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ON THE INVERSE POWER INEQUALITY FOR THE BEREZIN NUMBER OF OPERATORS

MUBARIZ GARAYEV, SUNA SALTAN, DILARA GUNDOGDU

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Abstract. The Berezin symbol $\widetilde{A}$ of operator A acting on the reproducing kernel Hilbert space $\mathscr{H} = \mathscr{H}(\Omega)$ over some set Ω is defined by

$$\widetilde{A}(\lambda) = \left\langle A\widehat{k}_{\mathscr{H},\lambda}, \widehat{k}_{\mathscr{H},\lambda} \right\rangle, \ \lambda \in \Omega,$$

where $\widehat{k}_{\mathscr{H},\lambda} = \frac{k_{\mathscr{H},\lambda}}{\|k_{\mathscr{H},\lambda}\|_{\mathscr{H}}}$ is the normalized reproducing kernel of $\mathscr{H}$. The Berezin number of operator A is the following number:

$$ber(A) := \sup\left\{\left|\widetilde{A}(\lambda)\right| : \lambda \in \Omega\right\}.$$

Clearly, $ber(A) \leqslant w(A)$, where $w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathscr{H},\ \|x\|_{\mathscr{H}} = 1\}$ is the numerical radius of A. The power inequality for the numerical radius of Hilbert space operator A is the following:

$$w(A^n) \leqslant (w(A))^n, \ \forall n \geqslant 1.$$

Since $ber(A) \leqslant w(A)$, the following question naturally arises: is it true that $ber(A^n) \leqslant (ber(A))^n$ for any operator A and any integer $n > 1$?

Although we do not solve this question, in this paper, by using some Hardy type inequality, we prove the inverse power inequality for $ber(A)$ for positive operators on $\mathscr{H}(\Omega)$; namely, we prove that $(ber(A))^n \leqslant C(n,m)ber(A^n)$ for any positive operator A on $\mathscr{H}(\Omega)$, where $C(n,m) > 1$ is the constant depending only on n and its conjugate m, where $\frac{1}{n} + \frac{1}{m} = 1$.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_n, $b_n \geqslant 0$ satisfy $0 < \sum_{n=1}^{\infty} a_n^p < +\infty$ and $0 < \sum_{n=1}^{\infty} b_n^q$, then the classical Hardy-Hilbert inequality [10, 11] asserts that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}} \tag{1}$$

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and an equivalent from is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \sum_{n=1}^{\infty} a_n^p, \tag{2}$$

where the constants $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p$ are the best possible in the sense that they can not be replaced by some smaller numbers such that the inequalities remain true for all (even finite) sequences of non-negative real numbers. The Hardy-Hilbert inequalities (1), (2) are important in analysis and its applications (see Mitrinovic et al. [16], Garayev et al. [7]).

Recently many generalizations and refinements of these inequalities have been also obtained, see [9, 15, 16, 18], and references therein.

Hardy et al. [10] proved an inequality, under the same condition of (1), similar to (1) as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}} \tag{3}$$

and an equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m,n\}}\right)^p < [pq]^p \sum_{n=1}^{\infty} a_n^p, \tag{4}$$

where the constant factors pq and $(pq)^p$ are the best possible. Some new inequalities similar to Hardy-Hilbert inequality, are given recently by Das and Sahoo [4, 5].

In the present article, we apply inequalities (3), (4) in one question of operator theory; namely, we will use these inequalities to estimate Berezin number of the powers of operators acting in the Reproducing Kernel Hilbert Space (shortly, RKHS).

Let Ω be a subset of some topological space. Recall that the Reproducing Kernel Hilbert Space is a Hilbert Space $\mathscr{H} := \mathscr{H}(\Omega)$ of complex valued functions on the set Ω such that the evaluation $f \to f(\lambda)$ at any point $\lambda \in \Omega$ is continuous on $\mathscr{H}$. Then the classical Riesz representation theorem ensures that a functional Hilbert space $\mathscr{H}$ has a reproducing kernel, that is, a function $k_{\mathscr{H},\lambda} : \Omega \times \Omega \to \mathbb{C}$ with defining property $\langle f, k_{\mathscr{H},\lambda}\rangle = f(\lambda)$ for all f in $\mathscr{H}$ and $\lambda \in \Omega$, where $k_{\mathscr{H},\lambda}^{(z)} = k_{\mathscr{H}}(z,\lambda) \in \mathscr{H}$. Let $\widehat{k}_{\mathscr{H},\lambda} = \frac{k_{\mathscr{H},\lambda}}{\|k_{\mathscr{H},\lambda}\|}$ be the normalized reproducing kernel of $\mathscr{H}$.

For any bounded linear operator A on $\mathscr{H}$, its Berezin symbol $\widetilde{A}$ is defined by (see Berezin [2, 3]):

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathscr{H},\lambda}, \widehat{k}_{\mathscr{H},\lambda}\right\rangle, \ \lambda \in \Omega.$$

The Berezin symbol of an operator provides important information about the operators. Namely, it is well known, in particular, that on the most familar RKHS, including the Hardy, Bergman and Fock spaces, the Berezin symbol uniquely determines the operator (i.e., $A = 0$ if and only if $\widetilde{A} = 0$).

Recall that if $\mathscr{B}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators on $\mathscr{H}$, then the numerical range and numerical radius of $A \in \mathscr{B}(\mathscr{H})$ is defined respectively by $W(A) := \{\langle Ax, x\rangle : x \in \mathscr{H},\ \|x\|_{\mathscr{H}} = 1\}$ and $w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathscr{H},\ \|x\|_{\mathscr{H}} = 1\}$. Since

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|,$$

the numerical radius $w(.)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|.\|$. An important inequality for $w(A)$ is the power inequality stating that $w(A^n) \leqslant w(A)^n$ $(n = 1, 2, \ldots)$ (see Halmos [8]). About numerical radius inequalities the readers can be found, for example, in [1, 6, 17]. Note that (see Karaev [12, 13]) the Berezin set and Berezin number for $A \in \mathscr{B}(\mathscr{H}(\Omega))$ is defined as

$$Ber(A) := Range(\widetilde{A}) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\} \text{ (Berezin set)}$$

and

$$ber(A) := \sup\left\{\left|\widetilde{A}(\lambda)\right| : \lambda \in \Omega\right\} \text{ (Berezin number).}$$

Clearly, $Ber(A) \subset W(A)$ and $ber(A) \leqslant w(A)$.

Note that there are many questions regarding to these new concepts (more detailly, see Karaev [13]). In particular, we do not know:

(i) *Is it true that*

$$ber(A^n) \leqslant (ber(A))^n \tag{5}$$

for any operator $A \in \mathscr{B}(\mathscr{H}(\Omega))$ and any integer $n > 1$?

(ii) *Is it true that*

$$(ber(A))^n \leqslant Cber(A^n) \tag{6}$$

for any operator $A \in \mathscr{B}(\mathscr{H}(\Omega))$, $n > 1$ and for some $C = C(n) > 0$?

The main goal of this article is to investigate inequality (6). Namely, we partially solve question (ii) for some special operators acting in RKHS (see Theorem 2 below).

2. Hardy-Hilbert type inequalities and Berezin number of some operators

We start to prove inequalities for some self-adjoint and positive operators on a RKHS $\mathscr{H} = \mathscr{H}(\Omega)$, which are similar to the inequalities (3) and (4).

THEOREM 1. *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be two continuous functions defined on an interval $\Delta \subset (0, +\infty)$ and $f, g \geqslant 0$. Then the following are true:*

(i)

$$\widetilde{(fg)(A)}(\lambda) + \frac{1}{2}\widetilde{g(B)}(\mu)\widetilde{f(A)}(\lambda) + \frac{1}{2}\widetilde{f(B)}(\mu)\widetilde{g(A)}(\lambda) + \frac{1}{2}\widetilde{(fg)(B)}(\mu)$$

$$< pq\left[(f(A)^p + f(B)^p)^{\frac{1}{p}}(g(A)^q + g(B)^q)^{\frac{1}{q}}\right]^{\sim}(\lambda)$$

for all self-adjoint operators $A, B \in \mathscr{B}(\mathscr{H}(\Omega))$ with spectra contained in Δ and for all $\mu,\ \lambda \in \Omega$.

(ii) $(ber(f(A)))^2 < 4(pq-1)ber(f(A)^2)$ *for any self-adjoint operator $A \in \mathscr{B}(\mathscr{H}(\Omega))$ with spectrum contained in Δ; in particular, $(ber(A))^2 < 4(pq-1)ber\left(A^2\right)$.*

Proof. (i) Let a_1, a_2, b_1, b_2 be positive scalars. Then using (3), we have

$$a_1b_1+\frac{a_1b_2}{2}+\frac{a_2b_1}{2}+\frac{a_2b_2}{2}<pq\left(a_1^p+a_2^p\right)^{\frac{1}{p}}\left(b_1^q+b_2^q\right)^{\frac{1}{q}}. \tag{7}$$

Let $x,y\in\Delta$. By the hypophyses of the theorem $f(x)\geqslant 0,\ g(x)\geqslant 0$ for all $x\in\Delta$. If we put $a_1=f(x)$, $a_2=f(y)$, $b_1=g(x)$, $b_2=g(y)$ in (7), then we have (see also [14])

$$\begin{aligned}&f(x)g(x)+\frac{1}{2}f(x)g(y)+\frac{1}{2}f(y)g(x)+\frac{1}{2}f(y)g(y)\\&<pq\left(f(x)^p+f(y)^p\right)^{\frac{1}{p}}\left(g(x)^q+g(y)^q\right)^{\frac{1}{q}}\end{aligned} \tag{8}$$

for all $x,y\in\Delta$. Let A be self-adjoint operator. Then by using the functional calculus and inequality (8) we have (for simplicity, we will identify λI with a scalar λ)

$$\begin{aligned}&f(A)g(A)+\frac{1}{2}g(y)f(A)+\frac{1}{2}f(y)g(A)+\frac{1}{2}f(y)g(y)\\&<pq\left(f(A)^p+f(y)^p\right)^{\frac{1}{p}}\left(g(A)^q+g(y)^q\right)^{\frac{1}{q}},\end{aligned}$$

and therefore

$$\begin{aligned}&\left\langle f(A)g(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{1}{2}g(y)\left\langle f(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle\\&+\frac{1}{2}f(y)\left\langle g(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{1}{2}f(y)g(y)\\&<pq\left\langle\left(f(A)^p+f(y)^p\right)^{\frac{1}{p}}\left(g(A)^q+g(y)^q\right)^{\frac{1}{q}}\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle\end{aligned}$$

for all $\lambda\in\Omega$ and any $y\in\Delta$.

Using the functional calculus once more to the self-adjoint operator B, we get

$$\begin{aligned}&\left\langle f(A)g(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{1}{2}g(B)\left\langle f(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle\\&+\frac{1}{2}f(B)\left\langle g(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{1}{2}f(B)g(B)\\&<pq\left\langle\left(f(A)^p+f(B)^p\right)^{\frac{1}{p}}\left(g(A)^q+g(B)^q\right)^{\frac{1}{q}}\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle\end{aligned} \tag{9}$$

Hence, we have from (9) that

$$\begin{aligned}&\left\langle f(A)g(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\tfrac{1}{2}\left\langle g(B)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle\left\langle f(A)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle\\&+\tfrac{1}{2}\left\langle f(B)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle\left\langle g(A)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle+\tfrac{1}{2}\left\langle f(B)g(B)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle\\&<pq\left\langle\left(f(A)^p+f(B)^p\right)^{\frac{1}{p}}\left(g(A)^q+g(B)^q\right)^{\frac{1}{q}}\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle\end{aligned}$$

which means that

$$\widetilde{f(A)g(A)}(\lambda)+\frac{1}{2}\widetilde{g(B)}(\mu)\widetilde{f(A)}(\lambda)+\frac{1}{2}\widetilde{f(B)}(\mu)\widetilde{g(A)}(\lambda)+\frac{1}{2}\widetilde{f(B)g(B)}(\mu)$$

$$< pq\left[\left(f(A)^p+f(B)^p\right)^{\frac{1}{p}}\left(g(A)^q+g(B)^q\right)^{\frac{1}{q}}\right]^{\sim}(\lambda) \tag{10}$$

for all self-adjoint operators $A,B\in\mathscr{B}(\mathscr{H}(\Omega))$ and for all $\lambda,\mu\in\Omega$. This proves (i).

(ii) In particular, for $B=A,\ g=f$ and $\mu=\lambda$, we have from inequality (10) that

$$\widetilde{f(A)^2}(\lambda)+\frac{1}{2}\widetilde{f(A)^2}(\lambda)+\frac{1}{2}\left[\widetilde{f(A)}(\lambda)\right]^2+\frac{1}{2}\widetilde{f(A)^2}(\lambda)$$

$$< pq2^{\frac{1}{p}+\frac{1}{q}}\widetilde{f(A)^2}(\lambda)=2pq\widetilde{f(A)^2}(\lambda)$$

or equivalently

$$\left[\widetilde{f(A)}(\lambda)\right]^2<4(pq-1)\widetilde{f(A)^2}(\lambda) \tag{11}$$

for all $\lambda\in\Omega$. Since, $pq-1>0$ and $\widetilde{f(A)}(\lambda)$ is the real number for all $\lambda\in\Omega$ (because $f(A)$ is self-adjoint), and therefore $\left[\widetilde{f(A)}(\lambda)\right]^2\geqslant 0,\ \forall\lambda\in\Omega,$ we deduce from (11) that $\widetilde{f(A)^2}(\lambda)\geqslant 0$ for all $\lambda\in\Omega$. Then we have that

$$[\widetilde{f(A)}(\lambda)]^2<4(pq-1)\sup_{\lambda\in\Omega}\widetilde{f(A)^2}(\lambda)=4(pq-1)ber(f(A)^2)$$

for all $\lambda\in\Omega$. This obviously implies that

$$(ber(f(A)))^2<4(pq-1)ber(f(A)^2);$$

in particular, for $f(x)=x$, we have that $(ber(A))^2<4(pq-1)ber(A^2)$, as desired. □

Our next result is the following theorem which partially solves the above raised question (ii).

THEOREM 2. *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let f be a continuous function defined on an interval $\triangle\subset(0,+\infty)$ and $f\geqslant 0$. Let $A:\mathscr{H}(\Omega)\to\mathscr{H}(\Omega)$ be a positive operator on a RKHS $\mathscr{H}(\Omega)$ with spectrum contained in $\triangle$. Then there exists a constant $C=C(p,q)>1$ such that*

$$[ber(f(A))]^p\leqslant Cber(f^p(A));$$

in particular, $ber(A)^p\leqslant Cber(A^p)$.

Proof. Let a_1, a_2, b_1, b_2 be positive numbers. Then using (4), we have that

$$\left(a_1+\frac{a_2}{2}\right)^p+\left(\frac{a_1}{2}+\frac{a_2}{2}\right)^p<[pq]^p[a_1^p+a_2^p]. \tag{12}$$

Let $x,y \in \triangle$. Since $f(x) \geqslant 0$ for all $x \in \triangle$, by putting $a_1 = f(x)$, $a_2 = f(y)$ in (12), we have

$$\left(f(x)+\frac{f(y)}{2}\right)^p+\left(\frac{f(x)}{2}+\frac{f(y)}{2}\right)^p<[pq]^p[f^p(x)+f^p(y)]. \tag{13}$$

So, by using the same functional calculus arguments as in the proof of Theorem 1, finally we get from (13) that

$$\begin{aligned}&\left[\left\langle f(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{1}{2}\left\langle f(B)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle\right]^p\\ &+\left[\frac{1}{2}\left\langle f(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{1}{2}\left\langle f(B)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle\right]^p\\ &<[pq]^p\left[\left\langle f^p(A)\widehat{k}_{\mathscr{H},\lambda},\widehat{k}_{\mathscr{H},\lambda}\right\rangle+\left\langle f^p(B)\widehat{k}_{\mathscr{H},\mu},\widehat{k}_{\mathscr{H},\mu}\right\rangle\right]\end{aligned}$$

and hence

$$\begin{aligned}&\left[\widetilde{f(A)}(\lambda)+\frac{1}{2}\widetilde{f(B)}(\mu)\right]^p+\left[\frac{1}{2}(\widetilde{f(A)}(\lambda)+\widetilde{f(B)}(\mu))\right]^p\\ &\qquad<[pq]^p\left[\widetilde{f^p(A)}(\lambda)+\widetilde{f^p(B)}(\mu)\right]\end{aligned} \tag{14}$$

for all positive operator B which spectrum contained in $\triangle$ and all $\lambda,\mu \in \Omega$. Now by replacing $B = A$ and $\mu = \lambda$, we have from (14) that

$$\left[\left(\frac{3}{2}\right)^p+1\right]\left[\widetilde{f(A)}(\lambda)\right]^p<2[pq]^p\left[\widetilde{f^p(A)}(\lambda)\right],$$

and hence

$$\left[\widetilde{f(A)}(\lambda)\right]^p<2\left[\left(\frac{3}{2}\right)^p+1\right]^{-1}[pq]^p\left[\widetilde{f^p(A)}(\lambda)\right]$$

for all $\lambda \in \Omega$. Since $\widetilde{f^p(A)}(\lambda) \geqslant 0$ for all $\lambda \in \Omega$ and for all $p > 1$, the last inequality shows that

$$\left[\widetilde{f(A)}(\lambda)\right]^p<2\left[\left(\frac{3}{2}\right)^p+1\right]^{-1}[pq]^p ber(f^p(A))$$

for all $\lambda \in \Omega$ and $p > 1$. This implies that

$$[ber(f(A))]^p\leqslant 2\left[\left(\frac{3}{2}\right)^p+1\right]^{-1}[pq]^p ber(f^p(A)),$$

in particular,

$$[ber(A)]^p\leqslant 2\left[\left(\frac{3}{2}\right)^p+1\right]^{-1}[pq]^p ber(A^p),$$

and since $C = C(p,q) := 2\left[\left(\frac{3}{2}\right)^p+1\right]^{-1}[pq]^p > 1$ for all p,q with $\frac{1}{p}+\frac{1}{q}=1$, this proves the theorem. □

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Mubariz Garayev
Institute of Mathematics and Mechanics
National Academy of Sciences of Azerbaijan
B. Vagabzade str. 9, Baku 370141, Azerbaijan
and
Department of Mathematics
College of Science, King Saud University
P. O. Box 2455, Riyadh 11451, Saudi Arabia
e-mail: mgarayev@ksu.edu.sa

Suna Saltan
Department of Mathematics
Suleyman Demirel University
32260, Isparta, Turkey
e-mail: sunasaltan@sdu.edu.tr

Dilara Gündoğdu
Department of Mathematics
Suleyman Demirel University
32260, Isparta, Turkey