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A LYAPUNOV–TYPE INEQUALITY FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH CAPUTO–FABRIZIO DERIVATIVE

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Abstract. In this work we obtain a Lyapunov-type inequality for a fractional differential equation with Caputo-Fabrizio operator subject to Dirichlet-type boundary conditions. As an application, we obtain a lower bound for the eigenvalues of corresponding equations.

1. Introduction and main results

Lyapunov's inequality is an outstanding result in mathematics with many applications – see [1, 2, 3] and references therein. The result, as proved by Lyapunov in [4], asserts that if $q \in C([a,b];\mathbb{R})$, then a necessary condition for the boundary value problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

to have a nontrivial solution is given by

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \tag{1.2}$$

Looking for a generalization for fractional differential equations, in [5], Ferreira investigated a Lyapunov-type inequality for the Riemann-Liouville fractional boundary value problem

$$\begin{cases} D_a^{\alpha} u(t) + q(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1.3}$$

where D_a^{α} is the (left) Riemann-Liouville derivative of order $\alpha \in (1,2]$ and $q \in C([a,b];\mathbb{R})$. He proved that, if (1.3) has a nontrivial solution, then

$$\int_a^b |q(s)|\,ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.4}$$

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Very recently, Ma and col. [6] considered the fractional boundary value problem with $\mathscr{D}_1^\alpha$ Hadamard derivative of order $1<\alpha\leqslant 2$

$$\begin{cases} \mathscr{D}_1^\alpha u(t)-q(t)u(t)=0, \ \ 1<t<e, \\ u(1)=u(e)=0, \end{cases} \tag{1.5}$$

and an interesting Lyapunov-type inequality was estabilished

$$\int_1^e |q(s)|\,ds > \Gamma(\alpha)\lambda^{\alpha-1}(1-\lambda)^{\alpha-1}\exp(\lambda) \tag{1.6}$$

where $\lambda=\frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$ and $q\in C([1,e];\mathbb{R})$.

Moreover, some Lyapunov-type inequalities for fractional boundary value problems have been obtained in [7, 8, 9, 10, 11, 12, 13].

In this paper we succeeded to generalize inequality (1.2) for the fractional boundary value problem.

Indeed, we stated here below consider the following fractional boundary value problem

$$\begin{cases} \mathfrak{D}_a^\alpha u(t)+q(t)u(t)=0, \ \ a<t<b, \\ u(a)=u(b)=0, \end{cases} \tag{1.7}$$

where $0\leqslant a<b<\infty$ and $\mathfrak{D}_a^\alpha$ is the Caputo-Fabrizio derivative of order $\alpha\in(1,2]$.

The main result of this paper is:

THEOREM 1.1. *If the fractional boundary value problem* (1.7) *has a nontrivial solution, where q is a real and continuous function in $[a,b]$, then*

$$\int_a^b |q(s)|\,ds > \frac{4(\alpha-1)(b-a)}{((\alpha-1)(b-a)-2+\alpha)^2}. \tag{1.8}$$

2. Definitions and some properties of Caputo-Fabrizio fractional operator

Recently, Caputo and Fabrizio introduced a new fractional derivative [14]:

$$\mathfrak{D}_a^\alpha f(t)=\frac{1}{1-\alpha}\int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)f'(s)ds, \tag{2.1}$$

where the order of the derivative $\alpha\in(0,1)$ and $f\in H^1(a,b)$, $-\infty<a<b<+\infty$. The interest of this new fractional derivative as justified by Caputo and Fabrizio [14] is due to the necessity of using it for a model describing the behavior of classical viscoelastic materials, thermal media, electromagnetic systems, etc.

In [15], Nieto and Losado have introduced an integral operator corresponding to the differential operator (2.1) as

$$\mathscr{I}_a^{\alpha} f(t) = (1-\alpha) f(t) + \alpha \int_a^t f(s)ds, \ \ s \geqslant a. \tag{2.2}$$

Here, we introduce the definition of Caputo-Fabrizio fractional operators of arbitrary order.

DEFINITION 2.1. Let $n-1 < \alpha < n$, $n \in \mathbb{N}$ and $f \in H^n(a,b)$. The Caputo-Fabrizio fractional derivative of order α is defined by

$$\mathfrak{D}_a^{\alpha} f(t) = \frac{1}{n-\alpha} \int_a^t \exp\left(-\frac{\alpha-n+1}{n-\alpha}(t-s)\right) f^{(n)}(s)ds. \tag{2.3}$$

DEFINITION 2.2. Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Caputo-Fabrizio fractional integral of order α is defined by

$$\mathscr{I}_a^{\alpha} f(t) = (n-\alpha)\mathscr{I}_a^{n-1} f(t) + (\alpha-n+1)\mathscr{I}_a^{n} f(s)ds, \ \ s \geqslant a. \tag{2.4}$$

Here $\mathscr{I}_a^k$ is the Cauchy integral

$$\mathscr{I}_a^k f(t) = \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} f(s)ds$$

with properties

$$\mathscr{I}_a^k f(t) = \mathscr{I}_a^{k-1} \mathscr{I}_a f(t) = \underbrace{\mathscr{I}_a \ldots \mathscr{I}_a}_{k} f(t), \quad \lim_{k\to 0} \mathscr{I}_a^k = \mathbf{I}, \tag{2.5}$$

where $\mathbf{I}$ is the identify operator $\mathbf{I}f(t) = f(t)$.

PROPERTY 2.3. Let $\alpha \in (n-1,n]$, it holds

$$\mathscr{I}^{\alpha}\mathfrak{D}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} u^{(k)}(a).$$

Proof. Applying the operator $\mathscr{I}^{\alpha}$ to $\mathfrak{D}^{\alpha} u(t)$, we obtain

$$\begin{aligned}
\mathscr{I}^{\alpha}\mathscr{D}^{\alpha} u(t) &= (n-\alpha)\mathscr{I}^{n-1}\mathscr{D}^{\alpha} u(t) + (\alpha-n+1)\mathscr{I}^{n}\mathscr{D}^{\alpha} u(t) \\
&= \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} \int_a^s \exp\left(-\frac{\alpha-n+1}{n-\alpha}(s-\tau)\right) u^{(n)}(\tau)d\tau ds \\
&\quad + \frac{1}{(n-1)!} \frac{\alpha-n+1}{n-\alpha} \int_a^t (t-s)^{n-1} \int_a^s \exp\left(-\frac{\alpha-n+1}{n-\alpha}(s-\tau)\right) u^{(n)}(\tau)d\tau ds.
\end{aligned}$$

By (2.5), we have

$$\mathscr{I}^{\alpha}\mathscr{D}^{\alpha}u(t) = (n-\alpha)\mathscr{I}^{n-1}\mathscr{D}^{\alpha}u(t) + (\alpha-n+1)\mathscr{I}^{n-1}\mathscr{I}\mathscr{D}^{\alpha}u(t). \tag{2.6}$$

We consider $\mathscr{I}\mathscr{D}^{\alpha}u(t),$ then by changing the order of integration, we have

$$\begin{aligned}
\mathscr{I}\mathscr{D}^{\alpha}u(t) &= \int_a^t\int_a^s \exp\left(-\frac{\alpha-n+1}{n-\alpha}(s-\tau)\right)u^{(n)}(\tau)d\tau ds \\
&= \int_a^t u^{(n)}(\tau)\int_\tau^t \exp\left(-\frac{\alpha-n+1}{n-\alpha}(s-\tau)\right)dsd\tau \\
&= -\frac{n-\alpha}{\alpha-n+1}\int_a^t u^{(n)}(\tau)\left[\exp\left(-\frac{\alpha-n+1}{n-\alpha}(t-\tau)\right)-1\right]d\tau.
\end{aligned}$$

Substituting into (2.6) the last expression we obtain

$$\begin{aligned}
\mathscr{I}^{\alpha}\mathscr{D}^{\alpha}u(t) &= (n-\alpha)\mathscr{I}^{n-1}\mathscr{D}^{\alpha}u(t) + (\alpha-n+1)\mathscr{I}^{n-1}\mathscr{I}\mathscr{D}^{\alpha}u(t) \\
&= \frac{1}{(n-2)!}\int_a^t (t-s)^{n-2}\int_a^s \exp\left(-\frac{\alpha-n+1}{n-\alpha}(s-\tau)\right)u^{(n)}(\tau)d\tau ds \\
&\quad - \frac{1}{(n-2)!}\int_a^t (t-s)^{n-2}\int_a^s \left[\exp\left(-\frac{\alpha-n+1}{n-\alpha}(s-\tau)\right)-1\right]u^{(n)}(\tau)d\tau ds \\
&= \frac{1}{(n-2)!}\int_a^t (t-s)^{n-2}\int_a^s u^{(n)}(\tau)d\tau ds = \mathscr{I}^{n-1}\mathscr{I}u^{(n)}(t) = \mathscr{I}^{n}u^{(n)}(t) \\
&= u(t) - \sum_{k=0}^{n-1}\frac{(t-a)^k}{k!}u^{(k)}(a).
\end{aligned}$$

The proof is complete. □

3. Proof of main results

LEMMA 3.1. *The function $u(t)$ is a solution of the boundary value problem* (1.7) *if, and only if, $u(t)$ satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds, \tag{3.1}$$

where

$$G(t,s) = \begin{cases} \frac{b-t}{b-a}\left((\alpha-1)(s-a)-2+\alpha\right), & a \leqslant s \leqslant x \leqslant b, \\ \frac{x-a}{b-a}\left((\alpha-1)(s-a)+2-\alpha\right), & a \leqslant x \leqslant s \leqslant b. \end{cases}$$

Proof. By the result in property 2.3, the solution of the Caputo-Fabrizio differential equation in (1.7) can be written as

$$u(t) = u(a) + u'(a)(t-a) - (2-\alpha)\int_a^t q(s)u(s)ds - (\alpha-1)\int_a^t (t-s)q(s)u(s)ds, \quad a < t < b.$$

Using to conditions $u(a) = u(b) = 0$ in (1.7) we get

$$u(t) = (2-\alpha)\frac{t-a}{b-a}\int_a^b q(s)u(s)ds - (\alpha-1)\frac{t-a}{b-a}\int_a^b (t-s)q(s)u(s)ds - (2-\alpha)\int_a^t q(s)u(s)ds - (\alpha-1)\int_a^t (t-s)q(s)u(s)ds, \quad a < t < b. \tag{3.2}$$

Whereupon

$$u(t) = \int_a^t \frac{b-t}{b-a}((\alpha-1)(s-a)-2+\alpha)q(s)u(s)ds + \int_t^b \frac{t-a}{b-a}((\alpha-1)(b-s)+2-\alpha)q(s)u(s)ds, \quad a < t < b.$$

This ends the proof. □

LEMMA 3.2. *Let Green function G be defined as in Lemma 3.1, then we have the estimate*

$$|G(t,s)| \leqslant \frac{((\alpha-1)(b-a)-2+\alpha)^2}{4(\alpha-1)(b-a)}. \tag{3.3}$$

Proof. Let

$$G_1(t,s) = G(t,s), \quad a \leqslant s \leqslant t \leqslant b$$

and

$$G_2(t,s) = G(t,s), \quad a \leqslant t \leqslant s \leqslant b.$$

For $s \leqslant t$, we observe that

$$G_1(t,s) \leqslant \frac{b-s}{b-a}((\alpha-1)(s-a)-2+\alpha). \tag{3.4}$$

It follows that we only need to get the maximum value of the function

$$g(s) = \frac{b-s}{b-a}((\alpha-1)(s-a)-2+\alpha).$$

We have

$$g'(s) = \frac{1}{b-a}((\alpha-1)(b+a-2s)+2-\alpha),$$

which implies that $g'(s)=0$ for $s=\frac{1}{2}\left(b+a+\frac{2-\alpha}{\alpha-1}\right)$. Since

$$g''(s) = -2\frac{\alpha-1}{b-a} \leqslant 0,$$

and by (3.4) and discussion above, we can conclude that the maximum value of the function G_2 is obtained at

$$t = s = \frac{1}{2}\left(b+a+\frac{2-\alpha}{\alpha-1}\right). \tag{3.5}$$

Similarly, the function $G_2(t,s)$ has a maximum value when (3.5) holds. □

Proof of Theorem 1.1. We equip $C[a,b]$ with the Chebyshev norm $\|u\| = \sup_{t\in[a,b]} |u|$. It follows from Lemma 3.1 that a solution to the fractional boundary value problem (1.7) satisfies the integral equation (3.1). Hence,

$$\|u\| \leqslant \max_{t\in[a,b]} \int_a^b \|u\| |G(t,s)q(s)|ds,$$

or, equivalently,

$$1 \leqslant \max_{t\in[a,b]} \int_a^b |G(t,s)q(s)|ds.$$

Using now to the properties of the Green function G proved in Lemma 3.2, we get

$$1 \leqslant \max_{t\in[a,b]} \int_a^b |G(t,s)q(s)|ds \leqslant \frac{((\alpha-1)(b-a)-2+\alpha)^2}{4(\alpha-1)(b-a)} \int_a^b |q(s)|ds,$$

from which the inequality (2.1) follows. □

REMARK 3.3. Note that if we set $\alpha = 2$ in (2.1), we obtain Lyapunov's classical inequality (1.2).

COROLLARY 3.4. *Let $\lambda \in \mathbb{R}$ be an eigenvalue of the problem*

$$\begin{cases} \mathfrak{D}_a^\alpha u(t) + \lambda u(t) = 0, \ \ a < t < b, \\ u(a) = u(b) = 0. \end{cases} \tag{3.6}$$

Then $|\lambda| > \frac{4(\alpha-1)(b-a)}{((\alpha-1)(b-a)-2+\alpha)^2}$.

COROLLARY 3.5. *If*

$$|\lambda| \leqslant \frac{4(\alpha-1)(b-a)}{((\alpha-1)(b-a)-2+\alpha)^2},$$

then the system of eigenfunctions of eigenvalue problem (3.6) *has no real zeros.*

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