Journal of Mathematical Inequalities
Volume 12, Number 4 (2018), 943–951 doi:10.7153/jmi-2018-12-71

A FAMILY OF MEROMORPHIC FUNCTIONS INVOLVING GENERALIZED MITTAG–LEFFLER FUNCTION

CAI-MEI YAN AND JIN-LIN LIU*

(*Communicated by H. M. Srivastava*)

Abstract. We introduce a new family of meromorphic functions defined by the second order differential subordination involving generalized Mittag-Leffler function. Certain convolution properties of the family are discussed.

1. Introduction

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\quad (z,\alpha,\beta\in\mathbb{C};\ \mathrm{Re}(\alpha)>0), \tag{1.1}$$

where $\Gamma(\cdot)$ is classical gamma function. It was first introduced in 1903 by Swedish mathematician Mittag-Leffler for $\beta=1$.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$, as well as its various further generalizations, arise naturally in the solution of fractional differential equations and fractional integro-differential equations which are associated with (for example) the kinetic equation, random walks, Lévy flights, super-diffusive transport problems and in the study of complex systems. In particular, it is an explicit formula for the resolvent of Riemann-Liouville fractional integrals by Hille and Tamarkin. The more properties and applications of the Mittag-Leffler functions, together with their generalizations, can be found in a number of recent works [1] to [3], and [8] to [12].

In [12], Yin and Huang considered a generalized Mittag-Leffler function as follows:

$$E_{\alpha,\beta,q}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_q(\alpha n+\beta)}\quad (z,\alpha,\beta\in\mathbb{C},\ q\in(0,+\infty);\ \mathrm{Re}(\alpha)>0), \tag{1.2}$$

where $\Gamma_q(x)$ is classical q-gamma function defined by

$$\Gamma_q(x)=\frac{q!q^x}{x(x+1)\cdots(x+q)}.$$

Mathematics subject classification (2010): 33E12, 30C45.
Keywords and phrases: Meromorphic function, Mittag-Leffler function, subordination, Hadamard product (convolution), convex univalent.

* Corresponding author.

It is known that $\lim_{q\to+\infty}\Gamma_q(x)=\Gamma(x)$. The logarithmic derivative of q-gamma function

$$\psi_q(x)=\frac{d}{dx}\log\Gamma_q(x)=\frac{\Gamma_q'(x)}{\Gamma_q(x)}$$

is known as generalized digamma function. Its derivatives $\psi_q^{(n)}(x)$ are known as the generalized polygamma function. These functions have the following representations of series

$$\psi_q^{(n)}(x)=\sum_{k=0}^{q}\frac{(-1)^{n-1}n!}{(x+k)^{n+1}}. \tag{1.3}$$

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_{n-p}z^{n-p}\quad (p\in\mathbb{N}=\{1,2,3,\cdots\}), \tag{1.4}$$

which are analytic in the punctured open unit disk $U_0=\{z\in\mathbb{C}:0<|z|<1\}$. The class $\Sigma(p)$ is closed under the Hadamard product (or convolution)

$$(f_1*f_2)(z)=z^{-p}+\sum_{n=1}^{\infty}a_{n-p,1}a_{n-p,2}z^{n-p}=(f_2*f_1)(z),$$

where

$$f_j(z)=z^{-p}+\sum_{n=1}^{\infty}a_{n-p,j}z^{n-p}\in\Sigma(p)\quad (j=1,2).$$

Now, for $f\in\Sigma(p)$, we consider the following operator $L_{\alpha,\beta}^{q}:\Sigma(p)\to\Sigma(p)$ associated with the generalized Mittag-Leffler function $E_{\alpha,\beta,q}(z)$:

$$\begin{aligned} L_{\alpha,\beta}^{q}f(z)&=\left(\Gamma_q(\beta)z^{-p}E_{\alpha,\beta,q}(z)\right)*f(z)\\ &=z^{-p}+\sum_{n=1}^{\infty}\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)}a_{n-p}z^{n-p}, \end{aligned} \tag{1.5}$$

where $z,\alpha,\beta\in\mathbb{C}$, $q\in(0,+\infty)$ and $\mathrm{Re}(\alpha)>0$.

For functions f and g analytic in U, we say that f is subordinate to g, written $f\prec g$, if g is univalent in U, $f(0)=g(0)$ and $f(U)\subset g(U)$.

Let P be the class of functions h with $h(0)=1$, which are analytic and convex univalent in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.

In this paper we introduce and investigate the following subclass of $\Sigma(p)$.

DEFINITION.. A function $f\in\Sigma(p)$ is said to be in the class $G_{\alpha,\beta,q}(\lambda;h)$ if it satisfies the second order differential subordination

$$\frac{\lambda-1}{p}z^{p+1}\left(L_{\alpha,\beta}^{q}f(z)\right)'+\frac{\lambda}{p(p+1)}z^{p+2}\left(L_{\alpha,\beta}^{q}f(z)\right)''\prec h(z), \tag{1.6}$$

where $\lambda,\alpha,\beta\in\mathbb{C}$, $q\in(0,+\infty)$, $\mathrm{Re}(\alpha)>0$ and $h\in P$.

Let A be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in U. A function $f \in A$ is said to be in the class $S^*(\gamma)$ if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma$ $(z \in U)$ for some γ $(\gamma < 1)$. When $0 \leqslant \gamma < 1$, $S^*(\gamma)$ is the class of starlike functions of order γ in U. A function $f \in A$ is said to be prestarlike of order γ in U if

$$\frac{z}{(1-z)^{2(1-\gamma)}} * f(z) \in S^*(\gamma) \quad (\gamma < 1).$$

We denote this class by $R(\gamma)$ (see [6]). It is obvious that a function $f \in A$ is in the class $R(0)$ if and only if f is convex univalent in U and $R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$.

The study of the Mittag-Leffler function and its various generalizations is a recent interesting topic in geometric function theory. In this paper we shall make a further contribution to the subject by showing some convolution properties for meromorphic functions involving the generalized Mittag-Leffler function.

The following lemmas will be used in our present investigation.

LEMMA 1. ([5]) *Let g be analytic in U and h be analytic and convex univalent in U with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{1.7}$$

where $\operatorname{Re}\mu \geqslant 0$ *and* $\mu \neq 0$, *then*

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z)$$

and $\widetilde{h}$ is the best dominant of (1.7).

LEMMA 2. ([6]) *Let $\gamma < 1$, $f \in S^*(\gamma)$ and $g \in R(\gamma)$. Then, for analytic function F in U,*

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the closed convex hull of $F(U)$.

LEMMA 3. ([4]) *Let $g(z) = 1 + \sum_{n=m}^{\infty} b_n z^n$ $(m \in \mathbb{N})$ be analytic in U. If* $\operatorname{Re}(g(z)) >$ 0 $(z \in U)$, *then*

$$\operatorname{Re}(g(z)) \geqslant \frac{1-|z|^m}{1+|z|^m} \quad (z \in U).$$

2. Properties of the class $G_{\alpha,\beta,q}(\lambda;h)$

In this section we shall derive several convolution properties for functions in the class $G_{\alpha,\beta,q}(\lambda;h)$.

THEOREM 1. *Let $\lambda < 0$, $\gamma > 0$ and $f \in G_{\alpha,\beta,q}(\lambda;\gamma h + 1 - \gamma)$. If $\gamma \leqslant \gamma_0$, where*

$$\gamma_0 = \frac{1}{2}\left(1 + \frac{p+1}{\lambda}\int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u} du\right)^{-1}, \tag{2.1}$$

then $f \in G_{\alpha,\beta,q}(0;h)$. *The bound* γ_0 *is sharp when* $h(z) = \frac{1}{1-z}$.

Proof. Let

$$g(z) = -\frac{1}{p} z^{p+1} \left(L^q_{\alpha,\beta} f(z) \right)' \tag{2.2}$$

for $f \in G_{\alpha,\beta,q}(\lambda; \gamma h + 1 - \gamma)$ with $\lambda < 0$ and $\gamma > 0$. Then we have

$$\begin{aligned} g(z) - \frac{\lambda}{p+1} z g'(z) &= \frac{\lambda - 1}{p} z^{p+1} \left(L^q_{\alpha,\beta} f(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L^q_{\alpha,\beta} f(z) \right)'' \\ &\prec \gamma h(z) + 1 - \gamma. \end{aligned}$$

Hence an application of Lemma 1 yields

$$g(z) \prec -\frac{\gamma(p+1)}{\lambda} z^{\frac{p+1}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda} - 1} h(t) dt + 1 - \gamma = (h * \psi)(z), \tag{2.3}$$

where

$$\psi(z) = -\frac{\gamma(p+1)}{\lambda} z^{\frac{p+1}{\lambda}} \int_0^z \frac{t^{-\frac{p+1}{\lambda} - 1}}{1-t} dt + 1 - \gamma. \tag{2.4}$$

If $0 < \gamma \leqslant \gamma_0$, where $\gamma_0 (> 1)$ is given by (2.1), then it follows from (2.3) that

$$\begin{aligned} \mathrm{Re}(\psi(z)) &= -\frac{\gamma(p+1)}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda} - 1} \mathrm{Re}\left(\frac{1}{1-uz} \right) du + 1 - \gamma \\ &> -\frac{\gamma(p+1)}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda} - 1}}{1+u} du + 1 - \gamma \\ &\geqslant \frac{1}{2} \quad (z \in U). \end{aligned}$$

Now, by using the Herglotz representation for the function ψ, from (2.2) and (2.3) we arrive at

$$-\frac{1}{p} z^{p+1} \left(L^q_{\alpha,\beta} f(z) \right)' \prec (h * \psi)(z) \prec h(z)$$

because h is convex univalent in U. This shows that $f \in G_{\alpha,\beta,q}(0;h)$.

For the function $h(z) = \frac{1}{1-z}$ and $f \in \Sigma(p)$ defined by

$$-\frac{1}{p} z^{p+1} \left(L^q_{\alpha,\beta} f(z) \right)' = -\frac{\gamma(p+1)}{\lambda} z^{\frac{p+1}{\lambda}} \int_0^z \frac{t^{-\frac{p+1}{\lambda} - 1}}{1-t} dt + 1 - \gamma,$$

it is easy to verify that

$$\frac{\lambda - 1}{p} z^{p+1} \left(L^q_{\alpha,\beta} f(z) \right)' + \frac{\lambda}{p(p+1)} z^{p+2} \left(L^q_{\alpha,\beta} f(z) \right)'' = \gamma h(z) + 1 - \gamma.$$

Thus $f \in G_{\alpha,\beta,q}(\lambda; \gamma h + 1 - \gamma)$. Also, for $\gamma > \gamma_0$, we have

$$\mathrm{Re}\left\{ -\frac{1}{p} z^{p+1} \left(L^q_{\alpha,\beta} f(z) \right)' \right\} \longrightarrow -\frac{\gamma(p+1)}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda} - 1}}{1+u} du + 1 - \gamma < \frac{1}{2} \quad (z \to -1),$$

which implies that $f \notin G_{\alpha,\beta,q}(0;h)$. Hence the bound γ_0 cannot be increased when $h(z) = \frac{1}{1-z}$. The proof of the theorem is completed. □

THEOREM 2. *Let* $f \in G_{\alpha,\beta,q}(\lambda;h)$, $g \in \Sigma(p)$ *and* $\mathrm{Re}\,(z^p g(z)) > \frac{1}{2}$ $(z \in U)$. *Then* $f * g \in G_{\alpha,\beta,q}(\lambda;h)$.

Proof. For $f \in G_{\alpha,\beta,q}(\lambda;h)$ and $g \in \Sigma(p)$, we have

$$\begin{aligned}
&\frac{\lambda-1}{p} z^{p+1}\left(L^q_{\alpha,\beta}(f*g)(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2}\left(L^q_{\alpha,\beta}(f*g)(z)\right)'' \\
&= \frac{\lambda-1}{p}\,(z^p g(z)) * \left(z^{p+1}(L^q_{\alpha,\beta}f(z))'\right) + \frac{\lambda}{p(p+1)}\,(z^p g(z)) * \left(z^{p+2}(L^q_{\alpha,\beta}f(z))''\right) \\
&= (z^p g(z)) * \psi(z),
\end{aligned} \tag{2.5}$$

where

$$\psi(z) = \frac{\lambda-1}{p} z^{p+1}\left(L^q_{\alpha,\beta}f(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2}\left(L^q_{\alpha,\beta}f(z)\right)'' \prec h(z). \tag{2.6}$$

In view of the conditions of the theorem, the function $z^p g(z)$ has the Herglotz representation:

$$z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.7}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x|=1$ and $\int_{|x|=1} d\mu(x) = 1$. Since the function h is convex univalent in U, it follows from (2.5), (2.6) and (2.7) that

$$\begin{aligned}
&\frac{\lambda-1}{p} z^{p+1}\left(L^q_{\alpha,\beta}(f*g)(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2}\left(L^q_{\alpha,\beta}(f*g)(z)\right)'' \\
&= \int_{|x|=1} \psi(xz)d\mu(x) \prec h(z).
\end{aligned}$$

This shows that $f * g \in G_{\alpha,\beta,q}(\lambda;h)$ and the theorem is proved. □

THEOREM 3. *Let* $f \in G_{\alpha,\beta,q}(\lambda;h)$ *be given by (1.4) and let*

$$s_k(z) = z^{-p} + \sum_{n=1}^{k-1} a_{n-p} z^{n-p} \quad (k \in \mathbb{N} \setminus \{1\}).$$

Then the function

$$p_k(z) = z^{-p-1} \int_0^z u^p s_k(u)du$$

is also in the class $G_{\alpha,\beta,q}(\lambda;h)$.

Proof. For $f \in G_{\alpha,\beta,q}(\lambda;h)$, we have

$$p_k(z) = z^{-p} + \sum_{n=1}^{k-1} \frac{a_{n-p}}{n+1} z^{n-p} = (f * g_k)(z) \quad (k \in \mathbb{N} \setminus \{1\}), \tag{2.8}$$

where

$$g_k(z) = z^{-p} + \sum_{n=1}^{k-1} \frac{z^{n-p}}{n+1} \in \Sigma(p).$$

Further, for $k \in \mathbb{N} \setminus \{1\}$, it is know from [7] that

$$\operatorname{Re}\{z^p g_k(z)\} = \operatorname{Re}\left\{1 + \sum_{n=1}^{k-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \quad (z \in U). \tag{2.9}$$

In view of (2.8) and (2.9), an application of Theorem 2 leads to $p_k(z) \in G_{\alpha,\beta,q}(\lambda;h)$. The proof of the theorem is completed. □

THEOREM 4. *Let* $f \in G_{\alpha,\beta,q}(\lambda;h)$, $g \in \Sigma(p)$ *and* $z^{p+1}g(z) \in R(\gamma)$ $(\gamma < 1)$. *Then* $f * g \in G_{\alpha,\beta,q}(\lambda;h)$.

Proof. From (2.5) we can write

$$\begin{aligned} &\frac{\lambda-1}{p} z^{p+1}\left(L_{\alpha,\beta}^q (f*g)(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2}\left(L_{\alpha,\beta}^q (f*g)(z)\right)'' \\ &= \frac{\left(z^{p+1}g(z)\right) * (z\psi(z))}{(z^{p+1}g(z)) * z}, \end{aligned} \tag{2.10}$$

where the function ψ is defined as in (2.6).

Since the function h is convex univalent in U,

$$\psi(z) \prec h(z), \quad z^{p+1}g(z) \in R(\gamma) \quad \text{and} \quad z \in S^*(\gamma)\ (\gamma < 1),$$

from (2.10) and Lemma 2, we obtain the desired result. The proof of the theorem is completed. □

Taking $\gamma = 0$ and $\gamma = \frac{1}{2}$ in Theorem 4, we have the following.

COROLLARY. *Let* $f \in G_{\alpha,\beta,q}(\lambda;h)$ *and let* $g \in \Sigma(p)$ *satisfy either of the following conditions:*

(i) $z^{p+1}g(z)$ *is convex univalent in* U

or

(ii) $z^{p+1}g(z) \in S^*\left(\frac{1}{2}\right)$.

Then $f * g \in G_{\alpha,\beta,q}(\lambda;h)$.

THEOREM 5. *Let* $\lambda \leqslant 0$ *and*

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in G_{\alpha,\beta,q}(\lambda;h_j) \quad (j = 1,2), \tag{2.11}$$

where

$$h_j(z) = \gamma_j + (1-\gamma_j)\frac{1+z}{1-z} \quad \text{and} \quad \gamma_j < 1. \tag{2.12}$$

If $f \in \Sigma(p)$ *is defined by*

$$\left(L^q_{\alpha,\beta} f(z)\right)' = -\frac{1}{p}\left(\left(L^q_{\alpha,\beta} f_1(z)\right)' * \left(L^q_{\alpha,\beta} f_2(z)\right)'\right), \tag{2.13}$$

then $f \in G_{\alpha,\beta,q}(\lambda;h)$, *where*

$$h(z) = \gamma + (1-\gamma)\frac{1+z}{1-z} \tag{2.14}$$

and γ *is given by*

$$\gamma = \begin{cases} 1-4(1-\gamma_1)(1-\gamma_2)\left(1+\frac{p+1}{\lambda}\int_0^1 \frac{u^{-\frac{p+1}{\lambda}-1}}{1+u}du\right) & (\lambda<0), \\ 1-4(1-\gamma_1)(1-\gamma_2) & (\lambda=0). \end{cases} \tag{2.15}$$

The bound γ *is the best possible.*

Proof. We consider the case when $\lambda < 0$. By setting

$$H_j(z) = \frac{\lambda-1}{p} z^{p+1}\left(L^q_{\alpha,\beta} f_j(z)\right)' + \frac{\lambda}{p(p+1)} z^{p+2}\left(L^q_{\alpha,\beta} f_j(z)\right)'' \quad (j=1,2)$$

for $f_j \ (j=1,2)$ given by (2.11), we find that

$$H_j(z) = 1 + \sum_{n=1}^{\infty} b_{n,j} z^n \prec \gamma_j + (1-\gamma_j)\frac{1+z}{1-z} \quad (j=1,2) \tag{2.16}$$

and

$$\left(L^q_{\alpha,\beta} f_j(z)\right)' = \frac{p(p+1)}{\lambda} z^{\frac{(1-\lambda)(p+1)}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda}-1} H_j(t)dt \quad (j=1,2). \tag{2.17}$$

Now, if $f \in \Sigma(p)$ is defined by (2.13), we find from (2.17) that

$$\begin{aligned} \left(L^q_{\alpha,\beta} f(z)\right)' &= -\frac{1}{p}\left(\left(L^q_{\alpha,\beta} f_1(z)\right)' * \left(L^q_{\alpha,\beta} f_2(z)\right)'\right) \\ &= -\frac{1}{p}\left(\frac{p(p+1)}{\lambda} z^{-p-1}\int_0^1 u^{-\frac{p+1}{\lambda}-1} H_1(uz)du\right) \\ &\quad * \left(\frac{p(p+1)}{\lambda} z^{-p-1}\int_0^1 u^{-\frac{p+1}{\lambda}-1} H_2(uz)du\right) \\ &= \frac{p(p+1)}{\lambda} z^{-p-1}\int_0^1 u^{-\frac{p+1}{\lambda}-1} H(uz)du, \end{aligned} \tag{2.18}$$

where

$$H(z) = -\frac{p+1}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}(H_1 * H_2)(uz)du. \tag{2.19}$$

Also, by using (2.16) and the Herglotz theorem, we see that

$$\mathrm{Re}\left\{\left(\frac{H_1(z)-\gamma_1}{1-\gamma_1}\right)*\left(\frac{1}{2}+\frac{H_2(z)-\gamma_2}{2(1-\gamma_2)}\right)\right\}>0 \quad (z\in U),$$

which leads to

$$\mathrm{Re}\{(H_1*H_2)(z)\}>\gamma_0=1-2(1-\gamma_1)(1-\gamma_2) \quad (z\in U).$$

According to Lemma 3, we have

$$\mathrm{Re}\{(H_1*H_2)(z)\}\geqslant \gamma_0+(1-\gamma_0)\frac{1-|z|}{1+|z|} \quad (z\in U). \tag{2.20}$$

Now it follows from (2.18) to (2.20) that

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{\lambda-1}{p}z^{p+1}\left(L^q_{\alpha,\beta}f(z)\right)'+\frac{\lambda}{p(p+1)}z^{p+2}\left(L^q_{\alpha,\beta}f(z)\right)''\right\}\\
&=\mathrm{Re}\{H(z)\}\\
&=-\frac{p+1}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\mathrm{Re}\{(H_1*H_2)(uz)\}du\\
&\geqslant -\frac{p+1}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\left(\gamma_0+(1-\gamma_0)\frac{1-u|z|}{1+u|z|}\right)du\\
&>\gamma_0-\frac{(p+1)(1-\gamma_0)}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\frac{1-u}{1+u}du\\
&=1-4(1-\gamma_1)(1-\gamma_2)\left(1+\frac{p+1}{\lambda}\int_0^1\frac{u^{-\frac{p+1}{\lambda}-1}}{1+u}du\right)\\
&=\gamma,
\end{aligned}$$

which proves that $f\in G_{\alpha,\beta,q}(\lambda;h)$ for the function h given by (2.14).

In order to show that the bound γ is sharp, we take the functions $f_j\in\Sigma(p)$ $(j=1,2)$ defined by

$$\left(L^q_{\alpha,\beta}f_j(z)\right)'=\frac{p(p+1)}{\lambda}z^{\frac{(1-\lambda)(p+1)}{\lambda}}\int_0^z t^{-\frac{p+1}{\lambda}-1}\left(\gamma_j+(1-\gamma_j)\frac{1+t}{1-t}\right)dt \quad (j=1,2), \tag{2.21}$$

for which we have

$$\begin{aligned}
H_j(z)&=\frac{\lambda-1}{p}z^{p+1}\left(L^q_{\alpha,\beta}f_j(z)\right)'+\frac{\lambda}{p(p+1)}z^{p+2}\left(L^q_{\alpha,\beta}f_j(z)\right)''\\
&=\gamma_j+(1-\gamma_j)\frac{1+z}{1-z} \quad (j=1,2)
\end{aligned}$$

and

$$(H_1*H_2)(z)=1+4(1-\gamma_1)(1-\gamma_2)\frac{z}{1-z}.$$

Hence, for the function f given by (2.13), we have

$$\begin{aligned}&\frac{\lambda-1}{p}z^{p+1}\left(L^{q}_{\alpha,\beta}f(z)\right)'+\frac{\lambda}{p(p+1)}z^{p+2}\left(L^{q}_{\alpha,\beta}f(z)\right)''\\&=-\frac{p+1}{\lambda}\int_0^1 u^{-\frac{p+1}{\lambda}-1}\left(1+4(1-\gamma_1)(1-\gamma_2)\frac{uz}{1-uz}\right)du\to\gamma\quad(\text{as } z\to-1).\end{aligned}$$

Finally, for the case when $\lambda=0$, the proof of the theorem is simple and we choose to omit the details involved. □

Acknowledgement. This work is supported by National Natural Science Foundation of China (Grant No. 11571299) and Natural Science Foundation of Jiangsu Province (Grant No. BK20151304).

REFERENCES

[1] A. A. ATTIYA, *Some applications of Mittag-Leffler function in the unit disk*, Filomat **30** (2016), 2075–2081.

[2] D. BANSAL AND J. K. PRAJAPAT, *Certain geometric properties of the Mittag-Leffler functions*, Complex Var. Elliptic Equ. **61** (2016), 338–350.

[3] M. GRAG, P. MANOHAR AND S. L. KALLA, *A Mittag-Leffler-type function of two variables*, Integral Transforms Spec. Funct. **24** (2013), 934–944.

[4] T. H. MACGREGOR, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), 532–537.

[5] S. S. MILLER AND P. T. MOCANU, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–171.

[6] S. RUSCHEWEYH, *Convolutions in Geometric Function Theory*, Les Presses de 1'Université de Montréal, Montréal, 1982.

[7] R. SINGH AND S. SINGH, *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc. **106** (1989), 145–152.

[8] H. M. SRIVASTAVA, *Some families of Mittag-Leffler type functions and associated operators of fractional calculus*, TWMS J. Pure Appl. Math. **7** (2016), 123–145.

[9] H. M. SRIVASTAVA AND D. BANSAL, *Close-to-convexity of a certain family of q-Mittag-Leffler functions*, J. Nonlinear Var. Anal. **1** (2017), 61–69.

[10] H. M. SRIVASTAVA, B. A. FRASIN AND V. PESCAR, *Univalence of integral operators involving Mittag-Leffler functions*, Appl. Math. Inf. Sci. **11** (2017), 635–641.

[11] H. M. SRIVASTAVA AND Ž. TOMOVSKI, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernal*, Appl. Math. Comput. **211** (2009), 198–210.

[12] L. YIN AND L.-G. HUANG, *Turán type inequalities for generalized Mittag-Leffler function*, J. Math. Inequal. **13** (2017), 667–672.

(Received July 22, 2017)

Cai-Mei Yan
Information Engineering College
Yangzhou University
Yangzhou, 225002, China
e-mail: cmyan@yzu.edu.cn

Jin-Lin Liu
Department of Mathematics
Yangzhou University
Yangzhou 225002, China
e-mail: jlliu@yzu.edu.cn