APPROXIMATION OF FUNCTIONS BY GENUINE BERNSTEIN–DURRMEYER TYPE OPERATORS

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Abstract. Very recently, in [4] Chen et. al introduced and considered a new generalization of Bernstein polynomials depending on a patameter α. As classical Bernstein operators, these operators also provide interpolation at the end points of $[0,1]$ and they have the linear precision property which means those reproduce the linear functions. In this paper we introduce genuine α-Bernstein-Durrmeyer operators. Some approximation results, which include local approximation, error estimation in terms of Ditzian-Totik modulus of smoothness are obtained. Also, the convergence of these operators to certain functions is shown by illustrative graphics using MAPLE algorithms.

1. Introduction and preliminaries

Bernstein polynomials have many useful properties, such as, positivity, interpolation property at the end points of $[0,1]$. With these properties, their simple structures and advantages in calculations make them interesting area of researches. These operators were introduced by S. N. Bernstein in 1912 (see [2]) and it was used to prove the fundamental theorem of Weierstrass. For more details on this topic we can refer the readers to excellent monographs [9] and [10].

The Bernstein operators are given by

$$B_n : C[0,1] \to C[0,1], \quad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ \ x \in [0,1].$$

The genuine Bernstein-Durrmeyer operators (see [3], [8]) are defined as follows:

$$U_n(f;x) = (1-x)^n f(0) + x^n f(1) + (n-1) \sum_{k=1}^{n-1} \left(\int_0^1 f(t) p_{n-2,k-1}(t) dt \right) p_{n,k}(x), \ \ f \in C[0,1].$$

These operators can be written as a composition of Bernstein operators and Beta operators, namely $U_n = B_n \circ \overline{\mathbb{B}}_n$. The Beta-type operators $\overline{\mathbb{B}}_n$ were introduced by A. Lupaş [12]. For $n = 1, 2, 3, \ldots$ and $f \in C[0,1]$, the explicit form of Beta operators is given by

$$
\overline{\mathbb{B}}_n(f;x) := \begin{cases} f(0), & x = 0, \\ \dfrac{1}{B(nx, n-nx)} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t) dt, & 0 < x < 1, \\ f(1), & x = 1, \end{cases}
$$

where $B(\cdot, \cdot)$ is the Euler's Beta function. These operators were studied widely by a numbers of authors (see [1], [6], [7], [13]).

Very recently, in [4] Chen et. al introduced and considered a new generalization of Bernstein polynomials depending on a parameter α as follows

$$
T_{n,\alpha}(f;x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x), \quad x \in [0,1], \quad f \in [0,1] \tag{2}
$$

for each positive integer n and any fixed real α, where the Bernstein type basis functions are considered

$$
p_{1,0}^{(\alpha)}(x) = 1 - x,
$$
$$
p_{1,1}^{(\alpha)}(x) = x,
$$
$$
p_{n,k}^{(\alpha)}(x) = \left[\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x) \right] x^{k-1}(1-x)^{n-k-1},
$$

for $n \geqslant 2$.

The operators (2) are called as α-Bernstein operators which are positive and monotone in the case of $\alpha \in [0,1]$. In the particular case, when $\alpha = 1$, α-Bernstein operators reduce to well-known Bernstein polynomials. As classical Bernstein operators, α-Bernstein operators provide interpolation at the end points of $[0,1]$ and they have the linear precision property which means those reproduce the linear functions. The authors of [4] have deeply studied many approximation properties of α-Bernstein operators such as uniform convergence, rate of convergence in terms of modulus of continuity, Voronovskaya type pointwise convergence, shape preserving properties, etc.

Our aim in this paper is to introduce genuine α-Bernstein-Durrmeyer operators as a composition of α-Bernstein operators and Beta operators, namely

$$
U_{n,\alpha} = T_{n,\alpha} \circ \overline{\mathbb{B}}_n.
$$

These operators are given in explicit form by

$$
U_{n,\alpha}(f;x) = p_{n,0}^{(\alpha)}(x)f(0) + p_{n,n}^{(\alpha)}(x)f(1) + (n-1) \sum_{k=1}^{n-1} p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n-2,k-1}(t)f(t) dt.
$$
$$\tag{3}$$

2. Basic results

This section is devoted to calculate moments and central moments of the operators (3). Also, the uniform convergence of the operators via Korovkin theorem is presented in this section.

LEMMA 2.1. *Let* $e_i(x) = x^i$, $i = 0,1,2,3,4$, *and* $n > 2$. *Then moments of the operators* $U_{n,\alpha}$ *are as follows:*

i) $U_{n,\alpha}(e_0;x) = 1$;

ii) $U_{n,\alpha}(e_1;x) = x$;

iii) $U_{n,\alpha}(e_2;x) = x^2 + \dfrac{2}{n+1}\left(1 + \dfrac{1-\alpha}{n}\right)x(1-x)$;

iv) $U_{n,\alpha}(e_3;x) = x^3 + \dfrac{6x(1-x)}{n(n+1)(n+2)}\left[n^2x + (1+x-\alpha x)n + 2(1-x)(1-\alpha)\right]$;

v) $U_{n,\alpha}(e_4;x) = x^4 + \dfrac{12x(1-x)}{n(n+1)(n+2)(n+3)}\left[n^3x^2 + (x^2 - \alpha x^2 + 3x)n^2 \right.$
$\left. -(4x^2 + 6\alpha x - 5x - 2 - 5\alpha x^2)n + 6(1-\alpha)(1-x)^2\right].$

COROLLARY 2.1. *As an immediate result of Lemma* 2.1, *we obtain central moments as following:*

i) $U_{n,\alpha}\left((t-x)^2;x\right) = \dfrac{2(n+1-\alpha)x(1-x)}{n(n+1)}$;

ii) $U_{n,\alpha}\left((t-x)^4;x\right) = \dfrac{12x(1-x)}{n(n+1)(n+2)(n+3)}\left[x(1-x)n^2 + (-5x+2\alpha x^2 - 2\alpha x + 2 + 5x^2)n \right.$
$\left. +24x^2 - 24x - 6\alpha - 24\alpha x^2 + 24\alpha x + 6\right];$

iii) $U_{n,\alpha}\left((t-x)^2;x\right) \leqslant \dfrac{2x(1-x)}{n}$.

LEMMA 2.2. *Let* $f \in C[0,1]$, $x \in [0,1]$ *and* $n \in \mathbb{N}$. *Then*

$$\|U_{n,\alpha}(f;\cdot)\| \leqslant \|f\|,$$

where $\|\cdot\|$ *is the uniform norm on* $[0,1]$.

Proof. Since $U_{n,\alpha}(e_0;x) = 1$, we get

$$|U_{n,\alpha}(f;x)| \leqslant U_{n,\alpha}(e_0;x)\|f\| = \|f\|. \quad \square$$

REMARK 2.1. According Bohman-Korovkin theorem, since $\lim\limits_{n\to\infty} U_{n,\alpha}(e_i;x) = x^i$, $i = 0,1,2$, the sequence $(U_{n,\alpha}(f))$ converges to f uniformly on $[0,1]$ for any $f \in C[0,1]$.

EXAMPLE 2.1. Let $f(x) = \sin(2\pi x)$ and $\alpha = 0.5$. In Figure 1 are given the graphs of function f and operator $U_{n,\alpha}$ for $n = 10$ and $n = 20$, respectively. This example explains the convergence of the operators $U_{n,\alpha}$ that are going to the function f if the values of n are increasing.
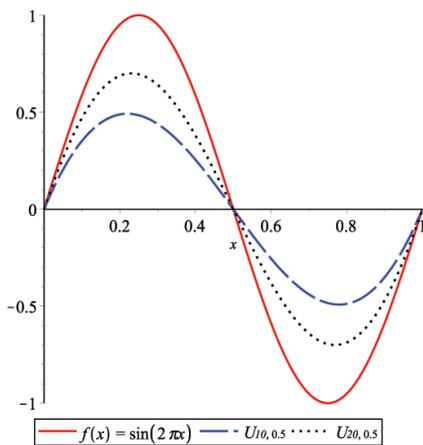

Figure 1: *The convergence of $U_{n,\alpha}(f;x)$ to $f(x)$*

EXAMPLE 2.2. Let $f(x) = \sin(2\pi x) + 2\sin\left(\dfrac{1}{2}\pi x\right)$ and $n = 10$. In Figure 2 we have seen that the choice of $\alpha = 0.9$ gives better approximation of f by genuine α-Bernstein-Durrmeyer operators than $\alpha = 0.1$.
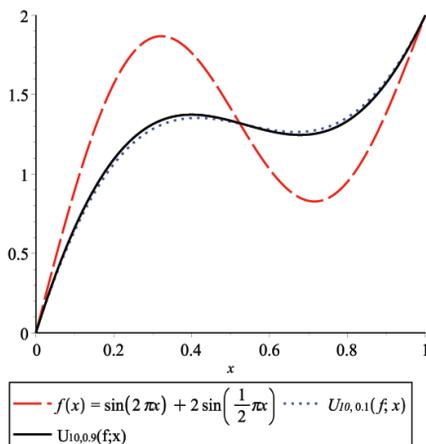

Figure 2: *The convergence of $U_{n,\alpha}(f;x)$ to $f(x)$*

Choosing $\alpha = 0.9$ we compute in Table 1 the error of approximation of genuine α-Bernstein-Durrmeyer operators.

TABLE 1. *Error of approximation for $U_{n,\alpha}$*

x	$|U_{20,\alpha}(f;x) - f(x)|$	$|U_{50,\alpha}(f;x) - f(x)|$	$|U_{100,\alpha}(f;x) - f(x)|$
0.10	0.119780164	0.046025979	0.022398022
0.15	0.201222945	0.083492574	0.042050802
0.20	0.272183862	0.118366586	0.060845800
0.25	0.317887961	0.142618970	0.074312981
0.30	0.328747703	0.150595558	0.079192881
0.35	0.301259559	0.139756237	0.073917663
0.40	0.238085339	0.110909279	0.058796858
0.45	0.147342879	0.067932809	0.035893486
0.50	0.041215809	0.017041614	0.008620233
0.55	0.065935177	0.034284832	0.018875513
0.60	0.159724367	0.078555151	0.042440029
0.65	0.227893334	0.109518925	0.058641675
0.70	0.262200455	0.123239161	0.065386128
0.75	0.259812435	0.118826266	0.062320649
0.80	0.224015403	0.098717529	0.050958557
0.85	0.164135892	0.068444202	0.034494237
0.90	0.094646991	0.035893873	0.017322996

3. Rate of convergence

In this section we study the rate of convergence of genuine α-Bernstein-Durrmeyer operators in terms of the Ditzian-Totik first order modulus of smoothness defined as follows:

$$\omega_\phi(f;t) = \sup_{0<h\leqslant t} \left\{ \left| f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right) \right|, x \pm \frac{h\phi(x)}{2} \in [0,1] \right\}, \quad (4)$$

where $\phi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$. The corresponding K-functional of the Ditzian-Totik first order modulus of smoothness is given by

$$K_\phi(f;t) = \inf_{g \in W_\phi[0,1]} \{\|f - g\| + t\|\phi g'\|\} \quad (t > 0), \quad (5)$$

where $W_\phi[0,1] = \{g : g \in AC_{loc}[0,1], \|\phi g'\| < \infty\}$ and $AC_{loc}[0,1]$ is the class of absolutely continuous functions on every interval $[a,b] \subset [0,1]$. Between K-functional and the Ditzian-Totik first order modulus of smoothness there is the following relation

$$K_\phi(f;t) \leqslant C\omega_\phi(f;t), \quad (6)$$

where $C > 0$ is a constant.

Now, we establish a global approximation theorem by means of Ditzian-Totik modulus of smoothness.

THEOREM 3.1. *Let $f \in C[0,1]$ and $\phi(x) = \sqrt{x(1-x)}$, then for every $x \in [0,1]$, we have*

$$|U_{n,\alpha}(f;x) - f(x)| \leqslant C\omega_\phi\left(f; \frac{1}{n^{1/2}}\right),$$

where C is a constant independent of n and x.

Proof. From the next representation

$$h(t) = h(x) + \int_x^t h'(u)du,$$

we get

$$|U_{n,\alpha}(h;x) - h(x)| = \left|U_{n,\alpha}\left(\int_x^t h'(u)du;x\right)\right|. \tag{7}$$

For any $x \in (0,1)$ and $t \in [0,1]$ we find that

$$\left|\int_x^t h'(u)du\right| \leqslant ||\phi h'|| \left|\int_x^t \frac{1}{\phi(u)}du\right|. \tag{8}$$

But,

$$\left|\int_x^t \frac{1}{\phi(u)}du\right| = \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \leqslant \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \tag{9}$$

$$\leqslant 2\left(|\sqrt{t} - \sqrt{x}| + |\sqrt{1-t} - \sqrt{1-x}|\right)$$

$$= 2|t-x|\left(\frac{1}{\sqrt{t} + \sqrt{x}} + \frac{1}{\sqrt{1-t} + \sqrt{1-x}}\right)$$

$$< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \leqslant \frac{2\sqrt{2}\,|t-x|}{\phi(x)}.$$

Combining (7)–(9) and using Cauchy-Schwarz inequality, we get

$$|U_{n,\alpha}(h;x) - h(x)| < 2\sqrt{2}||\phi h'||\phi^{-1}(x)U_{n,\alpha}(|t-x|;x)$$

$$\leqslant 2\sqrt{2}||\phi h'||\phi^{-1}(x)\left(U_{n,\alpha}((t-x)^2;x)\right)^{1/2}.$$

Now using Corollary 2.1, we obtain

$$|U_{n,\alpha}(h;x) - h(x)| \leqslant \frac{4}{n^{1/2}}||\phi h'||. \tag{10}$$

Using Lemma 2.2 and (10) we can write

$$|U_{n,\alpha}(f;x) - f(x)| \leqslant |U_{n,\alpha}(f-h;x)| + |f(x) - h(x)| + |U_{n,\alpha}(h;x) - h(x)|$$

$$\leqslant 4\left\{||f-h|| + \frac{1}{n^{1/2}}||\phi h'||\right\}.$$

From the definition of the K-functional (5), we get

$$|U_{n,\alpha}(f;x) - f(x)| \leqslant 4K_\phi\left(f;\frac{1}{n^{1/2}}\right),$$

and considering the relation (6), the proof is completed. $\quad\square$

The next result proposes a quantitative Voronovskaja type theorem by means of Ditzian-Totik modulus of smoothness.

THEOREM 3.2. *For any* $g \in C^2[0,1]$ *the following inequalities hold*

i) $\left|U_{n,\alpha}(g;x)-g(x)-\dfrac{n+1-\alpha}{n(n+1)}\phi^2(x)g^{(2)}(x)\right| \leqslant \dfrac{1}{n}C\omega_\phi\left(g^{(2)},\phi(x)n^{-1/2}\right),$

ii) $\left|U_{n,\alpha}(g;x)-g(x)-\dfrac{n+1-\alpha}{n(n+1)}\phi^2(x)g^{(2)}(x)\right| \leqslant \dfrac{1}{n}C\phi(x)\omega_\phi\left(g^{(2)},n^{-1/2}\right).$

Proof. Let $g \in C^2[0,1]$ be given and $t,x \in [0,1]$. Then by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g^{(2)}(u)du.$$

Hence

$$g(t) - g(x) - (t-x)g'(x) - \frac{1}{2}(t-x)^2 g^{(2)}(x) = \int_x^t (t-u)g^{(2)}(u)du - \int_x^t (t-u)g^{(2)}(x)du$$

$$= \int_x^t (t-u)[g^{(2)}(u) - g^{(2)}(x)]du.$$

Applying $U_{n,\alpha}(\cdot;x)$ to both sides of the above relation, we get

$$\left|U_{n,\alpha}(g;x)-g(x)-\frac{n+1-\alpha}{n(n+1)}\phi^2(x)g^{(2)}(x)\right| \leqslant U_{n,\alpha}\left(\left|\int_x^t |t-u||g^{(2)}(u)-g^{(2)}(x)|du\right|;x\right).$$

$$(11)$$

The quantity $\left|\int_x^t \left|g^{(2)}(u)-g^{(2)}(x)\right||t-u|du\right|$ was estimated in [11, p. 337] as follows:

$$\left|\int_x^t |g^{(2)}(u)-g^{(2)}(x)||t-u|du\right| \leqslant 2\|g^{(2)}-h\|(t-x)^2 + 2\|\phi h'\|\phi^{-1}(x)|t-x|^3, \quad (12)$$

where $h \in W_\phi[0,1]$.

Using Corallary 2.1 it follows that there exists a constant $C > 0$ such that

$$U_{n,\alpha}\left((t-x)^4;x\right) \leqslant \frac{C}{n^2}\phi^2(x). \qquad (13)$$

Therefore, combining (11)–(13) and applying Corollary 2.1 and the Cauchy-Schwarz inequality, we get

$$\left| U_{n,\alpha}(g;x) - g(x) - \frac{n+1-\alpha}{n(n+1)} \phi^2(x) g^{(2)}(x) \right|$$

$$\leqslant 2\|g^{(2)} - h\| U_{n,\alpha}\left((t-x)^2;x\right) + 2\|\phi h'\| \phi^{-1}(x) U_{n,\alpha}\left(|t-x|^3;x\right)$$

$$\leqslant \frac{4(n+1-\alpha)}{n(n+1)} \phi^2(x)\|g^{(2)} - h\| + 2\|\phi h'\| \phi^{-1}(x) \left\{ U_{n,\alpha}(t-x)^2;x \right\}^{1/2} \left\{ U_{n,\alpha}\left((t-x)^4;x\right) \right\}^{1/2}$$

$$\leqslant \frac{4(n+1-\alpha)}{n(n+1)} \phi^2(x)\|g^{(2)} - h\| + 4\sqrt{\frac{n+1-\alpha}{n(n+1)}} \phi(x)\frac{C}{n}\|\phi h'\|$$

$$\leqslant C\left\{ \frac{n+1-\alpha}{n(n+1)} \phi^2(x)\|g^{(2)} - h\| + \frac{1}{n}\sqrt{\frac{n+1-\alpha}{n(n+1)}} \phi(x)\|\phi h'\| \right\}$$

$$\leqslant \frac{C}{n}\left\{ \phi^2(x)\|g^{(2)} - h\| + n^{-1/2}\phi(x)\|\phi h'\| \right\}.$$

Since $\phi^2(x) \leqslant \phi(x) \leqslant 1, x \in [0,1]$, we obtain

$$\left| U_{n,\alpha}(g;x) - g(x) - \frac{n+1-\alpha}{n(n+1)} \phi^2(x) g^{(2)}(x) \right| \leqslant \frac{C}{n}\left\{ \|g^{(2)} - h\| + n^{-1/2}\phi(x)\|\phi h'\| \right\}.$$

Also, the following inequality can be obtained

$$\left| U_{n,\alpha}(g;x) - g(x) - \frac{n+1-\alpha}{n(n+1)} \phi^2(x) g^{(2)}(x) \right| \leqslant \frac{C}{n}\phi(x)\left\{ \|g^{(2)} - h\| + n^{-1/2}\|\phi h'\| \right\}.$$

Taking the infimum on the right hand side of the above relations over $h \in W_\phi[0,1]$, the theorem is proved.   □

4. Bézier variant of genuine α-Bernstein Durrmeyer type operators

In this section we propose a Bézier variant of genuine α-Bernstein Durrmeyer as

$$U_{n,\alpha}^{(v)}(g;x) = \sum_{k=0}^{n} F_{n,k} Q_{n,k}^{(v)}(x), \tag{14}$$

where

$$F_{n,k} = \begin{cases} (n-1)\int_0^1 p_{n-2,k-1}(t)g(t)dt, & 1 \leqslant k \leqslant n-1 \\ g(0), & k = 0, \\ g(1), & k = n, \end{cases}$$

$Q_{n,k}^{(v)}(x) = \left[S_{n,k}(x)\right]^v - \left[S_{n,k+1}(x)\right]^v$, $\alpha \geqslant 1$ with $S_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}^{(\alpha)}(x)$, when $k \leqslant n$ and 0 otherwise. Obviously, $U_{n,\alpha}^{(v)}$ is a sequence of linear positive operators and for $v = 1$, these operators reduce to the operators $U_{n,\alpha}$.

Next, our goal is to study the rate of convergence of the operators (14) for functions g whose derivative g' are of bounded variation on $[0,1]$. In order to prove the main results we will give the following two lemmas:

LEMMA 4.1. *a) If* $g \in C[0,1]$, *then* $\|U_{n,\alpha}^{(v)} g\| \leqslant \|g\|$.
b) If $x \in [0,1]$ *and* $g \in C[0,1]$ *such that* $g \geqslant 0$ *on* $[0,1]$ *then* $U_{n,\alpha}^{(v)}(g;x) \leqslant v U_{n,\alpha}(g;x)$.

An integral representation of the operators $U_{n,\alpha}^{(v)}$ can be given as follows:

$$U_{n,\alpha}^{(v)}(g;x) = \int_0^1 K_{n,\alpha}^{(v)}(x,t)g(t)dt, \tag{15}$$

where $K_{n,\alpha}^{(v)}$ is defined as

$$K_{n,\alpha}^{(v)}(x,t) = \sum_{k=1}^{n-1} Q_{n,k}^{(v)}(x)p_{n-2,k-1}(t) + Q_{n,0}^{(v)}(x)\delta(t) + Q_{n,n}^{(v)}(x)\delta(1-t),$$

$\delta(u)$ being the Dirac-delta function.

LEMMA 4.2. *For a sufficiently large* n *and a fixed* $x \in (0,1)$, *it follows*

(i) $\xi_{n,\alpha}^{(v)}(x,y) = \int_0^y K_{n,\alpha}^{(v)}(x,t)dt \leqslant \dfrac{2vx(1-x)}{n(x-y)^2}$, $0 \leqslant y < x$,

(ii) $1 - \xi_{n,\alpha}^{(v)}(x,z) = \int_z^1 K_{n,\alpha}^{(v)}(x,t)dt \leqslant \dfrac{2vx(1-x)}{n(z-x)^2}$, $x < z < 1$.

Denote $DBV[0,1]$ the class of differentiable functions g defined on $[0,1]$, whose derivatives g' are of bounded variation on $[0,1]$. Let $\bigvee_a^b g$ be the total variation of g on $[a,b]$ and g_x' is defined by

$$g_x'(t) = \begin{cases} g'(t) - g'(x-), & 0 \leqslant t < x \\ 0, & t = x \\ g'(t) - g'(x+) & x < t \leqslant 1. \end{cases} \tag{16}$$

The next results give us a rate of convergenge for a function $g \in DBV[0,1]$.

THEOREM 4.1. *If* $g \in DBV[0,1]$, *then for every* $x \in (0,1)$ *and sufficiently large* n, *the folowing inequality*

$$|U_{n,\alpha}^{(v)}(g;x) - g(x)| \leqslant \sqrt{\frac{vx(1-x)}{2n}} \{|g'(x+) + g'(x-)| + |g'(x+) - g'(x-)|\}$$

$$+ \frac{2(v + (1-v)_+)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (g_x')$$

holds.

Proof. Since $U_{n,\alpha}^{(v)}(1;x) = 1$, for every $x \in (0,1)$ we can write

$$
\begin{aligned}
U_{n,\alpha}^{(v)}(g;x) - g(x) &= \int_0^1 K_{n,\alpha}^{(v)}(x,t)(g(t) - g(x))dt \\
&= \int_0^x (g(t) - g(x)) K_{n,\alpha}^{(v)}(x,t)dt + \int_x^1 (g(t) - g(x)) K_{n,\alpha}^{(v)}(x,t)dt \\
&= -\int_0^x \left[\int_t^x g'(u)du \right] K_{n,\alpha}^{(v)}(x,t)dt + \int_x^1 \left[\int_x^t g'(u)du \right] K_{n,\alpha}^{(v)}(x,t)dt \\
&= -\mathscr{I}_1(x) + \mathscr{I}_2(x),
\end{aligned}
$$

where

$$
\mathscr{I}_1(x) = \int_0^x \left[\int_t^x g'(u)du \right] K_{n,\alpha}^{(v)}(x,t)dt, \quad \mathscr{I}_2(x) = \int_x^1 \left[\int_x^t g'(u)du \right] K_{n,\alpha}^{(v)}(x,t)dt.
$$

For any $g \in DBV[0,1]$, we decompose $g'(t)$ as follows:

$$
\begin{aligned}
g'(t) &= \frac{1}{2}(g'(x+) + g'(x-)) + g_x'(t) + \frac{1}{2}(g'(x+) - g'(x-))sgn(t-x) \qquad (17) \\
&\quad + \delta_x(t) \left[g'(x) - \frac{1}{2}(g'(x+) + g'(x-)) \right],
\end{aligned}
$$

where

$$
\delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x \end{cases}.
$$

Therefore, we get

$$
\begin{aligned}
\mathscr{I}_1(x) &= \int_0^x \left\{ \int_t^x \frac{1}{2}(g'(x+) + g'(x-)) + g_x'(u) + \frac{g'(x+) - g'(x-)}{2}sgn(u-x) \right. \\
&\quad \left. + \delta_x(u) \left[g'(x) - \frac{1}{2}(g'(x+) + g'(x-)) \right] du \right\} K_{n,\alpha}^{(v)}(x,t)dt \\
&= \frac{g'(x+) + g'(x-)}{2} \int_0^x (x-t) K_{n,\alpha}^{(v)}(x,t)dt + \int_0^x \left[\int_t^x g_x'(u)du \right] K_{n,\alpha}^{(v)}(x,t)dt \\
&\quad - \frac{g'(x+) - g'(x-)}{2} \int_0^x (x-t) K_{n,\alpha}^{(v)}(x,t)dt \\
&\quad + \left[g'(x) - \frac{g'(x+) + g'(x-)}{2} \right] \int_0^x \left[\int_t^x \delta_x(u)du \right] K_{n,\alpha}^{(v)}(x,t)dt.
\end{aligned}
$$

Since $\int_t^x \delta_x(u)du = 0$, we get

$$
\begin{aligned}
\mathscr{I}_1(x) &= \frac{g'(x+) + g'(x-)}{2} \int_0^x (x-t) K_{n,\alpha}^{(v)}(x,t)dt + \int_0^x \left[\int_t^x g_x'(u)du \right] K_{n,\alpha}^{(v)}(x,t)dt \\
&\quad - \frac{g'(x+) - g'(x-)}{2} \int_0^x (x-t) K_{n,\alpha}^{(v)}(x,t)dt.
\end{aligned}
$$

Using a similar method, we find that

$$\mathscr{I}_2(x) = \frac{g'(x+) + g'(x-)}{2} \int_x^1 (t-x) K_{n,\alpha}^{(v)}(x,t) dt + \int_x^1 \left[\int_x^t g_x'(u) \right] K_{n,\alpha}^{(v)}(x,t) dt$$
$$- \frac{g'(x+) - g'(x-)}{2} \int_x^1 (t-x) K_{n,\alpha}^{(v)}(x,t) dt.$$

Therefore,

$$- \mathscr{I}_1(x) + \mathscr{I}_2(x)$$
$$= \frac{g'(x+) + g'(x-)}{2} \int_0^1 (t-x) K_{n,\alpha}^{(v)}(x,t) dt + \frac{g'(x+) - g'(x-)}{2} \int_0^1 |t-x| K_{n,\alpha}^{(v)}(x,t) dt$$
$$- \int_0^x \left[\int_t^x g_x'(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt + \int_x^1 \left[\int_x^t g_x'(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt.$$

Then,

$$U_{n,\alpha}^{(v)}(g;x) - g(x)$$
$$= \frac{g'(x+) + g'(x-)}{2} \int_0^1 (t-x) K_{n,\alpha}^{(v)}(x,t) dt + \frac{g'(x+) - g'(x-)}{2} \int_0^1 |t-x| K_{n,\alpha}^{(v)}(x,t) dt$$
$$- \int_0^x \left[\int_t^x g_x'(u) du \right] K_n(x,t) dt + \int_x^1 \left[\int_x^t g_x'(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt.$$

From the above relation it follows

$$\left| U_{n,\alpha}^{(v)}(g;x) - g(x) \right|$$
$$\leqslant \left| \frac{g'(x+) + g'(x-)}{2} \right| \left| U_{n,\alpha}^{(v)}(t-x;x) \right| + \left| \frac{g'(x+) - g'(x-)}{2} \right| \left| U_{n,\alpha}^{(v)}(|t-x|;x) \right|$$
$$+ \left| - \int_0^x \left[\int_t^x g_x'(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt \right| + \left| \int_x^1 \left[\int_x^t g_x'(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt \right|. \qquad (18)$$

According to Lemma 4.2, we write

$$\int_0^x \left[\int_t^x g_x'(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt = \int_0^x \left[\int_t^x g_x'(u) du \right] \frac{\partial}{\partial t} \xi_{n,\alpha}^{(v)}(x,t) dt$$
$$= \int_0^x g_x'(t) \xi_{n,\alpha}^{(v)}(x,t) dt.$$

Thus,

$$\left| - \int_0^x \left[\int_t^x g_x'(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt \right| \leqslant \int_0^x |g_x'(t)| \xi_{n,\alpha}^{(v)}(x,t) dt$$
$$\leqslant \int_0^{x - \frac{x}{\sqrt{n}}} |g_x'(t)| \xi_{n,\alpha}^{(v)}(x,t) dt + \int_{x - \frac{x}{\sqrt{n}}}^x |g_x'(t)| \xi_{n,\alpha}^{(v)}(x,t) dt.$$

Since $g'_x(x) = 0$ and $\xi_{n,\alpha}^{(v)}(x,t) \leqslant 1$, one has

$$
\int_{x-\frac{x}{\sqrt{n}}}^{x} |g'_x(t)| \xi_{n,\alpha}^{(v)}(x,t) dt = \int_{x-\frac{x}{\sqrt{n}}}^{x} |g'_x(t) - g'_x(x)| \xi_{n,\alpha}^{(v)}(x,t) dt \leqslant \int_{x-\frac{x}{\sqrt{n}}}^{x} \bigvee_{t}^{x}(g'_x) dt
$$

$$
\leqslant \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(g'_x) \int_{x-\frac{x}{\sqrt{n}}}^{x} dt = \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(g'_x).
$$

From Lemma 4.2, we can write

$$
\int_{0}^{x-\frac{x}{\sqrt{n}}} |g'_x(t)| \xi_{n,\alpha}^{(v)}(x,t) dt \leqslant \frac{2vx(1-x)}{n} \int_{0}^{x-\frac{x}{\sqrt{n}}} |g'_x(t)| \frac{dt}{(x-t)^2}
$$

$$
= \frac{2vx(1-x)}{n} \int_{0}^{x-\frac{x}{\sqrt{n}}} |g'_x(t) - g'_x(x)| \frac{dt}{(x-t)^2}
$$

$$
\leqslant \frac{2vx(1-x)}{n} \int_{0}^{x-\frac{x}{\sqrt{n}}} \bigvee_{t}^{x}(g'_x) \frac{dt}{(x-t)^2}.
$$

Using the change of variables $t = x - \frac{x}{u}$, we have

$$
\frac{2vx(1-x)}{n} \int_{0}^{x-\frac{x}{\sqrt{n}}} \bigvee_{t}^{x}(g'_x) \frac{dt}{(x-t)^2} = \frac{2v(1-x)}{n} \int_{1}^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x}(g'_x) du \leqslant \frac{2v(1-x)}{n} \sum_{k=1}^{\sqrt{n}} \bigvee_{x-\frac{x}{k}}^{x}(g'_x)
$$

and hence, we get

$$
\left| -\int_{0}^{x} \left[\int_{t}^{x} g'_x(u) du \right] K_{n,\alpha}^{(v)}(x,t) dt \right| \leqslant \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(g'_x) + \frac{2v(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x}(g'_x)
$$

$$
\leqslant \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x}(g'_x) + \frac{2v(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x}(g'_x) \leqslant \frac{2(v+(1-v)_+)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x}(g'_x). \quad (19)
$$

Using a similar method, we get

$$
\left| \int_{x}^{1} \left[\int_{x}^{t} g'_x(u) \right] K_{n,\alpha}^{(v)}(x,t) dt \right| \leqslant \frac{1-x}{\sqrt{n}} \bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(g'_x) + \frac{2vx}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}}(g'_x)
$$

$$
\leqslant \frac{2(1-x)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}}(f'_x) + \frac{2vx}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}}(g'_x)
$$

$$
\leqslant \frac{2[v+(1-v)_+]}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{1-x}{k}}(g'_x). \quad (20)
$$

The relations (18), (19) and (20) complete the proof of the theorem. $\square$

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