1/x)} - \frac{127575x^6+328293x^4+346788x^2+133510}{9450x^7(9x^2+13)^3} := f\left(\frac{1}{x}\right).$$

Simplifying yields

$$f(t) = \frac{t^3}{2\sinh^2 t} - \frac{t}{9450}\frac{p_{19}(t)}{p_{13}(t)},$$

where

$$\begin{aligned} p_{19}(t) = {} & 56074200t^{19}+290784780t^{18}+806391950t^{17}+1630996354t^{16} \\ & +2573755772t^{15}-948850784t^{14}-5848462963t^{13}-3534213977t^{12} \\ & +16372329813t^{11}+57129403014t^{10}+109206482238t^{9}+152782165188t^{8} \\ & +164420940285t^{7}+152143429005t^{6}+108142112610t^{5}+70899015285t^{4} \\ & +32357867850t^{3}+14814902025t^{2}+3447969525t+985134150, \end{aligned}$$

$$p_{13}(t) = (t+2)(13t^2+9)^3\left(420t^6+1338t^5+2273t^4+2013t^3+1364t^2+429t+143\right).$$

Using the inequality

$$2\sinh^2 t = \cosh 2t - 1 > \sum_{n=0}^{6}\frac{2^{2n}}{(2n)!}t^{2n} - 1 = \sum_{n=1}^{6}\frac{2^{2n}}{(2n)!}t^{2n},$$

we have

$$\begin{aligned} f(t) &< \frac{t^3}{\sum_{n=1}^{6}\frac{2^{2n}}{(2n)!}t^{2n}} - \frac{t}{9450}\frac{p_{19}(t)}{p_{13}(t)} \\ &= \frac{t}{9450}\frac{9450p_{13}(t) - \left(\sum_{n=1}^{6}\frac{2^{2n}}{(2n)!}t^{2n-2}\right)p_{19}(t)}{\left(\sum_{n=1}^{6}\frac{2^{2n}}{(2n)!}t^{2n-2}\right)p_{13}(t)} \\ &:= -\frac{1}{4725}\frac{t^{13}\times p_{17}(t)}{\left(\sum_{n=1}^{6}\frac{2^{2n}}{(2n)!}t^{2n-2}\right)p_{13}(t)} < 0, \end{aligned}$$

where

$$\begin{aligned} p_{17}(t) = {} & \frac{213616}{891}t^{17} + \frac{1174888}{945}t^{16} + \frac{30363892}{2673}t^{15} + \frac{22453788188}{467775}t^{14} \\ & + \frac{20234223892}{66825}t^{13} + \frac{537563456096}{467775}t^{12} + \frac{360206516968}{66825}t^{11} \\ & + \frac{8397752582192}{467775}t^{10} + \frac{1379858652428}{22275}t^{9} + \frac{25906996096822}{155925}t^{8} \\ & + \frac{3144272626043}{7425}t^{7} + \frac{8209692140801}{10395}t^{6} + \frac{1214784299189}{825}t^{5} \\ & + \frac{7860969452488}{5775}t^{4} + \frac{94725155551}{55}t^{3} + \frac{353797595441}{385}t^{2} \\ & + \frac{3200114871}{5}t + \frac{6400229742}{35} > 0 \end{aligned}$$

for $t > 0$. This implies that $F''_{p_0}(x) < 0$ for $x > 0$, and the proof is complete. □

3. New bounds for gamma function

As a direct consequence of Theorem 1, we immediately get

COROLLARY 1. *For $p \geqslant -33/35$, let $\mathscr{W}_p(x)$ be defined by (1.8). Then the double inequality*

$$\lambda_p \mathscr{W}_p(x) < \Gamma(x+1) < \mathscr{W}_p(x), \tag{3.1}$$

or equivalently,

$$\begin{aligned} & \lambda_p \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35}\right) \\ & \leqslant \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} < \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\frac{x^2+p}{x^2+p+33/35}\right) \end{aligned} \tag{3.2}$$

holds for $x \geqslant 1$ if and only if $p \geqslant p_0 = 158/315$, where

$$\lambda_p = \frac{1}{\sqrt{2\pi\sinh 1}}\exp\left(\frac{1}{324}\frac{11333p+22025}{35p+68}\right) \tag{3.3}$$

is the best constant. In particular, we have

$$\lambda_{p_0}\mathscr{W}_{p_0}(x) < \Gamma(x+1) < \mathscr{W}_{p_0}(x),$$

namely,

$$\begin{aligned}&\lambda_{p_0}\left(x\sinh\frac{1}{x}\right)^{x/2}\exp\left(\frac{1}{1620x^5}\frac{x^2+158/315}{x^2+13/9}\right)\\ &\leqslant \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/2)^x} < \left(x\sinh\frac{1}{x}\right)^{x/2}\exp\left(\frac{1}{1620x^5}\frac{x^2+158/315}{x^2+13/9}\right)\end{aligned} \tag{3.4}$$

holds for $x \geqslant 1$ with the best constant

$$\lambda_{p_0} = \frac{1}{\sqrt{2\pi\sinh 1}}\exp\left(\frac{113357}{113400}\right) \approx 0.999963. \tag{3.5}$$

Proof. The necessary condition for the inequalities (3.2) to hold for $x \geqslant 1$ follows from $\lim_{x\to\infty} x^9 F_p(x) \leqslant 0$. This together with the limit relation (2.2) yields $p \geqslant p_0 = 158/315$.

The sufficient condition for the inequalities (3.2) to hold for $x \geqslant 1$ follows from the increasing property of F_p on $(0,\infty)$ if $p \geqslant p_0 = 158/315$.

Putting $p = p_0 = 158/315$ in inequalities (3.2) gives (3.4), which completes the proof. □

REMARK 1. We claim that the lower and upper bounds in inequalities (3.1) are decreasing and increasing with respect to the parameter p on $[-33/35,\infty)$ for $x \geqslant 1$. In fact, by differentiation we find that

$$\frac{\partial}{\partial p}\ln\mathscr{W}_p(x) = \frac{77}{108x^5(35x^2+35p+33)^2},$$

$$\frac{\partial}{\partial p}(\ln(\lambda_p\mathscr{W}_p(x))) = -\frac{77}{108}\frac{H_p(x)}{x^5(35p+68)^2(35x^2+35p+33)^2},$$

where

$$H_p(x) = 1225(x^5-1)p^2 + 70(35x^7+33x^5-68)p + (1225x^9+2310x^7+1089x^5-4624).$$

Clearly, $\partial\ln\mathscr{W}_p(x)/\partial p > 0$ for $p \geqslant -33/35$ and $x \geqslant 1$; while $\partial(\ln(\lambda_p\mathscr{W}_p(x)))/\partial p < 0$ is due to

$$H_p(x) = 1225\left[\left(x^5-1\right)\left(p+\frac{33}{35}\right)^2 + 2\left(x^7-1\right)\left(p+\frac{33}{35}\right) + \left(x^9-1\right)\right] > 0$$

for $p \geqslant -33/35$ and $x > 1$. Thus taking $p = 158/315$, $4/7$, ∞, we obtain the following corollary.

COROLLARY 2. *The inequalities*

$$\lambda_\infty\mathscr{W}_\infty(x) \leqslant \lambda_{4/7}\mathscr{W}_{4/7}(x) \leqslant \lambda_{p_0}\mathscr{W}_{p_0}(x) \leqslant \Gamma(x+1) < \mathscr{W}_{p_0}(x) < \mathscr{W}_{22/35}(x) < \mathscr{W}_\infty(x),$$

that are,

$$\lambda_\infty \exp\left(\frac{1}{1620x^5}\right) \leqslant \lambda_{4/7} \exp\left(\frac{1}{324}\frac{7x^2+4}{x^5(35x^2+53)}\right)$$
$$\leqslant \lambda_{p_0} \exp\left(\frac{1}{1620x^5}\frac{x^2+158/315}{x^2+13/9}\right) \leqslant \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/2)^x (x\sinh(1/x))^{x/2}}$$
$$< \exp\left(\frac{1}{1620x^5}\frac{x^2+158/315}{x^2+13/9}\right) < \exp\left(\frac{1}{324}\frac{7x^2+4}{x^5(35x^2+53)}\right) < \exp\left(\frac{1}{1620x^5}\right),$$

hold for $x \geqslant 1$*, where* $\lambda_{p_0} \approx 0.999963$ *given in (3.5) and*

$$\lambda_{4/7} = \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{2591}{2592}\right) \approx 0.999956,$$
$$\lambda_\infty = \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{1619}{1620}\right) \approx 0.999725$$

are the best constants.

It has been shown in [30] that the function $x \mapsto \ln\Gamma(x+1) - \ln W_2(x)$ is decreasing and concave on $(1,\infty)$. Since $W_2(x) = \mathscr{W}_0(x)$, by the same technique as the proof of Theorem 1 we can prove

THEOREM 2. *Let* $p \geqslant -33/35$*. Then the function* F_p *defined by (1.9) is decreasing and convex on* $[1,\infty)$ *if* $-33/35 \leqslant p \leqslant 0$*.*

Likewise, let $p = -33/35$, $-1/7$, 0 in Theorem 2. Then by using the monotonicity of F_p in x on $(1,\infty)$ and bounds in inequalities (3.2) we can obtain

COROLLARY 3. *The inequalities*

$$\mathscr{W}_{-33/35}(x) < \mathscr{W}_{-1/7}(x) < \mathscr{W}_0(x) < \Gamma(x+1) < \lambda_0 \mathscr{W}_0(x) \leqslant \lambda_{-1/7}\mathscr{W}_{-1/7}(x) \leqslant \mathscr{W}_{-33/35}(x),$$

namely,

$$\exp\left(\frac{1}{1620x^5} - \frac{11}{18\,900x^7}\right) < \exp\left(\frac{1}{2268}\frac{7x^2-1}{x^5(5x^2+4)}\right) < \exp\left(\frac{7}{324x^3(35x^2+33)}\right)$$
$$< \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/2)^x (x\sinh(1/x))^{x/2}} \leqslant \lambda_0 \exp\left(\frac{7}{324x^3(35x^2+33)}\right)$$
$$\leqslant \lambda_{-1/7} \exp\left(\frac{1}{2268}\frac{7x^2-1}{x^5(5x^2+4)}\right) \leqslant \lambda_{-33/35} \exp\left(\frac{1}{1620x^5} - \frac{11}{18\,900x^7}\right),$$

hold for $x \geqslant 1$ *with the best constants*

$$\lambda_0 = \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{22\,025}{22\,032}\right) \approx 1.000024,$$
$$\lambda_{-1/7} = \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{3401}{3402}\right) \approx 1.000048,$$
$$\lambda_{-33/35} = \frac{1}{\sqrt{2\pi \sinh 1}} \exp\left(\frac{28\,349}{28\,350}\right) \approx 1.000307.$$

4. Numerical comparisons

As we know, a remarkable approximation for gamma function is fairly accurate but relatively succinct. Such ones can see [5, p. 339], [8, Corollary 4.1], [7], [9], [11, Propositions 4 and 5], [13]. In this section, we list some more accurate but simple approximation formulas for gamma function and compare them with our family of approximation formulas $\mathscr{W}_p(x)$ defined by (1.8).

It is easy to check that Nemes' approximation formula [7] satisfies

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{210x^2+53}{360x(7x^2+2)}\right)\left(1+O\left(\frac{1}{x^7}\right)\right) := N_2(x), \tag{4.1}$$

as $x \to \infty$. It was showed in [13] that as $x \to \infty$,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^3+24x/7-1/2}\right)^{x^2+53/210}\left(1+O\left(\frac{1}{x^7}\right)\right) := C(x). \tag{4.2}$$

As shown in Introduction, we have

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}+\frac{1}{810x^6}\right)^{x/2}\left(1+O\left(\frac{1}{x^7}\right)\right) = W_1(x)\left(1+O\left(\frac{1}{x^7}\right)\right).$$

For our family of approximation formulas $\mathscr{W}_p(x)$ for gamma function, by (2.2) we see that

$$\lim_{x\to\infty}\frac{\ln\Gamma(x+1)-\ln\mathscr{W}_p(x)}{x^{-9}} = \lim_{x\to\infty}\frac{F_p(x)}{x^{-9}} = -\frac{11}{18900}\left(p-\frac{158}{315}\right),$$

which implies that

$$\Gamma(x+1) \sim \mathscr{W}_p(x)\left(1+O\left(\frac{1}{x^9}\right)\right) \text{ if } p \neq p_0 = \frac{158}{315}.$$

And, we easily verify that

$$\lim_{x\to\infty}\frac{\ln\Gamma(x+1)-\ln\mathscr{W}_{p_0}(x)}{x^{-11}} = -\frac{276277}{392931000},$$

which indicates that

$$\Gamma(x+1) \sim \mathscr{W}_{p_0}(x)\left(1+O\left(\frac{1}{x^{11}}\right)\right).$$

It thus can be seen that our new Windschitl type approximation formula $\mathscr{W}_{p_0}(x)$ are the best among ones listed above, which can be also found from the following Table 1.

Table 1: Comparison among N_2 (4.1), C (4.2), W_1 (1.3), $\mathscr{W}_0$ and $\mathscr{W}_{p_0}$

n	$\frac{N_2(n)-n!}{n!}$	$\frac{C(n)-n!}{n!}$	$\frac{W_1(n)-n!}{n!}$	$\frac{\mathscr{W}_0(n)-n!}{n!}$	$\frac{\mathscr{W}_{p_0}(n)-n!}{n!}$
1	1.11×10^{-4}	1.340×10^{-4}	1.83×10^{-4}	2.41×10^{-5}	3.74×10^{-5}
2	1.90×10^{-6}	2.22×10^{-6}	2.67×10^{-6}	2.31×10^{-7}	1.08×10^{-7}
5	4.35×10^{-9}	4.96×10^{-9}	5.74×10^{-9}	1.25×10^{-10}	1.13×10^{-11}
10	3.61×10^{-11}	4.09×10^{-11}	4.71×10^{-11}	2.79×10^{-13}	6.59×10^{-15}
20	2.86×10^{-13}	3.24×10^{-13}	3.73×10^{-13}	5.63×10^{-16}	3.38×10^{-18}
50	4.71×10^{-16}	5.33×10^{-16}	6.13×10^{-16}	1.49×10^{-19}	1.44×10^{-22}
100	3.68×10^{-18}	4.17×10^{-18}	4.79×10^{-18}	2.92×10^{-22}	7.01×10^{-26}

Acknowledgements. This work was supported by the Fundamental Research Funds for the Central Universities (No. 2015ZD29), and the Higher School Science Research Funds of Hebei Province of China (No. Z2015137).

REFERENCES

[1] W. BURNSIDE, *A rapidly convergent series for* $\log N!$, Messenger Math. **46** (1917), 157–159.

[2] R. W. GOSPER, *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA, **75** (1978), 40–42.

[3] N. BATIR, *Sharp inequalities for factorial* $n!$, Proyecciones **27**, 1 (2008), 97–102.

[4] C. MORTICI, *On the generalized Stirling formula*, Creative Math. Inf. **19**, 1 (2010), 53–56.

[5] S. RAMANUJAN, *The Lost Notebook and Other Unpublished Papers*, Springer, Berlin, 1988.

[6] W. D. SMITH, *The gamma function revisited*, `http://schule.bayernport.com/gamma/gamma05.pdf`, 2006.

[7] `http://www.rskey.org/gamma.htm`.

[8] G. NEMES, *New asymptotic expansion for the Gamma function*, Arch. Math. (Basel), **95** (2010), 161–169.

[9] C. MORTICI, *Improved asymptotic formulas for the gamma function*, Comput. Math. Appl. **61** (2011), 3364–3369.

[10] C. MORTICI, *A new fast asymptotic series for the gamma function*, Ramanujan J. **38**, 1 (2015), 549–559.

[11] ZH.-H. YANG AND Y.-M. CHU, *Asymptotic formulas for gamma function with applications*, Appl. Math. Comput. **270** (2015), 665–680.

[12] CH.-P. CHEN, *Asymptotic expansions of the gamma function related to Windschitl's formula*, Appl. Math. Comput. **245** (2014), 174–180.

[13] CH.-P. CHEN, *A more accurate approximation for the gamma function*, J. Number Theory **164** (2016), 417–428.

[14] D. LU, *A new sharp approximation for the Gamma function related to Burnside's formula*, Ramanujan J. **35**, 1 (2014), 121–129.

[15] D. LU, L. SONG AND C. MA, *Some new asymptotic approximations of the gamma function based on Nemes' formula, Ramanujan's formula and Burnside's formula*, Appl. Math. Comput. **253** (2015), 1–7.

[16] ZH. YANG AND J.-F. TIAN, *A comparison theorem for two divided differences and applications to special functions*, J. Math. Anal. Appl. **464** (2018), 580–595.

[17] N. BATIR, *Inequalities for the gamma function*, Arch. Math. **91** (2008), 554–563.

[18] C. MORTICI, *An ultimate extremely accurate formula for approximation of the factorial function*, Arch. Math. **93**, 1 (2009), 37–45.

[19] C. MORTICI, *New sharp inequalities for approximating the factorial function and the digamma functions*, Miskolc Math. Notes **11**, 1 (2010), 79–86.

[20] J.-L. ZHAO, B.-N. GUO AND F. QI, *A refinement of a double inequality for the gamma function*, Publ. Math. Debrecen **80**, 3–4 (2012), 333–342.

[21] C. MORTICI, *Further improvements of some double inequalities for bounding the gamma function*, Math. Comput. Model. **57** (2013), 1360–1363.

[22] F. QI, *Integral representations and complete monotonicity related to the remainder of Burnside's formula for the gamma function*, J. Comput. Appl. Math. **268** (2014), 155–167.

[23] ZH.-H. YANG AND J. TIAN, *Monotonicity and sharp inequalities related to gamma function*, J. Math. Inequal. **12**, 1 (2018), 1–22.

[24] H. ALZER, *On some inequalities for the gamma and psi functions*, Math. Comput. **66**, 217 (1997), 373–389.

[25] ZH.-H. YANG, *Approximations for certain hyperbolic functions by partial sums of their Taylor series and completely monotonic functions related to gamma function*, J. Math. Anal. Appl. **441** (2016), 549–564.

[26] ZH.-H. YANG AND J.-F. TIAN, *Two asymptotic expansions for gamma function developed by Windschitl's formula*, Open Math. **16** (2018), 1048–1060.

[27] C.-P. CHEN AND R. B. PARIS, *Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function*, Appl. Math. Comput. **250** (2015), 514–529.

[28] D. LU, L. SONG AND C. MA, *A generated approximation of the gamma function related to Windschitl's formula*, J. Number Theory, **140** (2014), 215–225.

[29] H. ALZER, *Sharp upper and lower bounds for the gamma function*, Proc. R. Soc. Edinb. **139A** (2009), 709–718.

[30] ZH.-H. YANG AND J.-F. TIAN, *An accurate approximation formula for gamma function*, J. Inequal. Appl. **2018** (2018): 56.

[31] J. TIAN, W. WANG, W.-S. CHEUNG, *Periodic boundary value problems for first-order impulsive difference equations with time delay*, Adv. Differ. Equ. **2018** (2018): 79.

[32] J.-F. TIAN, *Triple Diamond-Alpha integral and Hölder-type inequalities*, J. Inequal. Appl. **2018** (2018): 111.

[33] J.-F. TIAN, M.-H. HA AND CH. WANG, *Improvements of generalized Hölder's inequalities and their applications*, J. Math. Inequal. **12**, 2 (2018), 459–471.

[34] M. ABRAMOWTTZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1972.

(Received December 23, 2017)

Zhen-Hang Yang
College of Science and Technology
North China Electric Power University
Ruixiang Street 282, 071051, Baoding, P. R. China
and
State Grid Zhejiang Electric Power Company Research Institute
Department of Science and Technology
310014, Hangzhou, P. R. China
e-mail: yzhkm@163.com

Jing-Feng Tian
College of Science and Technology
North China Electric Power University
Ruixiang Street 282, 071051, Baoding, P. R. China
e-mail: tianjf@ncepu.edu.cn