COEFFICIENT PROBLEMS FOR UNIFIED STARLIKE AND CONVEX CLASSES OF $m$–FOLD SYMMETRIC BI–UNIVALENT FUNCTIONS

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Abstract. Let $\mathcal{T}_m$ denote the class of $m$-fold symmetric bi-univalent functions in the open unit disk. We obtain the coefficient bounds of $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in a new general subclass $\mathcal{C}_h^p(\alpha)$ of $\mathcal{T}_m$, where $h$ and $p$ are in Carathéodary class of functions. We investigate the initial Taylor-Maclaurin coefficients estimate problems associated with $\mathcal{C}_h^p(\alpha)$ also. Our conclusion improves some earlier related results.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are normalized analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{S}$ the class of all functions $f(z) \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Let $\mathcal{P}$ be the class of all analytic functions $p : \mathbb{U} \to \mathbb{C}$ satisfying $p(0) = 1$ and the real part $\Re p(z) > 0$ on $\mathbb{U}$.

The Koebe one-quarter theorem ensures that the image of $\mathbb{U}$ under every $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$ (see, Duren [11]). Thus, every function $f(z) \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\mathcal{T}$ denote the class of bi-univalent functions.

In 1967, Lewin [20] investigated the class $\mathcal{T}$ and showed that, for every function $f \in \mathcal{S}$ of the form (1), the second coefficient of $f$ satisfies the estimate $|a_2| < 1.51$. Mathematics subject classification (2010): 30C45, 30C50.

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Also, Brannan-Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \mathcal{T}$. Furthermore, Netanyahu [22] proved that $\max\{|a_2| : f \in \mathcal{T}\} = \frac{3}{4}$. In 1985, Kedzierawski [19] proved the Brannan-Clunie conjecture for bi-starlike functions and Tan [35] obtained the bound with $|a_2| < 1.485$, which is the best known estimate for functions in the class $\mathcal{T}$. In addition, Brannan-Taha [8] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the classes of bi-starlike functions of order $\beta$ ($0 \leq \beta < 1$) and bi-convex functions of order $\beta$ ($0 \leq \beta < 1$).

The study of bi-univalent functions was revived in recent years by Srivastava-Mishra-Gochhayat [24], and a considerably large number of sequels to Srivastava-Mishra-Gochhayat [24] have appeared in the literature since then (see, e.g., [3, 12, 15, 23, 25, 33, 36, 37, 38]). Recently, Çağlar-Deniz-Srivastava [10] studied the second Hankel determinant for certain subclasses of bi-univalent functions, Deniz [13] and Srivastava-Bansal [27] both extended and improved the results of Brannan–Taha [8] by the principle of subordination between analytic functions, and Srivastava-Gaboury-Ghanim [30] obtained the coefficient estimates for some general subclasses of analytic and bi-univalent functions.

Faber polynomials plays a considerable act in geometric function theory (see, e.g., [4, 6, 17]), which was introduced by Faber [16]. In particular, Srivastava-Eker-Ali [28] and Sakar-Güney [34] used the Faber polynomial expansion techniques to derive bounds for the general Taylor-Maclaurin coefficients $|a_n|$ of the functions in different subclasses of $\mathcal{T}$, and Srivastava-Eker-Hamidi-Jahangiri [31] studied the Faber polynomial coefficients for bi-univalent functions defined by the Tremblay fractional derivative operator.

Now, using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ can be expressed as (see, Airault-Bouali [4]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n,$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{2(-n-1)!(n-3)!} a_2^{-3} a_3$$

$$+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{-4} a_4 + \frac{(-n)!}{2(-n+2)!(n-5)!} a_2^{-5}[a_5 + (-n+2)a_3^2]$$

$$+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{-6}[a_6 + (-2n+5)a_3 a_4] + \sum_{j \geq 7} a_2^{-1} V_j,$$

in which $V_j$ ($7 \leq j \leq n$) is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ (see, Airault-Ren [5]). In particular, the first three terms of $K_{n-1}^{-n}$ are

$$\frac{1}{2} K_1^{-2} = -a_2, \quad \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \quad \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4).$$

Thus, the inverse function $f^{-1}$ may analytically continued to $\mathcal{U}$ as follows:

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.$$
For each $f \in \mathcal{S}$, the function
\[ h(z) = \sqrt[m]{f(z^m)}, \quad z \in \mathbb{D}, \quad m \in \mathbb{N}, \]
is univalent and maps the unit disk $\mathbb{D}$ into a region with $m$-fold symmetry. A function is said to be $m$-fold symmetric (see, e.g., [26, 29]) if it has the following normalized form:
\[ f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in \mathbb{D}, \quad m \in \mathbb{N}. \] (5)

We denote by $\mathcal{S}_m$ the class of $m$-fold symmetric univalent functions in $\mathbb{D}$. The functions in the class $\mathcal{S}$ are said to be one-fold symmetric.

Each bi-univalent function generates an $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of $f$ is given as in (5) and the series expansion for $f^{-1}$, which has been recently proven by Srivastava-Sivasubramanian-Sivakumar [26], is given as follows:
\[ g(w) = w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} + \cdots, \] (6)
where $f^{-1} = g$. We denote by $\mathcal{C}_m$ the class of $m$-fold symmetric bi-univalent functions in $\mathbb{D}$. Thus, when $m = 1$, the formula (6) coincides with the formula (4).

Here are some examples of $m$-fold symmetric bi-univalent functions (see, e.g., [26, 29])
\[ \left( \frac{z^m}{1 - z^m} \right)^{\frac{1}{m}}, \quad \left[ \frac{1}{2} \log \left( \frac{1 + z^m}{1 - z^m} \right) \right]^{\frac{1}{m}}, \quad [- \log(1 - z^m)]^{\frac{1}{m}} \]
with the corresponding inverse functions
\[ \left( \frac{w^m}{1 - w^m} \right)^{\frac{1}{m}}, \quad \left( \frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right)^{\frac{1}{m}}, \quad \left( \frac{e^{w^m} - 1}{e^{w^m}} \right)^{\frac{1}{m}}. \]

Srivastava-Gaboury-Ghanim [29] and Sivasubramanian-Sivakumar [32] deal with the coefficients problems for $f \in \mathcal{C}_m$. Bounds for the initial coefficients of different classes of $m$-fold symmetric bi-univalent functions were also investigated by the other authors (see, e.g., [14, 17, 26]).

**DEFINITION 1.** Let the function $h, p : \mathbb{D} \to \mathbb{C}$ be constrained that $h(0) = p(0) = 1$ and
\[ \min \{ \Re(h(z)), \Re((p(z)) \} > 0 \quad (z \in \mathbb{D}). \]
For a function $f \in \mathcal{C}_m$, we say $f \in \mathcal{C}_m^{h,p}(\alpha)$ if the following conditions are satisfied:
\[ \left( \frac{zf'(z)}{f(z)} \right)^{\alpha} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} \in h(\mathbb{D}) \quad (z \in \mathbb{D}, \quad 0 \leq \alpha \leq 1) \]
and
\[ \left( \frac{wg'(w)}{g(w)} \right)^{\alpha} \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} \in p(\mathbb{D}) \quad (w \in \mathbb{D}, \quad 0 \leq \alpha \leq 1), \]
where $g(w) = f^{-1}(w)$. 

REMARK 1. Obviously, $\mathcal{C}_{m}^{h, p}(\alpha)$ generalizes the class of $m$-fold symmetric bi-starlike and bi-convex functions. Specially, $\mathcal{C}_{1}^{h, p}(\alpha)$ was introduced and studied by Xiong-Liu [36] with $\mathcal{C}_{1}^{h, p}(\alpha)$. Some closely-related classes were investigated by Bulut [9] and Xu-Xiao-Srivastava [37] also.

If we let
\[
h(z) = m \frac{1 + (1 - 2\beta)z^m}{1 - z^m}, \quad p(z) = m \frac{1 - (1 - 2\beta)z^m}{1 + z^m}, \quad (0 \leq \beta < 1, \ z \in \mathbb{U})
\]
and
\[
h(z) = m \left( \frac{1 + z^m}{1 - z^m} \right)^\beta, \quad p(z) = m \left( \frac{1 - z^m}{1 + z^m} \right)^\beta, \quad (0 < \beta \leq 1, \ z \in \mathbb{U})
\]
in Definition 1 respectively, then we have the definition 2 and definition 3 as follows.

DEFINITION 2. For a function $f \in \mathcal{F}_m$, we say $f \in \mathcal{C}_{m}^{\beta}(\alpha)$ if the following conditions are satisfied:
\[
\Re \left\{ \left( \frac{zf''(z)}{f'(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1 - \alpha} \right\} > \beta \quad (z \in \mathbb{U})
\]
and
\[
\Re \left\{ \left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg'(w)}{g(w)} \right)^{1 - \alpha} \right\} > \beta \quad (w \in \mathbb{U}),
\]
where $g(w) = f^{-1}(w), \ 0 \leq \beta < 1, \ 0 \leq \alpha \leq 1$.

REMARK 2. (i) If $m = 1$ in Definition 2, then the class $\mathcal{C}_{1}^{\beta}(\alpha)$ was introduced and studied by Ali-Lee-Ravichandran-Supramaniama [3] with $\mathcal{C}_{1}^{\beta}(\alpha)$. Also the classes $\mathcal{C}_{1}^{\beta}(1) \equiv \mathcal{K}_{1}^{\beta}$ and $\mathcal{C}_{1}^{\beta}(0) \equiv \mathcal{K}_{1}^{0 \beta}$ were introduced by Brannan-Taha [8].

(ii) If $\alpha = 0$ in Definition 2, then the class $\mathcal{C}_{m}^{\beta}(0)$ was introduced and studied by Sivasubramanian-Sivakumar [32] with $\mathcal{K}_{1}^{\beta}$.

(iii) If $\alpha = 1$ in Definition 2, then the class $\mathcal{C}_{m}^{\beta}(1)$ was introduced and studied by Hamidi-Jahangiri [17] with $\mathcal{K}_{1}^{\beta}$.

DEFINITION 3. For a function $f \in \mathcal{F}_m$, we say $f \in \mathcal{C}_{m}^{\alpha \beta}(\alpha)$ if the following conditions are satisfied:
\[
\left| \arg \left[ \left( \frac{zf''(z)}{f'(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1 - \alpha} \right] \right| < \frac{\beta \pi}{2}, \quad z \in \mathbb{U}
\]
and
\[
\left| \arg \left[ \left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg'(w)}{g(w)} \right)^{1 - \alpha} \right] \right| < \frac{\beta \pi}{2}, \quad w \in \mathbb{U},
\]
where $g(w) = f^{-1}(w), \ 0 \leq \alpha \leq 1, \ 0 < \beta \leq 1$. 

Remark 3. (i) If \( m = 1 \) in Definition 3, then the class was introduced and studied by Ali-Lee-Ravichandran-Supramaniam [3] with \( C^* (\alpha) \). Also the classes \( C^* (1) \equiv \mathcal{S}^*_\beta \) and \( C^* (0) \equiv \mathcal{K}^*_\beta \) were introduced by Brannan-Taha [8].

(ii) If \( \alpha = 0 \) or \( \alpha = 1 \) in Definition 3, then the classes were introduced and studied by Sivasubramanian-Sivakumar [32] with \( \mathcal{K}^*_m \) or \( \mathcal{S}^*_m \), respectively.

Motivated and stimulated especially by the works of Srivastava–Mishra–Gochhayat [24], Xiong-Liu [36], Xu-Xiao-Srivastava [37] and Xu-Gui-Srivastava [38], we give the estimates on the initial coefficients \( |a_{m+1}| \) and \( |a_{2m+1}| \) for the subclass \( C^ {h,p}_m (\alpha) \) of \( m \)-fold symmetric bi-univalent functions in this paper. The corresponding results about the classes \( C^*_m (\alpha) \) and \( C^* (\alpha) \) were given also. Our results generalize and improve some earlier related works.

2. Main results

We begin by finding the estimates on the coefficients \( |a_{m+1}| \) and \( |a_{2m+1}| \) for functions in the class \( C^ {h,p}_m (\alpha) \).

**Theorem 1.** Let the function \( f(z) \) given by (5) be in the class \( C^ {h,p}_m (\alpha) \). Then

\[
|a_{m+1}| \leq \min \left\{ \sqrt{\frac{|h^{(2m)}(0)| + |p^{(2m)}(0)|}{(2m)!|L_m|}}, \sqrt{\frac{|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2}{2(m!)^2[(1 - \alpha)m^2 + 1]^2}} \right\}
\]

and

\[
|a_{2m+1}| \leq \min \left\{ \sqrt{\frac{|A + L_m|}{L_mB}} \frac{|h^{(2m)}(0)|}{(2m)!} + \sqrt{\frac{|A - L_m|}{L_mB}} \frac{|p^{(2m)}(0)|}{(2m)!}, \mathfrak{B} \right\},
\]

where \( A = m(m + 1)[2(1 - \alpha)(2m + 1) + 2\alpha] \), \( B = 4(1 - \alpha)m(2m + 1) + 4m\alpha \), \( L_m = (m + 1)[2(1 - \alpha)m(2m + 1) + 2m\alpha] + \alpha(\alpha - 1)m^2 + 2\alpha(1 - \alpha)m^2(m + 1) - 2m\alpha - (1 - \alpha)m^2(m + 1)^2 - 2m(1 - \alpha)(m + 1)^2 \) and

\[
\mathfrak{B} = \frac{A}{B} \frac{|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2}{2(m!)^2[(1 - \alpha)m^2 + 1]^2} + \frac{|h^{(2m)}(0)| + |p^{(2m)}(0)|}{(2m)!B}.
\]

**Proof.** For the function \( f \in C^ {h,p}_m (\alpha) \) and for the inverse map \( g = f^{-1} \), we obtain

\[
\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} = h(z) \quad (z \in \mathbb{U})
\]

and

\[
\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = p(w) \quad (w \in \mathbb{U}),
\]

where \( h \) and \( p \) satisfy the hypotheses in Definition 1. Now suppose that the functions \( h(z) \) and \( p(w) \) have the following series expansions:

\[
h(z) = 1 + h_m z^m + h_{2m} z^{2m} + \cdots
\]
and
\[ p(w) = 1 + p_m w + p_{2m} w^{2m} + \cdots, \tag{12} \]
respectively.

Following (5), we write:
\[ \left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} = 1 + T_m z^m + T_{2m} z^{2m} + \cdots, \]
where
\[ T_m = [(1 - \alpha)m^2 + 1]a_{m+1} \tag{13} \]
and
\[ T_{2m} = [2(1 - \alpha)m(2m + 1) + 2m\alpha]a_{2m+1} + \left[ \frac{\alpha(\alpha - 1)}{2}m^2 + \alpha(1 - \alpha)m^2(1 + m) - m\alpha - \frac{\alpha(1 - \alpha)}{2}m^2(m + 1)^2 - m(1 - \alpha)(m + 1)^2 \right]a_{m+1}. \]

Also from (5) and (6), we get
\[ \left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = 1 + G_m w^m + G_{2m} w^{2m} + \cdots, \tag{14} \]
where
\[ G_m = -[(1 - \alpha)m^2 + 1]a_{m+1} \]
and
\[ G_{2m} = \left[ m(m + 1)[2(1 - \alpha)(2m + 1) + 2\alpha] + \frac{\alpha(\alpha - 1)}{2}m^2 + \alpha(1 - \alpha)m^2(m + 1) - m\alpha - \frac{\alpha(1 - \alpha)}{2}m^2(m + 1)^2 - m(1 - \alpha)(m + 1)^2 \right]a_{m+1} - \left[ 2(1 - \alpha)m(2m + 1) + 2m\alpha \right]a_{2m+1}. \]

Now, combining (9)-(14), we have
\[ T_m = h_m, \tag{15} \]
\[ T_{2m} = h_{2m}, \tag{16} \]
\[ G_m = p_m, \tag{17} \]
\[ G_{2m} = p_{2m}. \tag{18} \]

From (15) and (17), it follows
\[ h_m = -p_m \tag{19} \]
and
\[ 2[(1 - \alpha)m^2 + 1]a_{m+1}^2 = h_m^2 + p_m^2. \tag{20} \]
Also from (16) and (18), we get

\[ L_m a_{m+1}^2 = h_{2m} + p_{2m}, \]  

(21)

where

\[ L_m = (m + 1)[2(1 - \alpha)m(2m + 1) + 2m\alpha] + \alpha(\alpha - 1)m^2 + 2\alpha(1 - \alpha)m^2(m + 1) - 2m(1 - \alpha)(m + 1)^2. \]

Therefore, from (20) and (21), we have

\[ a_{m+1}^2 = \frac{h_{2m}^2 + p_{2m}^2}{2[(1 - \alpha)m^2 + 1]^2}, \]  

(22)

and

\[ a_{m+1}^2 = \frac{h_{2m} + p_{2m}}{L_m}, \]  

(23)

which give the desired estimate on \(|a_{m+1}|\) as asserted in (7).

Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (18) from (16), we get

\[ 4[(1 - \alpha)m(2m + 1) + m\alpha]a_{2m+1} - m(m + 1)[2(1 - \alpha)(2m + 1) + 2\alpha]a_{m+1}^2 = h_{2m} - p_{2m}. \]  

(24)

By (22) and (24), it follows

\[ a_{2m+1} = \frac{m(m + 1)[2(1 - \alpha)(2m + 1) + 2\alpha]}{4[(1 - \alpha)m(2m + 1) + m\alpha]} \frac{h_{2m}^2 + p_{2m}^2}{2[(1 - \alpha)m^2 + 1]^2} + \frac{h_{2m} - p_{2m}}{4[(1 - \alpha)m(2m + 1) + m\alpha]}. \]  

(25)

On the other hand, from (23) and (24), it follows

\[ L_mB a_{2m+1} = A(h_{2m} + p_{2m}) + L_m(h_{2m} - p_{2m}), \]

where

\[ A = m(m + 1)[2(1 - \alpha)(2m + 1) + 2\alpha], \]

\[ B = 4(1 - \alpha)m(2m + 1) + 4m\alpha. \]

Thus we obtain

\[ a_{2m+1} = \frac{A + L_m}{L_mB} h_{2m} + \frac{A - L_m}{L_mB} p_{2m}, \]

which yields the desired estimate on \(|a_{2m+1}|\) as asserted in (8). \(\Box\)

**Theorem 2.** Let the function \(f(z)\) given by (5) be in the class \(C_{m+\beta}^\alpha(\alpha)\). Then

\[ |a_{m+1}| \leq \min \left\{ \frac{2\beta}{m} \sqrt{\frac{1}{(2m)!|L_m|}}, \frac{2\beta}{m(m!)[(1 - \alpha)m^2 + 1]} \right\} \]
and

$$|a_{2m+1}| \leq \min \left\{ \left( \left| \frac{A+L_m}{L_mB} \right| + \left| \frac{A-L_m}{L_mB} \right| \right) \frac{2\beta^2}{m^2(2m)!}, B_2 \right\},$$

where \( A = m(m+1)[2(1-\alpha)(2m+1) + 2\alpha], \ B = 4(1-\alpha)m(2m+1) + 4m\alpha, \ L_m = (m+1)[2(1-\alpha)m(2m+1) + 2m\alpha] + \alpha(\alpha-1)m^2 + 2\alpha(1-\alpha)m^2(m+1) - 2m\alpha - \alpha(1-\alpha)m^2(m+1)^2 - 2m(1-\alpha)(m+1)^2 \) and

$$B_2 = \frac{A}{B \cdot m^2(m!)^2[(1-\alpha)m^2+1]^2} + \frac{4\beta^2}{m^2(2m)!B}.$$

**Proof.** Let

$$h(z) = \sqrt[2m]{\frac{1+z}{1-z^{m}}} = 1 + 2 \frac{\beta}{m} + 2 \frac{\beta^2}{m^2} z^m + \cdots, \ z \in U$$

and

$$p(z) = \sqrt[2m]{\frac{1-z}{1+z^{m}}} = 1 - 2 \frac{\beta}{m} + 2 \frac{\beta^2}{m^2} z^m + \cdots, \ z \in U$$

in Theorem 1. Then we have Theorem 2. \( \square \)

**Theorem 3.** Let the function \( f(z) \) given by (5) be in the class \( \mathcal{C}_m^\beta(\alpha) \). Then

$$|a_{m+1}| \leq \min \left\{ \frac{2}{m} \sqrt{(1-\beta)[m+1-m(1-\beta)]}{(2m)!L_m}, \frac{2(1-\beta)}{m(m!)[(1-\alpha)m^2+1]} \right\}$$

and

$$|a_{2m+1}| \leq \min \left\{ \left( \left| \frac{A+L_m}{L_mB} \right| + \left| \frac{A-L_m}{L_mB} \right| \right) \frac{2(1-\beta)[m+1-m(1-\beta)]}{m^2(2m)!}, B_3 \right\},$$

where \( A = m(m+1)[2(1-\alpha)(2m+1) + 2\alpha], \ B = 4(1-\alpha)m(2m+1) + 4m\alpha, \ L_m = (m+1)[2(1-\alpha)m(2m+1) + 2m\alpha] + \alpha(\alpha-1)m^2 + 2\alpha(1-\alpha)m^2(m+1) - 2m\alpha - \alpha(1-\alpha)m^2(m+1)^2 - 2m(1-\alpha)(m+1)^2 \) and

$$B_3 = \frac{A}{B \cdot m^2(m!)^2[(1-\alpha)m^2+1]^2} + \frac{4(1-\beta)^2}{m^2(2m)!B}.$$

**Proof.** Let

$$h(z) = \sqrt[2m]{\frac{1+(1-2\beta)z^m}{1-z^m}}$$

$$= 1 + \frac{2}{m}(1-\beta)z^m + \left[ \frac{2}{m}(1-\beta) + \frac{1-m}{2m^2}(2-2\beta)^2 \right] z^m + \cdots, \ z \in U$$
and

\[ p(z) = \sqrt[2m]{\frac{1 - (1 - 2\beta)z^m}{1 + z^m}} = 1 - \frac{2m}{(1 - \beta)}z^m + \left[ \frac{2m}{m} (1 - \beta) \right] z^m + \cdots, \quad z \in \mathbb{U} \]

in Theorem 1. Then we have Theorem 3. \hfill \Box

### 3. Corollaries and consequences

In this section, we give some corollaries by using the above theorems.

**Corollary 1.** Let the function \( f(z) \) given by (5) be in the class \( \mathcal{C}^{h, p} (\alpha) \), then

\[
|a_2| \leq \min \left\{ \sqrt{\frac{|h''(0)| + |p''(0)|}{2|\alpha^2 - 3\alpha + 4|}}, \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(2 - \alpha)^2}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{|\alpha^2 - 11\alpha + 16|h''(0)|}{8(3 - 2\alpha)(\alpha^2 - 3\alpha + 4)}, \frac{|\alpha^2 + 5\alpha - 8| p''(0)|}{8(3 - 2\alpha)(\alpha^2 - 3\alpha + 4)} \right\} \mathfrak{B}_1,
\]

where

\[
\mathfrak{B}_1 = \frac{|h'(0)|^2 + |p'(0)|^2}{2(2 - \alpha)^2} + \frac{|h''(0)| + |p''(0)|}{8(3 - 2\alpha)}.
\]

**Proof.** By taking \( m = 1 \) in Theorem 1, we get Corollary 1, which is an improvement of the estimates given by Xiong-Liu [36]. \hfill \Box

**Corollary 2.** Let the function \( f(z) \) given by (5) be in the class \( \mathcal{K}^{*}_{m} \beta \). Then

\[
|a_{m+1}| \leq \min \left\{ \frac{2\beta}{m} \sqrt{\frac{1}{2(2m)!m^2(m + 1)}}, \frac{2\beta}{m(m!)(m^2 + 1)} \right\}
\]

and

\[
|a_{2m+1}| \leq \min \left\{ \frac{\beta^2}{m^4(2m)!}, \frac{2(m + 1)\beta^2}{m^2(m!)(2m^2 + 1)^2} + \frac{\beta^2}{m^3(2m + 1)!} \right\}.
\]

**Proof.** By letting \( \alpha = 0 \) in Theorem 2, we have Corollary 2. \hfill \Box

**Corollary 3.** Let the function \( f(z) \) given by (5) be in the class \( \mathcal{K}^{*}_{m} \beta \). Then

\[
|a_{m+1}| \leq \min \left\{ \frac{2\beta}{m} \sqrt{\frac{1}{(2m)!2m^2}}, \frac{2\beta}{m(m!)} \right\}
\]
and
\[ |a_{2m+1}| \leq \min \left\{ \frac{(m+1)\beta^2}{m^2(2m)!}, \frac{2(m+1)\beta^2}{m^2(m!)^2}, \frac{\beta^2}{m^3(2m)!} \right\}. \]

**Proof.** Let \( \alpha = 1 \) in Theorem 2. Then we have Corollary 3. \( \Box \)

**COROLLARY 4.** Let the function \( f(z) \) given by (5) be in the class \( \mathcal{K}_m^\beta \). Then
\[ |a_{m+1}| \leq \min \left\{ \frac{2}{m} \sqrt{\frac{(1-\beta)[m+(1-m)(1-\beta)]}{2(2m)!m^2(m+1)}}, \frac{2(1-\beta)}{m(m!)(m^2+1)} \right\} \]
and
\[ |a_{2m+1}| \leq \min \left\{ \frac{|(1-\beta)[m+(1-m)(1-\beta)]|}{m^4(2m)!}, \mathfrak{B}_4 \right\}, \]
where
\[ \mathfrak{B}_4 = \frac{2(m+1)(1-\beta)^2}{m^2(m!)^2(m^2+1)}, + \frac{(1-\beta)[m+(1-m)(1-\beta)]}{m^3(2m+1)!}. \]

**Proof.** Let \( \alpha = 0 \) in Theorem 3. Then we have Corollary 4. \( \Box \)

**COROLLARY 5.** Let the function \( f(z) \) given by (5) be in the class \( \mathcal{M}_m^\beta \). Then
\[ |a_{m+1}| \leq \min \left\{ \frac{2}{m} \sqrt{\frac{(1-\beta)[m+(1-m)(1-\beta)]}{2(2m)!m^2}}, \frac{2(1-\beta)}{m(m!)} \right\} \]
and
\[ |a_{2m+1}| \leq \min \left\{ \frac{(m+1)[(1-\beta)[m+(1-m)(1-\beta)]]}{m^4(2m)!}, \mathfrak{B}_5 \right\}, \]
where
\[ \mathfrak{B}_5 = \frac{2(m+1)(1-\beta)^2}{m^2(m!)^2} + \frac{(1-\beta)[m+(1-m)(1-\beta)]}{m^3(2m)!}. \]

**Proof.** Let \( \alpha = 1 \) in Theorem 3. Then we have Corollary 5. \( \Box \)

**REMARK 4.** In the case of one fold symmetric functions, Corollary 1 to Corollary 5 improve the estimates obtained by Brannan-Taha [8]. Sharp estimates for the coefficients \( |a_{m+1}| \), \( |a_{2m+1}| \) and other coefficients of functions belonging to the classes investigated in this paper are yet open problems.

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REFERENCES


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