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A CAUCHY–BUNYAKOVSKY–SCHWARZ TYPE INEQUALITY RELATED TO THE MÖBIUS ADDITION

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Abstract. We show a Cauchy-Bunyakovsky-Schwarz type inequality related to the Möbius addition in complex inner product spaces. The corresponding inequality in real inner product spaces can be derived easily as well.

1. Introduction and preliminaries

The celebrated Cauchy-Bunyakovsky-Schwarz (CBS in the sequel) inequality

$$|\langle u,v\rangle| \leqslant ||u||||v|| \quad (u,v\in\mathbb{V})$$

in any inner product space $\mathbb{V}$ is one of the most fundamental inequality in Mathematics.

Möbius addition is defined on the complex open unit disk $\mathbb{D}=\{z\in\mathbb{C};\,|z|<1\}$ by

$$a\oplus b=\frac{a+b}{1+\overline{a}b} \quad (a,b\in\mathbb{D}),$$

which appears in a wide variety of fields of mathematics. In particular, although Möbius addition is known in the literature as a *hyperbolic translation*, its group-like structure had gone unnoticed until it was uncovered by A. A. Ungar in 1988 [5], in the context of Einstein's special theory of relativity. Furthermore, Ungar extended the Möbius addition in the complex disk to the ball of an arbitrary real inner product space, and observed that the ball endowed with the Möbius addition is a gyrocommutative gyrogroup (see [3], [4], [5]).

Let us briefly recall the definition of the Möbius gyrogroup. For precise definitions and basic results of gyrocommutative gyrogroups and the Möbius gyrogroup, see [2]. For elementary facts on inner product spaces, one can refer [1].

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DEFINITION 1. [2, Definition 3.40] Let $\mathbb{V} = (\mathbb{V}, +, \langle\cdot,\cdot\rangle)$ be a real inner product space with a vector addition $+$ and a positive definite inner product $\langle\cdot,\cdot\rangle$. Let $\mathbb{V}_s$ be the open ball

$$\mathbb{V}_s = \{v \in \mathbb{V} : ||v|| < s\}$$

for any fixed $s > 0$, where $||v|| = \langle v, v\rangle^{\frac{1}{2}}$. The Möbius addition $\oplus_{\mathrm{M}}$ is given by the equation

$$u \oplus_{\mathrm{M}} v = \frac{\left(1 + \frac{2}{s^2}\langle u, v\rangle + \frac{1}{s^2}||v||^2\right) u + \left(1 - \frac{1}{s^2}||u||^2\right) v}{1 + \frac{2}{s^2}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}$$

for any $u, v \in \mathbb{V}_s$. The addition $\oplus_{\mathrm{M}}$ in the open interval $(-s, s)$ in the real line is defined by the equation

$$a \oplus_{\mathrm{M}} b = \frac{a+b}{1 + \frac{1}{s^2}ab}$$

for any $a, b \in (-s, s)$.

It is quite elementary to check that both denominators are positive, $u \oplus_{\mathrm{M}} v \in \mathbb{V}_s$ and $a \oplus_{\mathrm{M}} b \in (-s, s)$ are well-defined. We simply denote $\oplus_{\mathrm{M}}$ in the Möbius gyrogroup $\mathbb{V}_s$ or in the interval $(-s, s)$ by $\oplus_s$.

DEFINITION 2. [2, Definition 2.7, (2.1), (6.286), (6.293)] Obviously, the inverse element of u with respect to $\oplus_s$ coincides with $-u$ in the Möbius gyrogroup. We use the notation

$$u \ominus_s v = u \oplus_s (-v)$$

as in group theory. Moreover, the Möbius gyrodistance function d and Poincaré distance function h are defined by the equations

$$d(u, v) = ||v \ominus_s u||$$

$$h(u, v) = \tanh^{-1} \frac{d(u, v)}{s}.$$

Ungar showed that h satisfies the triangle inequality [2, (6.294)].

The following identities are easy consequence of the definition. One can refer [6, Lemma 14 (i), Lemma 12].

LEMMA 3. *Let $s > 0$. The following formulae hold*

(i) $\frac{u}{s} \oplus_1 \frac{v}{s} = \frac{u \oplus_s v}{s}$

(ii) $||u \oplus_s v||^2 = \frac{||u||^2 + 2\langle u, v\rangle + ||v||^2}{1 + \frac{2}{s^2}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}$

for any $u, v \in \mathbb{V}_s$.

In this article, we show a CBS type inequality related to the Möbius addition in complex inner product spaces. Our proof is based on just elementary calculus and algebra. The corresponding inequality in real inner product spaces can be easily derived from the proof as well. Nevertheless, it might be worthwhile to mention that the inequality (8) was found earlier than (1), in the study on real inner product gyrovector spaces.

2. Result

We begin with an elementary lemma related to a function of two real variables.

LEMMA 4. *Let* a, b, c *be real numbers,* $0 < a, b < 1$ *and* $0 < c \leqslant 1$. *Consider the following two real variable function*

$$f(X,Y) = 1 - (1-a^2)(1-b^2)c^2 - a^2c^2X - b^2c^2Y + a^2b^2c^4XY$$

on $[0,1] \times [0,1]$. *Then*

$$f(X,Y) \geqslant 0 \quad ((X,Y) \in [0,1] \times [0,1]).$$

Moreover, if $(X,Y) \in [0,1] \times [0,1]$, *then* $f(X,Y) = 0$ *if and only if* $c = 1$ *and* $(X,Y) = (1,1)$.

Proof. Obviously we have

$$f_X(X,Y) = -a^2c^2(1-b^2c^2Y) \qquad \text{and} \qquad f_Y(X,Y) = -b^2c^2(1-a^2c^2X).$$

Therefore, $f_X(X,Y) = f_Y(X,Y) = 0$ if and only if $X = \dfrac{1}{a^2c^2}$, $Y = \dfrac{1}{b^2c^2}$. It follows that $f(X,Y)$ takes its minimum value on the boundary of $[0,1] \times [0,1]$. It is easy to see that $f(1,Y)$ is monotone decreasing for $0 \leqslant Y \leqslant 1$ and

$$f(1,1) = (1 - a^2b^2c^2)(1-c^2) \geqslant 0,$$

which implies that $f(1,Y) \geqslant 0$ $(0 \leqslant Y \leqslant 1)$. Moreover, $f(0,Y)$ is also monotone decreasing and $f(0,1) > f(1,1) \geqslant 0$. Hence $f(0,Y) > 0$ $(0 \leqslant Y \leqslant 1)$. By symmetry, $f(X,0)$ and $f(X,1)$ is nonnegative as well. Thus $f(X,Y) \geqslant 0$ for $(X,Y) \in [0,1] \times [0,1]$. It is immediate to check that $f(X,Y) = 0$ on $[0,1] \times [0,1]$ if and only if $c = 1$ and $(X,Y) = (1,1)$. This completes the proof. $\square$

The following theorem is a CBS type inequality related to the Möbius addition in complex inner product spaces.

THEOREM 5. *Let $\mathbb{V}$ be a complex inner product space and let $w \in \mathbb{V}$ be a fixed element with $||w|| \leqslant 1$. For any $u, v \in \mathbb{V}$ and for any $s > \max\{||u||, ||v||\}$, the following inequality holds*

$$\left|\frac{\langle u,w\rangle - \langle v,w\rangle}{1-\frac{1}{s^2}\overline{\langle u,w\rangle}\langle v,w\rangle}\right| \leqslant \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u,v\rangle + ||v||^2}{1-\frac{2}{s^2}\mathrm{Re}\langle u,v\rangle + \frac{1}{s^4}||u||^2||v||^2}}. \tag{1}$$

The equality holds if and only if one of the following conditions is satisfied:

(i) $u = v$

(ii) $||w|| = 1$ *and* $u = \lambda w,\ v = \mu w$ *for some complex numbers* λ, μ.

Proof. If $u = 0$, then inequality (1) reduces to $|\langle v,w\rangle| \leqslant ||v||$, which follows from the classical CBS inequality and $||w|| \leqslant 1$. If $w = 0$, then inequality (1) trivially holds. Thus we may assume $u, v, w \neq 0$. At first, we show that if $s = 1$ and $||u||, ||v|| < 1$, then

$$\left|\frac{\langle u,w\rangle - \langle v,w\rangle}{1-\overline{\langle u,w\rangle}\langle v,w\rangle}\right|^2 \leqslant \frac{||u||^2 - 2\mathrm{Re}\langle u,v\rangle + ||v||^2}{1-2\mathrm{Re}\langle u,v\rangle + ||u||^2||v||^2}. \tag{2}$$

We can take real numbers $0 \leqslant \rho, r_1, r_2 \leqslant 1$ and $0 \leqslant t,\ x, y \leqslant 2\pi$ such that

$$\rho e^{it} = \frac{\langle u,v\rangle}{||u||||v||}, \qquad r_1 e^{ix} = \frac{\langle u,w\rangle}{||u||||w||} \qquad \text{and} \qquad r_2 e^{iy} = \frac{\langle v,w\rangle}{||v||||w||}. \tag{3}$$

Fix an arbitrary pair $\{u,v\}$, so ρ, t are also fixed, and change w according to the condition $0 < ||w|| \leqslant 1$. We denote by D the set of all tuples of 4 real variables (r_1, r_2, x, y) obtained by the procedure above, which is a subset of $[0,1]\times[0,1]\times[0,2\pi]\times[0,2\pi]$. Put

$$a = ||u||, b = ||v||, c = ||w||, u_R = u - ar_1e^{ix}\frac{w}{c} \quad \text{and} \quad v_R = v - br_2e^{iy}\frac{w}{c}.$$

Then it is immediate to see that

$$u = ar_1e^{ix}\frac{w}{c} + u_R, \qquad v = br_2e^{iy}\frac{w}{c} + v_R$$

are the orthogonal decompositions of u and v with respect to the closed linear subspace $\mathbb{C}w$, respectively. It follows from the Pythagorean identity that

$$a^2 = a^2r_1^2 + ||u_R||^2 \qquad b^2 = b^2r_2^2 + ||v_R||^2.$$

Therefore, we have

$$\rho e^{it} = \frac{\langle u,v\rangle}{||u||||v||} = \frac{\left\langle ar_1e^{ix}\frac{w}{c} + u_R, br_2e^{iy}\frac{w}{c} + v_R\right\rangle}{ab} = r_1r_2e^{i(x-y)} + \frac{\langle u_R, v_R\rangle}{ab}.$$

It follows from the CBS inequality that

$$\begin{aligned}\left|\frac{\langle u_R, v_R\rangle}{ab}\right|^2 &\leqslant \frac{||u_R||^2||v_R||^2}{a^2b^2}\\ &= \frac{(a^2-a^2r_1^2)(b^2-b^2r_2^2)}{a^2b^2}\\ &= (1-r_1^2)(1-r_2^2).\end{aligned}$$

Thus, if $(r_1, r_2, x, y) \in D$, then

$$|\rho e^{it} - r_1r_2e^{i(x-y)}|^2 \leqslant (1-r_1^2)(1-r_2^2),$$

in particular, we can obtain

$$\rho\cos t - \sqrt{(1-r_1^2)(1-r_2^2)} \leqslant r_1r_2\cos(x-y). \tag{4}$$

In order to prove the inequality (2), it is sufficient to show

$$\begin{aligned}&(a^2-2ab\rho\cos t+b^2)(1-2abc^2r_1r_2\cos(-x+y)+a^2b^2c^4r_1^2r_2^2)\\ &\quad-(1-2ab\rho\cos t+a^2b^2)(a^2c^2r_1^2-2abc^2r_1r_2\cos(-x+y)+b^2c^2r_2^2)\geqslant 0\end{aligned} \tag{5}$$

for any $(r_1, r_2, x, y) \in D$. The left hand side of (5) can be calculated as follows:

$$\begin{aligned}&a^2-2a^3bc^2r_1r_2\cos(-x+y)+a^4b^2c^4r_1^2r_2^2\\ &\quad-2ab\rho\cos t+4a^2b^2c^2\rho r_1r_2\cos t\cos(-x+y)-2a^3b^3c^4\rho r_1^2r_2^2\cos t\\ &\quad+b^2-2ab^3c^2r_1r_2\cos(-x+y)+a^2b^4c^4r_1^2r_2^2\\ &\quad-a^2c^2r_1^2+2abc^2r_1r_2\cos(-x+y)-b^2c^2r_2^2\\ &\quad+2a^3bc^2\rho r_1^2\cos t-4a^2b^2c^2\rho r_1r_2\cos t\cos(-x+y)+2ab^3c^2\rho r_2^2\cos t\\ &\quad-a^4b^2c^2r_1^2+2a^3b^3c^2r_1r_2\cos(-x+y)-a^2b^4c^2r_2^2\\ &=a^2(1-b^4c^2r_2^2)(1-c^2r_1^2)+b^2(1-a^4c^2r_1^2)(1-c^2r_2^2)\\ &\quad-2ab\rho\cos t(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)\\ &\quad+2abc^2r_1r_2\cos(-x+y)(1-a^2)(1-b^2).\end{aligned}$$

By using (4), we can continue to estimate:

$$\begin{aligned}
&\geqslant a^2(1-b^4c^2r_2^2)(1-c^2r_1^2)+b^2(1-a^4c^2r_1^2)(1-c^2r_2^2)\\
&\quad -2ab\rho\cos t(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)\\
&\quad +2abc^2\left\{\rho\cos t-\sqrt{(1-r_1^2)(1-r_2^2)}\right\}(1-a^2)(1-b^2)\\
&= a^2(1-b^4c^2r_2^2)(1-c^2r_1^2)+b^2(1-a^4c^2r_1^2)(1-c^2r_2^2)\\
&\quad -2ab\rho\cos t\{(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)-c^2(1-a^2)(1-b^2)\}\\
&\quad -2abc^2\sqrt{(1-r_1^2)(1-r_2^2)}(1-a^2)(1-b^2).
\end{aligned}$$

Since $(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)-c^2(1-a^2)(1-b^2)\geqslant 0$ by Lemma 4, so we can continue our estimation:

$$\begin{aligned}
&\geqslant a^2(1-b^4c^2r_2^2)(1-c^2r_1^2)+b^2(1-a^4c^2r_1^2)(1-c^2r_2^2)\\
&\quad -2ab\{(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)-c^2(1-a^2)(1-b^2)\}\\
&\quad -2abc^2\sqrt{(1-r_1^2)(1-r_2^2)}(1-a^2)(1-b^2).
\end{aligned}$$

It follows from Arithmetic-Geometric mean inequality that

$$\begin{aligned}
&\geqslant a^2(1-b^4c^2r_2^2)(1-c^2r_1^2)+b^2(1-a^4c^2r_1^2)(1-c^2r_2^2)\\
&\quad -2ab\{(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)-c^2(1-a^2)(1-b^2)\}\\
&\quad -\left\{a^2c^2(1-r_1^2)+b^2c^2(1-r_2^2)\right\}(1-a^2)(1-b^2)\\
&= a^2+b^2-2ab\\
&\quad +(2ab-2a^3b-2ab^3+2a^3b^3-a^2+a^4+2a^2b^2-a^4b^2-b^2+b^4-a^2b^4)c^2\\
&\quad -a^2c^2(a-b)^2r_1^2-b^2c^2(a-b)^2r_2^2+a^2b^2c^4(a-b)^2r_1^2r_2^2\\
&= (a-b)^2\{(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)-c^2(1-a^2)(1-b^2)\}\\
&\geqslant 0.
\end{aligned}$$

Here, we used Lemma 4 again for the last inequality. Thus the desired inequality (2) is shown.

It is immediate to see that the equality in (5) holds provided the condition (i) or (ii) is satisfied. Conversely, suppose that the equality in (5) holds. We may assume $a\neq 0$ or $b\neq 0$. If $a=0$, then we have $u=0$ and $c=r_2=1$, the latter and the equality condition of the classical CBS inequality implies that $v=\mu w$ for some complex number μ.

Similarly, if $b=0$, then $v=0$ and $u=\lambda w$ for some complex number λ. If $a,b\neq 0$, then the proof of the inequality (5) above shows that

$$(\rho\cos t-1)\{(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)-c^2(1-a^2)(1-b^2)\}=0 \tag{6}$$

$$(a-b)^2\{(1-a^2c^2r_1^2)(1-b^2c^2r_2^2)-c^2(1-a^2)(1-b^2)\}=0. \tag{7}$$

If $c=r_1=r_2=1$, then $||w||=1$ and $u=\lambda w$, $v=\mu w$ for some complex numbers λ,μ. If $c=r_1=r_2=1$ doesn't hold, then the formula (6) and the equality condition in Lemma 4 yield that $\rho\cos t=1$, while (7) shows that $a=b$. In this case, we can conclude that $u=v$.

Next, let $u,v\in\mathbb{V}$ be arbitrary elements, and let $s>\max\{||u||,||v||\}$. Since

$$\left|\left|\frac{u}{s}\right|\right|,\ \left|\left|\frac{v}{s}\right|\right|<1,$$

the inequality (2) shows

$$\left|\frac{\left\langle\frac{u}{s},w\right\rangle-\left\langle\frac{v}{s},w\right\rangle}{1-\overline{\left\langle\frac{u}{s},w\right\rangle}\left\langle\frac{v}{s},w\right\rangle}\right|^2\leqslant\frac{\left|\left|\frac{u}{s}\right|\right|^2-2\mathrm{Re}\left\langle\frac{u}{s},\frac{v}{s}\right\rangle+\left|\left|\frac{v}{s}\right|\right|^2}{1-2\mathrm{Re}\left\langle\frac{u}{s},\frac{v}{s}\right\rangle+\left|\left|\frac{u}{s}\right|\right|^2\left|\left|\frac{v}{s}\right|\right|^2},$$

from which we obtain

$$\left|\frac{\langle u,w\rangle-\langle v,w\rangle}{1-\frac{1}{s^2}\overline{\langle u,w\rangle}\langle v,w\rangle}\right|^2\leqslant\frac{||u||^2-2\mathrm{Re}\langle u,v\rangle+||v||^2}{1-\frac{2}{s^2}\mathrm{Re}\langle u,v\rangle+\frac{1}{s^4}||u||^2||v||^2}.$$

Thus the inequality (1) is shown. It is obvious that the equality condition in (1) is identical to that in (2). This completes the proof. □

REMARK 6. By taking $v=0$, it is immediate to see that Theorem 5 is an extension of the classical CBS inequality.

REMARK 7. One can easily modify the proof of Theorem 5 to obtain the corresponding CBS type inequality in real inner product spaces. Indeed, we can take real numbers $-1\leqslant\rho$, $r_1,r_2\leqslant 1$ such that

$$\rho=\frac{\langle u,v\rangle}{||u||||v||},\qquad r_1=\frac{\langle u,w\rangle}{||u||||w||}\qquad\text{and}\qquad r_2=\frac{\langle v,w\rangle}{||v||||w||},$$

instead of the polar forms (3). It is not necessary to deal with arguments t,x,y, and the rest of the proof is similar to and easier than the complex case. We state the following theorem for the real inner product spaces, showing relation between the Möbius addition (subtraction) and inner product.

THEOREM 8. *Let $\mathbb{V}$ be a real inner product space and let $w \in \mathbb{V}$ be a fixed element with $||w|| \leqslant 1$. For any $u, v \in \mathbb{V}$ and for any $s > \max\{||u||, ||v||\}$, the following inequality holds*

$$|\langle u, w\rangle \ominus_s \langle v, w\rangle| \leqslant ||u \ominus_s v||.$$

That is,

$$\left| \frac{\langle u, w\rangle - \langle v, w\rangle}{1 - \frac{1}{s^2}\langle u, w\rangle\langle v, w\rangle} \right| \leqslant \sqrt{\frac{||u||^2 - 2\langle u, v\rangle + ||v||^2}{1 - \frac{2}{s^2}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}}. \tag{8}$$

The equality holds if and only if one of the following conditions is satisfied

(i) $u = v$

(ii) $||w|| = 1$ *and* $u = \lambda w,\ v = \mu w$ *for some real numbers* λ, μ.

REFERENCES

[1] JOHN B. CONWAY, *A Course in Functional Analysis*, 2nd edition Springer-Verlag, Graduate Texts in Mathematics **96**, New York, 1990.

[2] ABRAHAM A. UNGAR, *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2008.

[3] A. A. UNGAR, *Extension of the unit disk gyrogroup into the unit ball of any real inner product space*, J. Math. Anal. Appl. **202**, 3 (1996), 1040–1057.

[4] A. A. UNGAR, *Group-like structure underlying the unit ball in real inner product spaces*, Results Math. **18**, 3–4 (1990), 355–364.

[5] A. A. UNGAR, *Thomas rotation and the parametrization of the Lorentz transformation group*, Found. Phys. Lett. **1**, 1 (1988), 57–89.

[6] K. WATANABE, *Orthogonal Gyroexpansion in Möbius Gyrovector Spaces*, J. Funct. Spaces **2017**, (2017), Article ID 1518254, doi:10.1155/2017/1518254, 13 pages.

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