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DURRMEYER TYPE (p,q)–BASKAKOV OPERATORS PRESERVING LINEAR FUNCTIONS

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Abstract. The present paper deals with the construction of Baskakov Durrmeyer operators, which preserve the linear functions, in (p,q)-calculus. More precisely, using (p,q)-Gamma function we introduce genuine mixed type Baskakov Durrmeyer operators having Baskakov and Szász basis functions. After construction of the operators and calculations of their moments and central moments, rate of convergence of the operators by means of appropriate modulus of continuity, approximation behaviors for functions belong to Lipschitz class and weighted approximation are explored.

1. Introduction

In the theory of approximation by linear positive operators, no doubtly, Bernstein polynomials have a crucial role. Beyond the fact that they are first sample of the theory, they have been still taken much attention because of their simple, useful and applicable structure. However, Bernstein polynomials have role for approximation of functions on bounded intervals. As a generalization of Bernstein polynomials, in 1932 Chlodowsky [10] introduced Bernstein type polynomials, however the intervals of approximating functions are depend on increasing sequence of positive numbers. To approximate the functions by linear positive operators on unbounded intervals, in 1957 Baskakov [8] introduced the sequence of linear positive operators, defined for functions $f \in C[0,\infty)$, by

$$B_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \in [0,\infty).$$

The Baskakov operators and their generalizations have been intensively studied. Afterward the construction of Bernstein type polynomials by Durrmeyer [12], which allows us to approximate the Lebesgue integrable functions, this construction was carried over to other many well-known sequence of linear positive operators. We can mention, for instance, the Szász-Durrmeyer operators were introduced by Mazhar and Totik [19] and the Baskakov-Durrmeyer operators were introduced by Sagai and Prasad [26].

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Beside the extensions of the class of approximating functions, another main propose of the researches in approximation theory is to construct modification of corresponding operators which presents better approach than the existing one. At this point, the construction of Bernstein polynomials by Phillips [24] is the pioneer of approximation by linear positive operators in quantum calculus. Along last two decades, q-analogues of the sequence of linear positive operators have been studied deeply and the advantages of q-analogues of the operators have been investigated. The researches in quantum calculus show that the rate of convergence of q-analogues of corresponding operators is at least as good as the classical ones.

Very recently, around two years, approximation by linear positive operators have been intensively started to study in post-quantum calculus. As q-analogues of linear positive operators present better degree of approximation than classical ones, their (p,q)-analogues also present better degree than q-analogues one. The pioneer of the studies of approximation by linear positive operators is Mursaleen and his research group. For more details of the comprehensive literature information about (p,q)-analogues of linear positive operators, among the others, we can refer the readers to: (p,q)-Bernstein operators by Mursaleen et al. [21] (see also [22]), (p,q)-Lorentz polynomials on a compact disk by Mursaleen et al. [23], (p,q)-Szász-Mirakyan operators by Acar [1], (p,q)-Baskakov-Kantorovich operators by Acar et al. [2], King type (p,q)-Szász-Mirakyan operators by Acar et al. [4], bivariate (p,q)-Bernstein-Kantorovich operators by Acar et al. [3], (p,q)-Meyer-König-Zeller operators by Maurya et al. [18] and references therein.

One of the other recent study in this direction is (p,q)-analogue of Baskakov-Durrmeyer operators having Baskakov and Szász basis functions by Acar et al. [5]. The aim of this paper is to construct the another modification of (p,q)-analogue of Baskakov-Durrmeyer operators which preserve the linear functions. Such a construction of Baskakov-Durrmeyer operators present a better degree of approximation than classical ones and q-analogues ones. The paper is organized as follows. Section 2 contains the basic concepts of (p,q)-calculus, construction of the operators and some preliminary results such as moments and central moments of the operators. The fundamental concepts of approximation theory which will be used throughout the paper are presented as well. The Section 3 is devoted to approximation properties of new operators. The uniform convergence of the operators, rate of convergence by means of modulus of continuity, weighted approximation properties and approximation behaviors of the operators for the functions belong to Lipschitz classes are studied.

2. Preliminaries and construction of the operators

Let us first recall some fundamental notations of (p,q)-calculus which can be found in the papers [14, 15, 25]. Set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, The (p,q)-numbers are defined for $n \in \mathbb{N}$ as

$$[n]_{p,q} := p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}.$$

The (p,q)-factorial of a natural number n is given by

$$[n]_{p,q}! := \begin{cases} [n]_{p,q}[n-1]_{p,q}\dots 1, & n \geqslant 1, \\ 1, & n = 0 \end{cases}$$

and the (p,q)-binomial is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}, \quad n \geqslant k \geqslant 0.$$

Further, the (p,q)-power basis is defined

$$(x \oplus a)^n_{p,q} = (x+a)(px+qa)(p^2x+q^2a)\cdots(p^{n-1}x+q^{n-1}a),$$

and

$$(x \ominus a)^n_{p,q} = (x-a)(px-qa)(p^2x-q^2a)\cdots(p^{n-1}x-q^{n-1}a).$$

Also the (p,q)-derivative of a function f, denoted by $D_{p,q}f$, is defined by

$$(D_{p,q}f)(x) := \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0, \quad (D_{p,q}f)(0) := f'(0)$$

provided that f is differentiable at 0. The formula for the (p,q)-derivative of a product is

$$D_{p,q}(u(x)v(x)) := D_{p,q}(u(px))v(x) + D_{p,q}(u(x))v(qx).$$

There are two (p,q)-analogues of the exponential function, see [15],

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!}, \tag{2.1}$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions. Further, it is obvious that by (p,q)-derivative formula that

$$\begin{aligned} D_{p,q}E_{p,q}(x) &= E_{p,q}(qx), \\ D_{p,q}E_{p,q}(ax) &= aE_{p,q}(aqx). \end{aligned}$$

To introduce Durrmeyer type generalizations of (p,q)-operators, one would need to (p,q)-analogues of well-known Beta and Gamma functions. The (p,q)-analogue of well-known Gamma function has been recently introduced in [6] as

$$\Gamma_{p,q}(n) = \int_0^{\infty} p^{(n-1)(n-2)/2} x^{n-1} E_{p,q}(-qx)\, d_{p,q}x \tag{2.2}$$

and it was shown in [6] that

$$\Gamma_{p,q}(n+1) = [n]_{p,q}!. \tag{2.3}$$

Considering the (p,q)-Gamma function, they introduced Durrmeyer type generalization of (p,q)-Szász-Mirakyan operators defined in [1].

Further, another (p,q) analogue of most important operators of approximation theory, that is, Baskakov operators were introduced by Aral et al. [7] as

$$B_{n,p,q}(f;x)=\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) f\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \tag{2.4}$$

where $x\in[0,\infty)$, $0<q<p\leqslant 1$ and

$$b_{n,k}^{p,q}(x)=\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} p^{k+n(n-1)/2} q^{k(k-1)/2} \frac{x^k}{(1\oplus x)_{p,q}^{n+k}}.$$

In the recent paper, Acar et al. [5] introduced Durrmeyer modification of (p,q)-Baskakov operators having Baskakov and Szász basis functions as

$$\mathscr{B}_n^{p,q}(f;x)=[n]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)\int_0^{\infty}\frac{\left(p^{(k-1)/2}[n]_{p,q}t\right)^k}{[k]_{p,q}!}E_{p,q}(-q[n]_{p,q}t)\, f\left(\frac{p^{k+n-1}t}{q^{k-1}}\right) d_{p,q}t. \tag{2.5}$$

In the present paper, we consider the following modifications of the operators (2.5) by

$$\begin{aligned} G_n^{p,q}(f;x) = {} & [n]_{p,q}\sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\int_0^{\infty} p^{(k-1)(k-2)/2}\frac{\left([n]_{p,q}t\right)^{k-1}}{[k-1]_{p,q}!}E_{p,q}(-q[n]_{p,q}t) \\ & \times f\left(q^{1-k}p^{k+n-2}t\right) d_{p,q}t + \frac{p^{n(n-1)/2}}{(1\oplus x)_{p,q}^{n}} f(0), \end{aligned} \tag{2.6}$$

which reproduce the linear functions.

3. Auxiliary results

In this section, we calculate the moments, and upper estimate for second central moment. We recall some notations which will be used throughout the paper. Some lemmas are also proved.

LEMMA 1. ([7]) *Starting with the following relations between (p,q)-calculus and q-calculus:*

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} = p^{k(n-1)} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q/p}$$

and

$$(x\oplus a)_{p,q}^n = p^{n(n-1)/2}(x+a)_{q/p}^n$$

and using moments of q-Baskakov operators, it can easily be verified by simple computation that

$$B_{n,p,q}(1;x)=1, B_{n,p,q}(t;x)=x, B_{n,p,q}(t^2;x)=x^2+\frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right).$$

LEMMA 2. *For* $x \in [0,\infty)$, $0 < q < p \leqslant 1$, *we have*

$$G_n^{p,q}(1;x) = 1,$$
$$G_n^{p,q}(t;x) = x,$$
$$G_n^{p,q}\left(t^2;x\right) = x^2\left(1+\frac{p^n}{q[n]_{p,q}}\right)+x\frac{p^{n-2}q}{[n]_{p,q}}(p+q)$$

Proof. Using the definition of the operators (2.6) we immediately have

$$\begin{aligned}
G_n^{p,q}(1;x) &= [n]_{p,q}\sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\int_0^{\infty} p^{(k-1)(k-2)/2}\frac{([n]_{p,q}t)^{k-1}}{[k-1]_{p,q}!}E_{p,q}(-q[n]_{p,q}t)\,d_{p,q}t \\
&\quad+\frac{p^{n(n-1)/2}}{(1\oplus x)_{p,q}^n} \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{\Gamma(k)}{[k-1]_{p,q}!}+\frac{p^{n(n-1)/2}}{(1\oplus x)_{p,q}^n} \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)+\frac{p^{n(n-1)/2}}{(1\oplus x)_{p,q}^n} \\
&= \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) = 1, \\
G_n^{p,q}(t;x) &= [n]_{p,q}\sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\int_0^{\infty} p^{(k-1)(k-2)/2}\frac{([n]_{p,q}t)^{k-1}}{[k-1]_{p,q}!} \\
&\quad\times E_{p,q}(-q[n]_{p,q}t)\left(q^{1-k}p^{k+n-2}t\right)d_{p,q}t \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)q^{1-k}p^{n-1}\int_0^{\infty} p^{k(k-1)/2}\frac{([n]_{p,q}t)^{k}}{[k-1]_{p,q}!}E_{p,q}(-q[n]_{p,q}t)\,d_{p,q}t \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)q^{1-k}p^{n-1}\frac{\Gamma(k+1)}{[n]_{p,q}[k-1]_{p,q}!} \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)q^{1-k}p^{n-1}\frac{[k]_{p,q}}{[n]_{p,q}} \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{p^{n-1}}{q^{k-1}}\frac{[k]_{p,q}}{[n]_{p,q}} \\
&= x.
\end{aligned}$$

Also using the equality $[k+1]_{p,q} = q^k + p[k]_{p,q}$ and Lemma 1 we have

$$\begin{aligned}
G_n^{p,q}\left(t^2;x\right) &= [n]_{p,q}\sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\int_0^{\infty} p^{(k-1)(k-2)/2}\frac{([n]_{p,q}t)^{k-1}}{[k-1]_{p,q}!} \\
&\quad\times E_{p,q}(-q[n]_{p,q}t)\left(q^{2-2k}p^{2k+2n-4}t^2\right)d_{p,q}t \\
&= \frac{1}{[n]_{p,q}}\sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)q^{2-2k}p^{2n-3}\int_0^{\infty} p^{k(k+1)/2}\frac{([n]_{p,q}t)^{k+1}}{[k-1]_{p,q}!}E_{p,q}(-q[n]_{p,q}t)\,d_{p,q}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{[n]_{p,q}^2} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-2k} p^{2n-3} \frac{\Gamma(k+2)}{[k-1]_{p,q}!} \\
&= \frac{1}{[n]_{p,q}^2} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-2k} p^{2n-3} [k]_{p,q} [k+1]_{p,q} \\
&= \frac{1}{[n]_{p,q}^2} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-2k} p^{2n-3} [k]_{p,q} \left(q^k + p[k]_{p,q}\right) \\
&= \frac{1}{[n]_{p,q}^2} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-2k} p^{2n-3} [k]_{p,q} q^k \\
&\quad + \frac{1}{[n]_{p,q}^2} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-2k} p^{2n-3} [k]_{p,q} p[k]_{p,q} \\
&= \frac{1}{[n]_{p,q}^2} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-k} p^{2n-3} [k]_{p,q} + \frac{p}{[n]_{p,q}^2} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-2k} p^{2n-3} [k]_{p,q}^2 \\
&= \frac{1}{[n]_{p,q}} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-k} p^{2n-3} \frac{[k]_{p,q}}{[n]_{p,q}} + p \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) q^{2-2k} p^{2n-3} \frac{[k]_{p,q}^2}{[n]_{p,q}^2} \\
&= \frac{p^{n-2} q}{[n]_{p,q}} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) \frac{p^{n-1}}{q^{k-1}} \frac{[k]_{p,q}}{[n]_{p,q}} + \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) \frac{p^{2n-2}}{q^{2k-2}} \frac{[k]_{p,q}^2}{[n]_{p,q}^2} \\
&= \frac{p^{n-2} q}{[n]_{p,q}} x + x^2 + \frac{p^{n-1} x}{[n]_{p,q}} \left(1 + \frac{p}{q} x\right). \quad \square
\end{aligned}
$$

COROLLARY 1. *Using Lemma 2 we set*

$$
G_n^{p,q}(t-x;x) = 0, \tag{3.1}
$$

$$
\begin{aligned}
G_n^{p,q}\left((t-x)^2;x\right) &= x^2 \frac{p^n}{q[n]_{p,q}} + x \frac{p^{n-2} q}{[n]_{p,q}} (p+q) \\
&\leqslant \frac{2x^2}{q[n]_{p,q}} + \frac{2x}{[n]_{p,q}} \\
&\leqslant \frac{2x(1+x)}{q[n]_{p,q}} \leqslant \frac{2(1+x)^2}{q[n]_{p,q}}.
\end{aligned} \tag{3.2}
$$

4. Quantitative results

Let $C_B[0,\infty)$ denote the space of all real valued continuous and bounded functions on $[0,\infty)$. In this space we consider the norm

$$
\|f\|_{C_B} = \sup_{x \in [0,\infty)} |f(x)|.
$$

Let us consider the following $\mathscr{K}$-functional:

$$
\mathscr{K}_2(f,\delta) = \inf_{g \in W^2} \left\{ \|f-g\|_{C_B} + \delta \|g''\|_{C_B} \right\},
$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [11, p. 177, Theorem 2.4] there exists an absolute constant $C > 0$ such that

$$\mathscr{K}_2(f,\delta) \leqslant C\omega_2\left(f,\sqrt{\delta}\right), \tag{4.1}$$

where

$$\omega_2(f,\delta) = \sup_{0<h\leqslant\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leqslant\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

LEMMA 3. *Let $p,q \in (0,1)$ such that $0 < q < p \leqslant 1$. Then the inequality*

$$|G_n^{p,q}(f;x)| \leqslant \|f\|_{C_B}$$

holds, which implies the sequence of the operators $G_n^{p,q}$ acts from $C_B[0,\infty)$ into $C_B[0,\infty)$.

Proof. Using the definition of the operators $G_n^{p,q}$ and Lemma 2 we obtain

$$\begin{aligned}
|G_n^{p,q}(f;x)| &\leqslant [n]_{p,q} \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x) \int_0^{\infty} p^{(k-1)(k-2)/2} \frac{([n]_{p,q}t)^{k-1}}{[k-1]_{p,q}!} \\
&\quad \times E_{p,q}(-q[n]_{p,q}t) \left| f\left(q^{1-k}p^{k+n-2}t\right)\right| d_{p,q}t + \frac{p^{n(n-1)/2}}{(1\oplus x)_{p,q}^n} |f(0)| \\
&\leqslant \sup_{x\in[0,\infty)} |f(x)|\, G_n^{p,q}(1;x) \\
&= \|f\|_{C_B}
\end{aligned}$$

which is desired. $\square$

For $f \in C_B[0,\infty)$ the Steklov mean is defined by

$$f_h(x) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} [2f(x+u+v) - f(x+2(u+v))]\, du dv.$$

Hence the following holds:

$$\|f_h - f\|_{C_B} \leqslant \tilde{\omega}_2(f,h) \tag{4.2}$$

If f is continuous, then $f_h', f_h'' \in C_B[0,\infty)$ and

$$\|f_h'\|_{C_B} \leqslant \frac{5}{h}\omega(f;h), \tag{4.3}$$

$$\|f_h''\|_{C_B} \leqslant \frac{9}{h^2}\tilde{\omega}_2(f,h), \tag{4.4}$$

where $\omega(\cdot;h)$ and $\tilde{\omega}_2(\cdot,h)$ are the first and second order modulus of continuity and respectively given by

$$\omega(f;h) = \sup_{\substack{x,u,v\geqslant 0\\ |u-v|<h}} |f(x+u) - f(x+v)|,$$

$$\tilde{\omega}_2(\cdot,h) = \sup_{\substack{x,u,v\geqslant 0\\ |u-v|<h}} |f(x+2u)-2f(x+u+v)+f(x+2v)|, \quad h\geqslant 0.$$

THEOREM 1. *Let* $p,q\in(0,1)$. *Then the inequality*

$$|G_n^{p,q}(f;x)-f(x)| \leqslant \frac{9}{2}\tilde{\omega}_2\left(f,\sqrt{1/[n]_{p,q}}\right)\left[2+\frac{2(1+x)^2}{q}\right],$$

holds for $f\in C_B[0,\infty)$.

Proof. Using Steklov function, for $x\geqslant 0$ and $n\in\mathbb{N}$ we have

$$|G_n^{p,q}(f;x)-f(x)| \leqslant G_n^{p,q}(|f-f_h|;x)+|G_n^{p,q}(f-f_h(x);x)|+|f_h(x)-f(x)|.$$

Considering the inequality (4.2) we get

$$\begin{aligned} G_n^{p,q}(|f-f_h|;x) &\leqslant \|G_n^{p,q}(f-f_h)\|_{C_B}\\ &\leqslant \|f-f_h\|_{C_B}\\ &\leqslant \tilde{\omega}_2(f,h). \end{aligned}$$

On the other hand, since $G_n^{p,q}$ are linear positive operators, by Taylor's expansion we obtain

$$|G_n^{p,q}(f-f_h(x);x)| \leqslant \left|f_h'(x)\right| G_n^{p,q}((t-x);x)+\frac{1}{2}\left\|f_h''\right\|_{C_B} G_n^{p,q}\left((t-x)^2;x\right).$$

By Lemma 2 we have

$$|G_n^{p,q}(f-f_h(x);x)| \leqslant \frac{9}{2h^2}\tilde{\omega}_2(f,h)\,G_n^{p,q}\left((t-x)^2;x\right)$$

and using the inequality (3.2), choosing $h=\sqrt{1/[n]_{p,q}}$ we obtain

$$|G_n^{p,q}(f-f_h(x);x)| \leqslant \frac{9}{2}\tilde{\omega}_2\left(f,\sqrt{1/[n]_{p,q}}\right)\left[2+\frac{2(1+x)^2}{q}\right],$$

which is desired. $\square$

THEOREM 2. *Let* $f\in C_B[0,\infty)$. *Then for every* $x\in[0,\infty)$, *there exists a constant* $C>0$ *such that*

$$|G_n^{p,q}(f;x)-f(x)| \leqslant C\omega_2\left(f,\frac{\sqrt{2}(1+x)}{\sqrt{q[n]_{p,q}}}\right).$$

Proof. Since $g\in C_B^2[0,\infty)$, using the Taylor's expansion for $x\in[0,\infty)$ we have

$$g(t)=g(x)+g(x)(t-x)+\int_x^t (t-u)g''(u)\,du.$$

Applying the operators $G_n^{p,q}$ to both sides of above equality and considering the fact (3.1) we obtain

$$G_n^{p,q}(g;x)-g(x)=G_n^{p,q}\left(\int_x^t (t-u)\,g''(u)\,du;x\right). \tag{4.5}$$

Also we get

$$\left|\int_x^t (t-u)\,g''(u)\,du\right|\leqslant\left|\int_x^t |t-u|\,|g''(u)|\,du\right|\leqslant\|g''\|_{C_B}\left|\int_x^t |t-u|\,du\right|\leqslant\|g''\|_{C_B}(t-x)^2. \tag{4.6}$$

Using the inequalities (4.6) in (4.5) and considering the inequality (3.2) we immediately have

$$|G_n^{p,q}(g;x)-g(x)|\leqslant\|g''\|_{C_B}\frac{2(1+x)^2}{q[n]_{p,q}}.$$

On the other hand, using Lemma 3 we get

$$\begin{aligned}|G_n^{p,q}(f;x)-f(x)| &\leqslant |G_n^{p,q}(f-g;x)-(f-g)(x)|+|G_n^{p,q}(g;x)-g(x)|\\ &\leqslant 2\|f-g\|_{C_B}+\|g''\|_{C_B}\frac{2(1+x)^2}{q[n]_{p,q}}\end{aligned}$$

and passing to infimum over all $g\in W^2$ on the right hand side, the desired results is obtained. □

THEOREM 3. *Let $0<\alpha\leqslant 1$ and E be any subset of the interval $[0,\infty)$. Then, if $f\in C_B[0,\infty)$ is locally $Lip(\alpha)$, i.e., the condition*

$$|f(y)-f(x)|\leqslant L|y-x|^{\alpha},\ \ y\in E \text{ and } x\in[0,\infty) \tag{4.7}$$

holds, then for each $x\in[0,\infty)$ we have

$$|G_n^{p,q}(f;x)-f(x)|\leqslant L\left\{\frac{2^{\alpha/2}(1+x)^{\alpha}}{(q[n]_{p,q})^{\alpha/2}}+2(d(x,E))^{\alpha}\right\},$$

where L is a constant depending on α and f; and $d(x,E)$ is the distance between x and E defined by

$$d(x,E)=\inf\{|t-x|:t\in E\}.$$

Proof. Let $\bar{E}$ denote the closure of E in $[0,\infty)$. Then, there exists a point $x_0\in\bar{E}$ such that $|x-x_0|=d(x,E)$. Using the triangle inequality

$$|f(t)-f(x)|\leqslant|f(t)-f(x_0)|+|f(x)-f(x_0)|$$

we immediately have by (4.7) that

$$\begin{aligned}|G_n^{p,q}(f;x)-f(x)| &\leqslant G_n^{p,q}(|f(t)-f(x_0)|;x)+G_n^{p,q}(|f(x)-f(x_0)|;x)\\ &\leqslant L\left\{G_n^{p,q}\left(|t-x_0|^{\alpha};x\right)+|x-x_0|^{\alpha}\right\}\end{aligned}$$

$$\begin{aligned}&\leqslant L\left\{G_n^{p,q}\left(|t-x|^\alpha+|x-x_0|^\alpha;x\right)+|x-x_0|^\alpha\right\}\\&=L\left\{G_n^{p,q}\left(|t-x|^\alpha;x\right)+2|x-x_0|^\alpha\right\}.\end{aligned}$$

Using Hölder inequality with $p=2/\alpha,\ q=2/(2-\alpha)$, we obtain

$$\begin{aligned}\left|G_n^{p,q}(f;x)-f(x)\right|&\leqslant L\left\{\left[G_n^{p,q}\left(|t-x|^{\alpha p};x\right)\right]^{\frac{1}{p}}+2\left(d(x,E)\right)^\alpha\right\}\\&=L\left\{\left[G_n^{p,q}\left(|t-x|^2;x\right)\right]^{\frac{\alpha}{2}}+2\left(d(x,E)\right)^\alpha\right\}\\&\leqslant L\left\{\frac{2^{\alpha/2}(1+x)^\alpha}{\left(q[n]_{p,q}\right)^{\alpha/2}}+2\left(d(x,E)\right)^\alpha\right\}.\qquad\square\end{aligned}$$

Next we obtain the local direct estimate of the operators $\mathscr{B}_n^{p,q}$, using the Lipcshitz type maximal function of order α introduced by Lenze [17] as

$$\tilde{\omega}_a(f,x)=\sup_{t\neq x,t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^\alpha},\quad x\in[0,\infty)\text{ and }\alpha\in(0,1].\tag{4.8}$$

THEOREM 4. *Let $f\in C_B[0,\infty)$ and $0<\alpha\leqslant 1$. Then, for all $x\in[0,\infty)$ we have*

$$\left|G_n^{p,q}(f;x)-f(x)\right|\leqslant\tilde{\omega}_a(f,x)\frac{2^{\alpha/2}(1+x)^\alpha}{\left(q[n]_{p,q}\right)^{\alpha/2}}.$$

Proof. From the Eq. (4.8), we have

$$\left|G_n^{p,q}(f;x)-f(x)\right|\leqslant\tilde{\omega}_a(f,x)\,G_n^{p,q}\left(|t-x|^\alpha;x\right).$$

Applying the Hölder inequality with $p=2/\alpha,\ q=2/(2-\alpha)$, we get

$$\begin{aligned}\left|G_n^{p,q}(f;x)-f(x)\right|&\leqslant\tilde{\omega}_a(f,x)\left[G_n^{p,q}\left(|t-x|^2;x\right)\right]^{\frac{\alpha}{2}}\\&\leqslant\tilde{\omega}_a(f,x)\frac{2^{\alpha/2}(1+x)^\alpha}{\left(q[n]_{p,q}\right)^{\alpha/2}}.\qquad\square\end{aligned}$$

5. Weighted approximation

First, let us recall the definitions of weighted spaces and corresponding modulus of continuity. Let $C[0,\infty)$ be the set of all continuous functions f defined on $[0,\infty)$ and $B_{x^2}[0,\infty)$ the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)|\leqslant M\left(1+x^2\right)$ with some positive constant M which may depend only on f. $C_{x^2}[0,\infty)$ denotes the subspace of all continuous functions in $B_{x^2}[0,\infty)$. By $C_{x^2}^*[0,\infty)$, we denote the subspace of all functions $f\in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. $B_{x^2}[0,\infty)$ is a linear normed space with the norm $\|f\|_{x^2}=\sup_{x\geqslant 0}\frac{|f(x)|}{1+x^2}$.

Throughout this section we assume $p=p_n$ and $q=q_n$ satisfies $0<q_n<p_n\leqslant 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $q_n^n\to 1$ and $p_n^n\to 1$.

THEOREM 5. *Let* $p_n, q_n \in (0,1)$ *such that* $0 < q_n < p_n \leqslant 1$ *and for* n *sufficiently large* $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$, $q_n^n \to 1$. *Then,*

$$\lim_{n\to\infty} \|G_n^{p_n,q_n} f - f\|_{x^2} = 0,$$

for all $f \in C_{x^2}[0,\infty)$.

Proof. By weighted Korovkin theorem, it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|G_n^{p_n,q_n}(e_i) - e_i\|_{x^2} = 0, \;\; i = 0,1,2.$$

By Lemma 2, we get

$$G_n^{p_n,q_n}(e_0,x) - e_0(x) = 0,$$
$$G_n^{p_n,q_n}(e_1,x) - e_1(x) = 0.$$

Also we obtain

$$\begin{aligned}\|G_n^{p_n,q_n}(e_2) - e_2\|_{x^2} &= \frac{p_n^n}{q_n[n]_{p_n,q_n}} \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} + \frac{p_n^{n-2}q_n}{[n]_{p_n,q_n}}(p_n+q_n) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &\leqslant \frac{p_n^n}{q_n[n]_{p_n,q_n}} + \frac{p_n^{n-2}q_n}{[n]_{p_n,q_n}}(p_n+q_n)\end{aligned}$$

which implies

$$\|G_n^{p,q}(e_2) - e_2\|_{x^2} \to 0 \text{ as } n \to \infty.$$

Thus the proof is completed. □

THEOREM 6. *For each* $f \in C_{x^2}^*[0,\,\infty)$, *we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|G_n^{p_n,q_n}(f,x) - f(x)\right|}{(1+x^2)^{1+\alpha}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\left|G_n^{p_n,q_n}(f,x) - f(x)\right|}{(1+x^2)^{1+\alpha}} &\leqslant \sup_{x\leqslant x_0} \frac{\left|G_n^{p_n,q_n}(f,x) - f(x)\right|}{(1+x^2)^{1+\alpha}} + \sup_{x\geqslant x_0} \frac{\left|G_n^{p_n,q_n}(f,x) - f(x)\right|}{(1+x^2)^{1+\alpha}} \\ &\leqslant \|G_n^{p_n,q_n}(f) - f\|_{C[0,x_0]} + \|f\|_{x^2} \sup_{x\geqslant x_0} \frac{\left|G_n^{p_n,q_n}(1+t^2,x)\right|}{(1+x^2)^{1+\alpha}} \\ &\quad + \sup_{x\geqslant x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}},\end{aligned}$$

where $\|\cdot\|_{C[0,x_0]}$ is the uniform norm on $[0,x_0]$. Since $|f(x)| \leqslant M\left(1+x^2\right)$, we have $\sup_{x\geqslant x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leqslant \frac{\|f\|_{x^2}}{(1+x_0^2)^{1+\alpha}}$. Let $\varepsilon > 0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{\|f\|_{x^2}}{\left(1+x_0^2\right)^{1+\alpha}} < \frac{\varepsilon}{3}. \tag{5.1}$$

On the other hand, in view of Lemma 2 we get

$$\|f\|_{x^2} \lim_{n\to\infty} \frac{\left|G_n^{p_n,q_n}\left(1+t^2,x\right)\right|}{\left(1+x^2\right)^{1+\alpha}} \to 0. \tag{5.2}$$

Hence we can choose $x_0 > 0$ so large that the inequality

$$\sup_{x\geqslant x_0} \frac{\left|G_n^{p_n,q_n}\left(1+t^2,x\right)\right|}{\left(1+x^2\right)^{1+\alpha}} < \frac{\varepsilon}{3}.$$

Also, the first term of the above inequality tends to zero by well known Korovkin's theorem, that is,

$$\|G_n^{p_n,q_n}(f) - f\|_{C[0,x_0]} < \frac{\varepsilon}{3}. \tag{5.3}$$

Therefore, combining (5.1)-(5.3) we get the desired result. □

REMARK 1. The further properties of the operators such as convergence properties via summability methods (see, for example, [9], [13], [20]) can be studied.

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