

SOME EXTENDED INTEGRAL INEQUALITIES ON TIME SCALES

BOQUN OU

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Abstract. In this paper, we establish an integral inequality on time scales. As an applications, we study the stability of a kind of difference equations.

1. Introduction and preliminaries

As we all know, some of the integral inequality used in dynamic equations on time scales has attracted much attention to many scholars. In 2004, Pachpatte has established the useful linear Volterra-Fredholm type integral inequalities in [4, 5, 6]. In 2010, in [10] Ma has established the useful nonlinear Volterra-Fredholm type integral inequalities and discrete inequalities. However, it seems to us that is very little known results about Volterra-Fredholm type dynamic integral inequalities on time scales.

Suppose that $x(t) : \mathbb{T} \rightarrow \mathbb{R}_+$ is rd -continuous and satisfies

$$x(t) \leq b(t) + \int_0^{\alpha_1(t)} K_1 f_1(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \omega(x(s)) \Delta s \quad (1.1)$$

$$+ \int_0^{\alpha_3(t)} K_3 f_3(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_4(t)} K_4 f_4(t, s) \omega(x(s)) \Delta s;$$

where $f_i(t, s); f_j^{\Delta t}(t, s) \in C_{rd}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$; K_i are nonnegative constants ($i = 1, 2, 3, 4$). $b(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$, $\omega(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ are non-decreasing functions and $\exists A > 0$, $\sup_{s \in [0, \infty)} \omega'(s) = A$; $\alpha_j(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ are strictly increasing function, and $\alpha_j(0) = 0$, $\alpha_j(t) \leq t$, for $t \in \mathbb{R}_+$ ($j = 1, 2, 3, 4$).

When $\mathbb{T} = \mathbb{R}_+$, $\alpha_1(t) = \alpha_3(t) = \alpha(t)$, $\alpha_2(t) = \alpha_4(t) = t$, (1.1) reduce the following inequality.

$$x(t) \leq b(t) + \int_0^{\alpha(t)} K_1 f_1(t, s) \omega(x(s)) ds + \int_0^t K_2 f_2(t, s) \omega(x(s)) ds \quad (1.2)$$

$$+ \int_0^{\alpha(t)} K_3 f_3(t, s) \omega(x(s)) ds + \int_0^t K_4 f_4(t, s) \omega(x(s)) ds.$$

In [1], the authors proved the following theorem.

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THEOREM A. *If (1.2) holds and $A\omega[b(t)] \int_0^t R(s) \exp\left(\int_0^s AQ(r)dr\right) ds < 1$ for all $t \in \mathbb{R}_+$; then*

$$x(t) \leq \omega^{-1} \left[\frac{\omega[b(t)] \exp\left(\int_0^t AQ(s)ds\right)}{1 - A\omega[b(t)] \int_0^t R(s) \exp\left(\int_0^s AQ(r)dr\right) ds} \right], \quad t \geq 0 \tag{1.3}$$

where

$$Q(t) = \frac{d}{dt} \left(\int_0^{\alpha(t)} K_1 f_1(t, s) ds + \int_0^t K_2 f_2(t, s) ds \right);$$

$$R(t) = \frac{d}{dt} \left(\int_0^{\alpha(t)} K_3 f_3(t, s) ds + \int_0^t K_4 f_4(t, s) ds \right).$$

$t \in S$ is chosen in such a way that $\frac{\omega[b(t)] \exp(\int_0^t AQ(s)ds)}{1 - A\omega[b(t)] \int_0^t R(s) \exp(\int_0^s AQ(r)dr)ds} \in \text{Dom}(\omega^{-1})$, ω^{-1} is the inverse of ω .

Unfortunately, the proof of Theorem A is wrong.

In fact, In [1] in page 856, line 5, the inequality

$$z'(t) + z(t) \left(\frac{d}{dt} q(t) \right) \geq -R(t). \tag{1.4}$$

should be

$$\frac{z'(t)}{\omega'[b(T) + x(t)]} + z(t) \left(\frac{d}{dt} q(t) \right) \geq -R(t). \tag{1.5}$$

The inequality (1.5) is more complicated to handle than the inequality (1.4). The proof method in [1] can not be used to handle the inequality (1.5).

In this paper, we made the corrections of the proof of the Theorem A and prove the following theorem.

THEOREM B. *Suppose (1.1) is holds, $\omega'(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ is bounded, let $\sup_{s \in [0, \infty)} \omega'(s) = A > 0$ and $A\omega[b(t)] \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s), 0) \Delta s < 1$ for all $t \in \mathbb{R}_+$; then*

$$x(t) \leq \omega^{-1} \left[\frac{\omega[b(t)] e_{\ominus(-AQ)}(t, 0)}{1 - A\omega[b(t)] \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s), 0) \Delta s} \right], \quad t \geq 0 \tag{1.6}$$

where

$$Q(t) = \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \Delta s \right)^\Delta;$$

$$R(t) = \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \Delta s + \int_0^{\alpha_4(t)} K_4 f_4(t, s) \Delta s \right)^\Delta.$$

As an applications, we study the stability of a kind of difference equations.

We recall some basic results for dynamic equations and the calculus on time scales.

DEFINITION 1.1. (see [7]) A function $h : \mathbb{T} \rightarrow \mathbb{R}$ is said to be regressive provided $1 + \mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \rightarrow \mathbb{R}$ is denoted by \mathfrak{R} , while the set \mathfrak{R}^+ is given by $\mathfrak{R}^+ = \{h \in \mathfrak{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Let $\varphi \in \mathfrak{R}$. The exponential function on \mathbb{T} is defined by $e_\varphi(t, s) = \exp\left(\int_s^t \xi_{\mu(r)}(\varphi(r))\Delta r\right)$.

Here $\xi_{\mu(s)}$ is the cylinder transformation given by

$$\xi_{\mu(r)}(\varphi(r)) := \begin{cases} \frac{1}{\mu(r)}\text{Log}\left(1 + \mu(r)\varphi(r)\right), & \mu(r) > 0, \\ \varphi(r), & \mu(r) = 0. \end{cases}$$

and the exponential function $e_\varphi(t, s)$, $\varphi \in \mathfrak{R}^+$ has the following properties:

$$\left\{ \begin{aligned} e_0(s, t) \equiv 1, \quad e_\varphi(t, t) \equiv 1 \text{ and } e_\varphi(\sigma(t), s) &= (1 + \mu(t)\varphi(t))e_\varphi(t, s), \\ \frac{1}{e_\varphi(t, s)} = e_{\ominus\varphi}(t, s) \text{ where } \ominus\varphi(t) &= -\frac{\varphi(t)}{1 + \mu(t)\varphi(t)}. \end{aligned} \right. \tag{1.7}$$

LEMMA 1.1. (see [2]) (contains Chain Rule 1 and Chain Rule 2)

Chain Rule 1: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable and the formula

$$(f \circ g)^\Delta(t) = f'(g(c))g^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh \right\} g^\Delta(t); \tag{1.8}$$

holds; there exists c in the real interval $[t, \sigma(t)]$.

Chain Rule 2: Assume that $v : \mathbb{T} \rightarrow \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $\omega : \tilde{\mathbb{T}} \rightarrow \mathbb{R}$. If $v^\Delta(t)$ and $\omega^\Delta(v(t))$ exist for $t \in \mathbb{T}^k$, then

$$(\omega \circ v)^\Delta = (\omega^\Delta \circ v)v^\Delta. \tag{1.9}$$

LEMMA 1.2. (see [2]) If $f \in C_{rd}$ and $t \in \mathbb{T}^{\mathbb{K}}$, then

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t) = f(t)(\sigma(t) - t). \tag{1.10}$$

LEMMA 1.3. (see [2]) Let $a \in \mathbb{T}^{\mathbb{K}}, b \in \mathbb{T}$ and assume $f : \mathbb{T} \times \mathbb{T}^{\mathbb{K}} \rightarrow \mathbb{R}$ is continuous at (t, t) , where $t \in \mathbb{T}^{\mathbb{K}}$ with $t > a$. Also assume that $f^\Delta(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t , independent of $\tau \in [a, \sigma(t)]$, such that

$$|f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \geq U. \tag{1.11}$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

$$\left\{ \begin{aligned} (i) \quad g(t) &:= \int_a^t f(t, \tau)\Delta\tau \text{ implies } g^\Delta(t) = \int_a^t f^\Delta(t, \tau)\Delta\tau + f(\sigma(t), t); \\ (ii) \quad h(t) &:= \int_t^b f(t, \tau)\Delta\tau \text{ implies } h^\Delta(t) = \int_t^b f^\Delta(t, \tau)\Delta\tau - f(\sigma(t), t). \end{aligned} \right. \tag{1.12}$$

LEMMA 1.4. (see [11]) *Let f be an integrable function on $[a, b]$ and $m = \inf\{f(t) : t \in [a, b]\}$; $M = \sup\{f(t) : t \in [a, b]\}$. Then there exists a number $\Lambda \in [m, M]$ such that*

$$\int_a^b f(t)\Delta t = \Lambda(b - a). \tag{1.13}$$

2. Main results

In what follows, \mathbb{T} is an arbitrary time scale, $\mathbb{R}_+ = [0, +\infty)_{\mathbb{T}}$; C_{rd} denotes the set of rd continuous functions.

First, we give an extended form of Lemma 1.2 and its proof.

LEMMA 2.1. *If $f \in C_{rd}$ and $t \in \mathbb{T}^{\mathbb{K}}$, $\alpha(t) \in C_{rd}$ is strictly increasing function for $t \in \mathbb{T}$, then*

$$\int_{\alpha(t)}^{\alpha(\sigma(t))} f(\sigma(t), \tau)\Delta\tau = f(\sigma(t), \alpha(t))(\alpha(\sigma(t)) - \alpha(t)). \tag{2.1}$$

Proof. There exists an antiderivative F of f , and $F^{\Delta\tau}(\sigma(t), \tau) = f(\sigma(t), \tau)$

$$\begin{aligned} \int_{\alpha(t)}^{\alpha(\sigma(t))} f(\sigma(t), \tau)\Delta\tau &= F(\sigma(t), \alpha(\sigma(t))) - F(\sigma(t), \alpha(t)) \\ &= \frac{F(\sigma(t), \alpha(\sigma(t))) - F(\sigma(t), \alpha(t))}{\alpha(\sigma(t)) - \alpha(t)}(\alpha(\sigma(t)) - \alpha(t)) \\ &= F^{\Delta\tau}(\sigma(t), \tau)\Big|_{\tau=\alpha(t)}(\alpha(\sigma(t)) - \alpha(t)) \\ &= f(\sigma(t), \alpha(t))(\alpha(\sigma(t)) - \alpha(t)). \end{aligned}$$

The proof of Lemma 2.1 is completed. \square

Then, we give an extended form of Lemma 1.3 and its proof.

LEMMA 2.2. *Assume $f(t, s)$; $f^{\Delta t}(t, s) \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{T})$; $\alpha(t)$ is strictly increasing function, $\alpha^{\Delta}(t)$ is rd-continuous on \mathbb{T} . Then*

$$\left\{ \begin{array}{l} (i) \ g(t) := \int_a^{\alpha(t)} f(t, \tau)\Delta\tau \text{ implies } g^{\Delta}(t) = \int_a^{\alpha(t)} f^{\Delta}(t, \tau)\Delta\tau + f(\sigma(t), \alpha(t))\alpha^{\Delta}(t); \\ (ii) \ h(t) := \int_{\alpha(t)}^b f(t, \tau)\Delta\tau \text{ implies } h^{\Delta}(t) = \int_{\alpha(t)}^b f^{\Delta}(t, \tau)\Delta\tau - f(\sigma(t), \alpha(t))\alpha^{\Delta}(t). \end{array} \right. \tag{2.2}$$

Proof. Here we only prove (i), (ii) similar.

We will discuss two cases:

(1) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$\begin{aligned}
 g^\Delta(t) &= \frac{g(\sigma(t)) - g(t)}{\mu(t)} = \frac{\int_a^{\alpha(\sigma(t))} f(\sigma(t), \tau) \Delta \tau - \int_a^{\alpha(t)} f(t, \tau) \Delta \tau}{\mu(t)}; \tag{2.3} \\
 &= \int_a^{\alpha(t)} \frac{f(\sigma(t), \tau) - f(t, \tau)}{\mu(t)} \Delta \tau + \frac{1}{\mu(t)} \int_{\alpha(t)}^{\alpha(\sigma(t))} f(\sigma(t), \tau) \Delta \tau; \\
 &= \int_a^{\alpha(t)} f^\Delta(t, \tau) \Delta \tau + f(\sigma(t), \alpha(t)) \frac{\alpha(\sigma(t)) - \alpha(t)}{\mu(t)}; \\
 &= \int_a^{\alpha(t)} f^\Delta(t, \tau) \Delta \tau + f(\sigma(t), \alpha(t)) \alpha^\Delta(t).
 \end{aligned}$$

(2) If t is right-dense, let's assume $s \leq \sigma(t)$, Clearly $f(t, \tau)$; $f(\sigma(t), \tau)$ are continuous and bounded function for $(t, \tau) \in \mathbb{T} \times [\alpha(s), \alpha(\sigma(t))]\mathbb{T}$.

Let

$$m = \inf\{f(\sigma(t), \tau) : \tau \in [\alpha(s), \alpha(\sigma(t))]\}; \quad M = \sup\{f(\sigma(t), \tau) : \tau \in [\alpha(s), \alpha(\sigma(t))]\}.$$

By Lemma 1.5, there exists a number $\Lambda \in [m, M]$ such that

$$\int_{\alpha(s)}^{\alpha(\sigma(t))} f(\sigma(t), \tau) \Delta \tau = \Lambda (\alpha(\sigma(t)) - \alpha(s)). \tag{2.4}$$

By the function α, f continuity

$$\begin{aligned}
 \lim_{s \rightarrow t} (\alpha(\sigma(t)) - \alpha(s)) &= 0; \tag{2.5} \\
 \implies \lim_{s \rightarrow t} (\sup\{f(\sigma(t), \tau) : \tau \in [\alpha(s), \alpha(\sigma(t))]\} - \inf\{f(\sigma(t), \tau) : \tau \in [\alpha(s), \alpha(\sigma(t))]\}) \\
 &= \lim_{s \rightarrow t} (M - m) = 0; \\
 \implies \lim_{s \rightarrow t} \Lambda &= f(\sigma(t), \alpha(t)).
 \end{aligned}$$

Then f is differentiable at t if and only if the limit

$$\begin{aligned}
 g^\Delta(t) &= \lim_{s \rightarrow t} \frac{g(\sigma(t)) - g(s)}{\sigma(t) - s} = \lim_{s \rightarrow t} \frac{\int_a^{\alpha(\sigma(t))} f(\sigma(t), \tau) \Delta \tau - \int_a^{\alpha(s)} f(s, \tau) \Delta \tau}{\sigma(t) - s}; \tag{2.6} \\
 &= \lim_{s \rightarrow t} \int_a^{\alpha(s)} \frac{f(\sigma(t), \tau) - f(s, \tau)}{\sigma(t) - s} \Delta \tau + \lim_{s \rightarrow t} \frac{1}{\sigma(t) - s} \int_{\alpha(s)}^{\alpha(\sigma(t))} f(\sigma(t), \tau) \Delta \tau; \\
 &= \int_a^{\alpha(t)} f^\Delta(t, \tau) \Delta \tau + \lim_{s \rightarrow t} \frac{\Lambda (\alpha(\sigma(t)) - \alpha(s))}{\sigma(t) - s}; \\
 &= \int_a^{\alpha(t)} f^\Delta(t, \tau) \Delta \tau + f(\sigma(t), \alpha(t)) \alpha^\Delta(t).
 \end{aligned}$$

The proof of Lemma 2.2 is completed. \square

With these results, we can get the following Theorem 2.1.

THEOREM 2.1. *We assume that the following conditions hold:*

- (H₁): *binary function: $f_i(t, s); f_i^{\Delta t}(t, s) \in C_{rd}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$; ($i = 1, 2, 3, 4$);*
- (H₂): *$b(t), \omega(t)$ are non-decreasing functions; $b(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$, $\omega(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\exists A > 0, \sup_{s \in [0, \infty)} \omega'(s) = A$;*
- (H₃): *$\alpha_j(t)$ is strictly increasing function, $\alpha_j(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha_j(0) = 0, \alpha_j(t) \leq t$, for $t \in \mathbb{R}_+$; ($j = 1, 2, 3, 4$).*
- (H₄): *$x(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$x(t) \leq b(t) + \int_0^{\alpha_1(t)} K_1 f_1(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \omega(x(s)) \Delta s \tag{2.7}$$

$$+ \int_0^{\alpha_3(t)} K_3 f_3(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_4(t)} K_4 f_4(t, s) \omega(x(s)) \Delta s;$$

and $A\omega[b(t)] \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s), 0) \Delta s < 1$ for all $t \in \mathbb{R}_+$; where K_i ($i = 1, 2, 3, 4$) are nonnegative constants. Then

$$x(t) \leq \omega^{-1} \left[\frac{\omega[b(t)] e_{\ominus(-AQ)}(t, 0)}{1 - A\omega[b(t)] \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s), 0) \Delta s} \right], \quad t \geq 0 \tag{2.8}$$

where

$$Q(t) = \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \Delta s \right)^\Delta;$$

$$R(t) = \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \Delta s + \int_0^{\alpha_4(t)} K_4 f_4(t, s) \Delta s \right)^\Delta.$$

$t \in S$ is chosen in such a way that $\frac{\omega[b(t)] e_{\ominus(-AQ)}(t, 0)}{1 - A\omega[b(t)] \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s), 0) \Delta s} \in \text{Dom}(\omega^{-1})$, ω^{-1} is the inverse of ω .

Proof. First we define the function $y(t)$ by

$$y(t) = \int_0^{\alpha_1(t)} K_1 f_1(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \omega(x(s)) \Delta s \tag{2.9}$$

$$+ \int_0^{\alpha_3(t)} K_3 f_3(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_4(t)} K_4 f_4(t, s) \omega(x(s)) \Delta s.$$

Since (H₁), (H₂), (H₃), we know that $y(t)$ is non-decreasing function for $t \in \mathbb{R}_+$ and $y(0) = 0$.

Let $T \in S, T \geq 0$ be fixed, $\forall t \in [0, T]: x(t) \leq b(t) + y(t) \leq b(T) + y(t)$.

By (2.2) and (H₂), (H₃), we have

$$\left\{ \begin{aligned} & \left(\int_0^{\alpha_j(t)} f_j(t, s) \omega(x(s)) \Delta s \right)^\Delta \\ & = \int_0^{\alpha_j(t)} f_j^\Delta(t, s) \omega(x(s)) \Delta s + f_j(\sigma(t), \alpha_j(t)) \omega(x(\alpha_j(t))) \alpha_j^\Delta(t); \\ & x(\alpha_j(t)) \leq b(\alpha_j(t)) + y(\alpha_j(t)) \leq b(t) + y(t) \leq b(T) + y(t), \quad (j = 1, 2, 3, 4), t \in [0, T]. \end{aligned} \right. \tag{2.10}$$

Therefore, $y(t)$ seeking derivative in $[0, T]$, we have

$$y^\Delta(t) = \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \omega(x(s)) \Delta s \right)^\Delta + \left(\int_0^{\alpha_2(t)} K_2 f_2(t, s) \omega(x(s)) \Delta s \right)^\Delta \quad (2.11)$$

$$+ \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \omega(x(s)) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) \omega(x(s)) \Delta s \right)^\Delta.$$

By (2.10) we have

$$\left(\int_0^{\alpha_j(t)} K_j f_j(t, s) \omega(x(s)) \Delta s \right)^\Delta \quad (2.12)$$

$$= \int_0^{\alpha_j(t)} K_j f_j^\Delta(t, s) \omega(x(s)) \Delta s + K_j f_j(\sigma(t), \alpha_j(t)) \omega(x(\alpha_j(t))) \alpha_j^\Delta(t);$$

$$\leq \omega[b(T) + y(t)] \left(\int_0^{\alpha_j(t)} K_j f_j^\Delta(t, s) \Delta s + K_j f_j(\sigma(t), \alpha_j(t)) \alpha_j^\Delta(t) \right)$$

$$= \omega[b(T) + y(t)] \left(\int_0^{\alpha_j(t)} K_j f_j(t, s) \Delta s \right)^\Delta; \quad (j = 1, 2)$$

and

$$\left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \omega(x(s)) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) \omega(x(s)) \Delta s \right)^\Delta \quad (2.13)$$

$$= \left(\int_0^{\alpha_3(t)} K_3 f_3^\Delta(t, s) \omega(x(s)) \Delta s + K_3 \alpha_3^\Delta(t) f_3(\sigma(t), \alpha_3(t)) \omega(x(\alpha_3(t))) \right)$$

$$\times \int_0^{\alpha_4(t)} K_4 f_4(t, s) \omega(x(s)) \Delta s$$

$$+ \left(\int_0^{\alpha_4(t)} K_4 f_4^\Delta(t, s) \omega(x(s)) \Delta s + K_4 \alpha_4^\Delta(t) f_4(\sigma(t), \alpha_4(t)) \omega(x(\alpha_4(t))) \right)$$

$$\times \int_0^{\alpha_3(\sigma(t))} K_3 f_3(\sigma(t), s) \omega(x(s)) \Delta s;$$

$$\leq \omega^2[b(T) + y(t)] \left(\int_0^{\alpha_3(t)} K_3 f_3^\Delta(t, s) \Delta s + K_3 \alpha_3^\Delta(t) f_3(\sigma(t), \alpha_3(t)) \right)$$

$$\times \int_0^{\alpha_4(t)} K_4 f_4(t, s) \Delta s$$

$$+ \omega[b(T) + y(t)] \left(\int_0^{\alpha_4(t)} K_4 f_4^\Delta(t, s) \Delta s + K_4 \alpha_4^\Delta(t) f_4(\sigma(t), \alpha_4(t)) \right)$$

$$\times \int_0^{\alpha_3(\sigma(t))} K_3 f_3(\sigma(t), s) \omega(x(s)) \Delta s.$$

By lemma 2.1, then

$$\int_0^{\alpha_3(\sigma(t))} K_3 f_3(\sigma(t), s) \omega(x(s)) \Delta s; \quad (2.14)$$

$$= \int_0^{\alpha_3(t)} K_3 f_3(\sigma(t), s) \omega(x(s)) \Delta s + \int_{\alpha_3(t)}^{\alpha_3(\sigma(t))} K_3 f_3(\sigma(t), s) \omega(x(s)) \Delta s;$$

$$\begin{aligned}
 &\leq \omega[b(T) + y(t)] \int_0^{\alpha_3(t)} K_3 f_3(\sigma(t), s) \Delta s \\
 &\quad + K_3 f_3(\sigma(t), \alpha_3(t)) \omega(x(\alpha_3(t))) (\alpha_3(\sigma(t)) - \alpha_3(t)); \\
 &\leq \omega[b(T) + y(t)] \left[\int_0^{\alpha_3(t)} K_3 f_3(\sigma(t), s) \Delta s + K_3 f_3(\sigma(t), \alpha_3(t)) (\alpha_3(\sigma(t)) - \alpha_3(t)) \right]; \\
 &= \omega[b(T) + y(t)] \left[\int_0^{\alpha_3(t)} K_3 f_3(\sigma(t), s) \Delta s + \int_{\alpha_3(t)}^{\alpha_3(\sigma(t))} K_3 f_3(\sigma(t), s) \Delta s \right]; \\
 &= \omega[b(T) + y(t)] \int_0^{\alpha_3(\sigma(t))} K_3 f_3(\sigma(t), s) \Delta s.
 \end{aligned}$$

Put (2.14) into (2.13), we have

$$\begin{aligned}
 &\left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \omega(x(s)) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) \omega(x(s)) \Delta s \right)^\Delta; \tag{2.15} \\
 &\leq \omega^2[b(T) + y(t)] \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) \Delta s \right)^\Delta.
 \end{aligned}$$

Next we put (2.12), (2.15) into (2.11), and so we obtain

$$\begin{aligned}
 y^\Delta(t) &\leq \omega[b(T) + y(t)] \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \Delta s \right)^\Delta \tag{2.16} \\
 &\quad + \omega^2[b(T) + y(t)] \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) \Delta s \right)^\Delta.
 \end{aligned}$$

Let

$$\begin{aligned}
 Q(t) &= \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \Delta s \right)^\Delta; \\
 R(t) &= \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) \Delta s \right)^\Delta. \tag{2.17}
 \end{aligned}$$

By (H_1) , (H_2) , (H_3) , we obtain $Q(t)$, $R(t)$ is non-decreasing function for $t \in \mathbb{R}_+$.

Suppose $b(t) > 0$ (if $b(t) = 0$, carry out the following arguments with $b(t) + \varepsilon$ instead of $b(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \mapsto 0$ to complete the proof) then we put (2.17) into (2.16)

$$\frac{y^\Delta(t)}{\omega^2[b(T) + y(t)]} - \frac{Q(t)}{\omega[b(T) + y(t)]} \leq R(t). \tag{2.18}$$

Let

$$\begin{aligned}
 z(t) &:= \frac{1}{\omega[b(T) + y(t)]} \tag{2.19} \\
 \Rightarrow z^\Delta(t) &= - \frac{\omega'(\zeta) y^\Delta(t)}{\omega[b(T) + y(t)] \omega[b(T) + y(\sigma(t))]} \leq 0, \zeta \in [b(T), b(T) + y(T)].
 \end{aligned}$$

Then

$$\begin{aligned} \frac{y^\Delta(t)}{\omega^2[b(T) + y(t)]} &= -\frac{z^\Delta(t)}{\omega'(\zeta)} \cdot \frac{\omega[b(T) + y(\sigma(t))]}{\omega[b(T) + y(t)]} \\ &= -\frac{z^\Delta(t)}{\omega'(\zeta)} \cdot \frac{z(t)}{z(\sigma(t))} \geq -\frac{z^\Delta(t)}{\omega'(\zeta)} \geq -\frac{z^\Delta(t)}{A}. \end{aligned} \tag{2.20}$$

Put (2.19), (2.20) into (2.18), we have

$$z^\Delta(t) + A Q(t) z(t) \geq -AR(t). \tag{2.21}$$

Then we have

$$\left(\frac{z(t)}{e_{-AQ}(t,0)}\right)^\Delta = \frac{z^\Delta(t) + A Q(t) z(t)}{e_{-AQ}(\sigma(t),0)} \geq -\frac{AR(t)}{e_{-AQ}(\sigma(t),0)} = -AR(t) e_{\ominus(-AQ)}(\sigma(t),0). \tag{2.22}$$

Integrating both sides of (2.22) from 0 to t and we obtain that

$$\frac{z(t)}{e_{-AQ}(t,0)} \geq z(0) - A \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s),0) \Delta s. \tag{2.23}$$

Hence from (2.23) we obtain

$$\begin{aligned} z(t) &= \frac{1}{\omega[b(T) + y(t)]} \geq e_{-AQ}(t,0) \left(\frac{1}{\omega[b(T)]} - A \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s),0) \Delta s \right); \\ &= \frac{e_{-AQ}(t,0)}{\omega[b(T)]} \left(1 - A \omega[b(T)] \int_0^t R(s) e_{\ominus(-AQ)}(\sigma(s),0) \Delta s \right), \quad \text{for } 0 \leq t \leq T. \end{aligned} \tag{2.24}$$

Let $t = T$ and note that $A \omega[b(T)] \int_0^T R(s) e_{\ominus(-AQ)}(\sigma(s),0) \Delta s < 1$, we have

$$\omega[b(T) + y(T)] \leq \frac{\omega[b(T)] e_{\ominus(-AQ)}(T,0)}{1 - A \omega[b(T)] \int_0^T R(s) e_{\ominus(-AQ)}(\sigma(s),0) \Delta s}. \tag{2.25}$$

Considering that the function ω is non-decreasing on \mathbb{R}_+ , then

$$x(T) \leq b(T) + y(T) \leq \omega^{-1} \left[\frac{\omega[b(T)] e_{\ominus(-AQ)}(T,0)}{1 - A \omega[b(T)] \int_0^T R(s) e_{\ominus(-AQ)}(\sigma(s),0) \Delta s} \right], \quad T \in \mathbb{S}. \tag{2.26}$$

Since $T \geq 0$ is arbitrarily chosen, we get (2.8), The proof of Theorem 2.1 is completed. \square

Different choices of $K_i, f_i, \alpha_i, (i = 1, 2, 3, 4)$ in Theorem 2.1, we can derive many forms of inequalities.

COROLLARY 2.1. Let $K_i, f_i, \alpha_i, (i = 1, 2), \omega$ be as in Theorem 2.1. Assume in addition that $b_i(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+), (i = 1, 2)$ is non-decreasing. If $x(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$x(t) \leq \left(b_1(t) + \int_0^{\alpha_1(t)} K_1 f_1(t, s) \omega(x(s)) \Delta s \right) \left(b_2(t) + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \omega(x(s)) \Delta s \right) \tag{2.27}$$

and $A\omega[b_1(t)b_2(t)] \int_0^t R(s) e_{\ominus(-A Q)}(\sigma(s), 0) \Delta s < 1$ for all $t \in \mathbb{R}_+$; where $K_i (i = 1, 2)$ are nonnegative constants then

$$x(t) \leq \omega^{-1} \left[\frac{\omega[b_1(t)b_2(t)] e_{\ominus(-A Q)}(t, 0)}{1 - A\omega[b_1(t)b_2(t)] \int_0^t R(s) e_{\ominus(-A Q)}(\sigma(s), 0) \Delta s} \right], \quad t \geq 0 \tag{2.28}$$

where

$$\begin{cases} Q(t) = \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) b_2(t) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) b_1(t) \Delta s \right)^\Delta; \\ R(t) = \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \Delta s \int_0^{\alpha_2(t)} K_2 f_2(t, s) \Delta s \right)^\Delta. \end{cases}$$

$t \in S$ is chosen in such a way that $\frac{\omega[b_1(t)b_2(t)] e_{\ominus(-A Q)}(t, 0)}{1 - A\omega[b_1(t)b_2(t)] \int_0^t R(s) e_{\ominus(-A Q)}(\sigma(s), 0) \Delta s} \in \text{Dom}(\omega^{-1}), \omega^{-1}$ is the inverse of ω .

We expand (2.27) and compare (2.7) and apply Theorem 2.1, we can get (2.28).

In the nonlinear case, let $\omega(x) = x$ in the theorems 2.1. We get the results of linear case.

COROLLARY 2.2. Let $K_i, f_i, \alpha_i (i = 1, 2, 3, 4), b(t)$ be as in Theorem 2.1. If $x(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\begin{aligned} x(t) \leq & b(t) + \int_0^{\alpha_1(t)} K_1 f_1(t, s) x(s) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) x(s) \Delta s \\ & + \int_0^{\alpha_3(t)} K_3 f_3(t, s) x(s) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) x(s) \Delta s; \end{aligned} \tag{2.29}$$

and $Ab(t) \int_0^t R(s) e_{\ominus(-A Q)}(\sigma(s), 0) \Delta s < 1$ for all $t \in \mathbb{R}_+$; where $K_i (i = 1, 2, 3, 4)$ are nonnegative constants. Then

$$x(t) \leq \frac{b(t) e_{\ominus(-A Q)}(t, 0)}{1 - Ab(t) \int_0^t R(s) e_{\ominus(-A Q)}(\sigma(s), 0) \Delta s}, \quad t \geq 0 \tag{2.30}$$

where

$$\begin{cases} Q(t) = \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \Delta s \right)^\Delta; \\ R(t) = \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \Delta s \int_0^{\alpha_4(t)} K_4 f_4(t, s) \Delta s \right)^\Delta. \end{cases}$$

Theorem 2.1 can be extended to a more general form.

COROLLARY 2.3. We assume that the following conditions hold:

(H₁): binary function: $f_i(t, s); f_i^{\Delta t}(t, s) \in C_{rd}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$; ($i = 0, 1, 2, 3, 4, 5, 6$);

(H₂): $b(t), \omega(t)$ are non-decreasing functions; $b(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$, $\omega(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\exists A > 0, \sup_{s \in [0, \infty)} \omega'(s) = A$;

(H₃): $\alpha_j(t)$ is strictly increasing function, $\alpha_j(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha_j(0) = 0, \alpha_j(t) \leq t$, for $t \in \mathbb{R}_+$; ($j = 1, 2, 3, 4, 5, 6$).

(H₄): $x(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\begin{aligned}
 x(t) \leq & b(t) + \int_0^{\alpha_1(t)} K_1 f_1(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \omega(x(s)) \Delta s \quad (2.31) \\
 & + \int_0^{\alpha_3(t)} K_3 f_3(t, s) \omega(x(s)) \left(K_0 f_0(t, s) + \int_0^{\alpha_4(s)} K_4 f_4(t, r) \omega(x(r)) \Delta r \right) \Delta s \\
 & + \int_0^{\alpha_5(t)} K_5 f_5(t, s) \omega(x(s)) \Delta s \int_0^{\alpha_6(t)} K_6 f_6(t, s) \omega(x(s)) \Delta s;
 \end{aligned}$$

and $A\omega[b(t)] \int_0^t R(s) e_{\ominus(-A\mathcal{Q})}(\sigma(s), 0) \Delta s < 1$ for all $t \in \mathbb{R}_+$; where K_i ($i=0, 1, 2, 3, 4, 5, 6$) are nonnegative constants. Then

$$x(t) \leq \omega^{-1} \left[\frac{\omega[b(t)] e_{\ominus(-A\mathcal{Q})}(t, 0)}{1 - A\omega[b(t)] \int_0^t R(s) e_{\ominus(-A\mathcal{Q})}(\sigma(s), 0) \Delta s} \right], \quad t \geq 0 \quad (2.32)$$

where

$$\left\{ \begin{aligned}
 Q(t) &= \left(\int_0^{\alpha_1(t)} K_1 f_1(t, s) \Delta s + \int_0^{\alpha_2(t)} K_2 f_2(t, s) \Delta s + \int_0^{\alpha_3(t)} K_0 K_3 f_0(t, s) f_3(t, s) \Delta s \right) \Delta; \\
 R(t) &= \left(\int_0^{\alpha_3(t)} K_3 f_3(t, s) \int_0^{\alpha_4(s)} K_4 f_4(t, r) \Delta r \Delta s \right. \\
 &\quad \left. + \int_0^{\alpha_5(t)} K_5 f_5(t, s) \Delta s \int_0^{\alpha_6(t)} K_6 f_6(t, s) \Delta s \right) \Delta.
 \end{aligned} \right.$$

$t \in S$ is chosen in such a way that $\frac{\omega[b(t)] e_{\ominus(-A\mathcal{Q})}(t, 0)}{1 - A\omega[b(t)] \int_0^t R(s) e_{\ominus(-A\mathcal{Q})}(\sigma(s), 0) \Delta s} \in \text{Dom}(\omega^{-1})$, ω^{-1} is the inverse of ω .

In fact, according to Theorem 2.1 the same way, we can get the proof of Corollary 2.3. Since the proof is similar, the details are left to the readers.

3. Applications and Examples

EXAMPLE 1. Consider the integral equation

$$\begin{aligned}
 x(t) = & x(0) + \int_0^{\alpha_1(t)} f_1(t, s) \omega(x(s)) \Delta s + \int_0^{\alpha_2(t)} f_2(t, s) \omega(x(s)) \Delta s \quad (3.1) \\
 & + \int_0^{\alpha_3(t)} f_3(t, s) \omega(x(s)) \Delta s \int_0^{\alpha_4(t)} f_4(t, s) \omega(x(s)) \Delta s;
 \end{aligned}$$

(H₁): binary function: $f_i(t, s); f_i^{\Delta t}(t, s) \in C_{rd}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$; ($i = 1, 2, 3, 4$);

(H₂): $\omega(t)$ is non-decreasing functions, $\omega(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\exists A > 0$,

$\sup_{s \in [0, \infty)} \omega'(s) = A$;

(H₃): $\alpha_j(t)$ is strictly increasing function, $\alpha_j(t) \in C_{rd}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha_j(0) = 0$, $\alpha_j(t) \leq t$, for $t \in \mathbb{R}_+$; ($j = 1, 2, 3, 4$).

(H₄): $x(t) \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\begin{aligned}
 x(t) &\leq x(0) + \int_0^t f_1(t, s)\omega(x(s))\Delta s + \int_0^t f_2(t, s)\omega(x(s))\Delta s \\
 &\quad + \int_0^t f_3(t, s)\omega(x(s))\Delta s + \int_0^t f_4(t, s)\omega(x(s))\Delta s
 \end{aligned} \tag{3.2}$$

and $A\omega[x(0)] \int_0^t R(s)e_{\ominus(-AQ)}(\sigma(s), 0)\Delta s < 1$ for all $t \in \mathbb{R}_+$; then

$$x(t) \leq \omega^{-1} \left[\frac{\omega[x(0)]e_{\ominus(-AQ)}(t, 0)}{1 - A\omega[x(0)] \int_0^t R(s)e_{\ominus(-AQ)}(\sigma(s), 0)\Delta s} \right], \quad t \geq 0 \tag{3.3}$$

where

$$\begin{aligned}
 Q(t) &= \left(\int_0^t (f_1(t, s) + f_2(t, s))\Delta s \right)^\Delta; \\
 R(t) &= \left(\int_0^t f_3(t, s)\Delta s + \int_0^t f_4(t, s)\Delta s \right)^\Delta.
 \end{aligned}$$

$t \in S$ is chosen in such a way that $\frac{\omega[x(0)]e_{\ominus(-AQ)}(t, 0)}{1 - A\omega[x(0)] \int_0^t R(s)e_{\ominus(-AQ)}(\sigma(s), 0)\Delta s} \in \text{Dom}(\omega^{-1})$, ω^{-1} is the inverse of ω .

In the following, we let $\mathbb{T} = \mathbb{Z}$ and choose some explicit functions for $f_i(t, s)$, $\alpha_i(t) \in \mathbb{N}$, ($i = 1, 2, 3, 4$); $\omega(x(t))$; and (3.1), (3.2) implies

$$\begin{aligned}
 x(n) &= x(0) + \sum_{i=0}^{\alpha_1(n)-1} f_1(n, i)\omega(x(i)) + \sum_{i=0}^{\alpha_2(n)-1} f_2(n, i)\omega(x(i)) \\
 &\quad + \sum_{i=0}^{\alpha_3(n)-1} \left(f_3(n, i)\omega(x(i)) \sum_{j=0}^{\alpha_4(i)-1} f_4(i, j)\omega(x(j)) \right).
 \end{aligned} \tag{3.4}$$

$$\begin{aligned}
 x(n) &\leq x(0) + \sum_{i=0}^{n-1} f_1(n, i)\omega(x(i)) + \sum_{i=0}^{n-1} f_2(n, i)\omega(x(i)) \\
 &\quad + \sum_{i=0}^{n-1} f_3(n, i)\omega(x(i)) \cdot \sum_{i=0}^{n-1} f_4(n, i)\omega(x(i)).
 \end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}
 f_1(n, i) &= \frac{1}{2^{2^i} + 2}, \quad f_2(n, i) = \frac{1}{(2^{2^i} + 1)(2^{2^i} + 2)}, \quad f_3(n, i) = \frac{11}{12(i+1)(i+2)(i+3)}, \\
 f_4(n, i) &= \frac{(n+1)12^n}{(n+3)(12^n - 1)} \cdot \frac{1}{12^i}; \quad \omega(x(n)) = \sqrt{x^2(n) + 1} - 1, \quad n, i \in \mathbb{N}.
 \end{aligned}$$

Clearly, f_1, f_2, f_3 are non-decreasing on n ; $\omega(x(n))$ is non-linear.

Let $\delta(t) := \frac{(t+1)12^t}{(t+3)(12^t-1)}, t \geq 1 \implies \delta'(t) = \frac{[2(12^t-1)-(t+1)(t+3)\ln 12]12^t}{(t+3)^2(12^t-1)^2} > 0, t \geq 1$;
 which implies $f_4(n, i)$ is incrementing on n .

Therefore, $f_1, f_2, f_3, f_4, \omega$ satisfy the conditions of Theorem 2.1; then we have

$$\omega[x(0)] = \sqrt{x^2(0) + 1} - 1; \quad \omega^{-1}(x(n)) = (x(n) + 1)^2 - 1;$$

$$\Delta\omega(n) = \omega(n+1) - \omega(n) = \frac{2n+1}{\sqrt{n^2+1} + \sqrt{(n+1)^2+1}} \implies \sup_{n \in \mathbb{N}} \{\Delta\omega(n)\} = 1 = A;$$

$$\varphi(n) := \sum_{i=0}^{n-1} (f_1(n, i) + f_2(n, i)) = \sum_{i=0}^{n-1} \frac{1}{2^{2^i} + 1};$$

$$Q(n) = \Delta\varphi(n) = \varphi(n+1) - \varphi(n) = \frac{1}{2^{2^n} + 1};$$

$$\begin{aligned} \psi(n) &:= \sum_{i=0}^{n-1} f_3(n, i) \cdot \sum_{i=0}^{n-1} f_4(n, i) \\ &= \frac{(n+1)12^n}{(n+3)(12^n-1)} \sum_{i=0}^{n-1} \frac{11}{12(i+1)(i+2)(i+3)} \cdot \sum_{i=0}^{n-1} \frac{1}{12^i} = \frac{n}{4(n+2)}; \end{aligned}$$

$$R(n) = \Delta\psi(n) = \psi(n+1) - \psi(n) = \frac{1}{2(n+2)(n+3)}.$$

Since

$$\begin{aligned} &(2^{2^0} + 1)(2^{2^1} + 1)(2^{2^2} + 1) \cdots (2^{2^n} + 1) \\ &= [(2^{2^0} - 1)(2^{2^0} + 1)](2^{2^1} + 1)(2^{2^2} + 1) \cdots (2^{2^n} + 1); \\ &= [(2^{2^1} - 1)(2^{2^1} + 1)](2^{2^2} + 1) \cdots (2^{2^n} + 1) = \cdots = 2^{2^{n+1}} - 1; \end{aligned}$$

there is

$$\begin{aligned} e_{\ominus(-AQ)}(n, 0) &= e_{\frac{AQ}{1-AQ}}(n, 0) = e_{\frac{1}{2^{2^n}}}(n, 0) \\ &= \left(1 + \frac{1}{2^{2^1}}\right) \left(1 + \frac{1}{2^{2^2}}\right) \left(1 + \frac{1}{2^{2^3}}\right) \cdots \left(1 + \frac{1}{2^{2^n}}\right) e_{\frac{1}{2^{2^n}}}(0, 0) \\ &= \frac{4}{3} - \frac{1}{3 \cdot 2^{2^{k+1}-2}} \leq \frac{4}{3}; \end{aligned}$$

$$e_{\ominus(-AQ)}(k+1, 0) = e_{\frac{1}{2^{2^k}}}(k+1, 0) = \frac{8}{3} - \frac{1}{3 \cdot 2^{2^{k+1}-1}} \leq \frac{8}{3};$$

$$\sum_{k=0}^{n-1} R(k) e_{\ominus(-AQ)}(k+1, 0) \leq \sum_{k=0}^{n-1} \frac{1}{2(k+2)(k+3)} \cdot \frac{8}{3} = \frac{2n}{3(n+2)};$$

$$A\omega[x(0)] \sum_{k=0}^{n-1} R(k) e_{\ominus(-AQ)}(k, 0) \leq \frac{2n}{3(n+2)} (\sqrt{x^2(0) + 1} - 1).$$

For $A\omega[x(0)] \sum_{k=0}^n R(k)e_{\ominus(-AQ)}(k, 0) < 1$ establishment, as long as $\sqrt{x^2(0) + 1} - 1 < \frac{3}{2} + \frac{3}{n}$, then we let $\sqrt{x^2(0) + 1} - 1 \leq 1 < \frac{3}{2} + \frac{3}{n}$; where $x(0) \leq \sqrt{3}$.

$$\theta := \frac{\omega[x(0)]e_{\ominus(-AQ)}(n, 0)}{1 - A\omega[x(0)] \sum_{k=0}^{n-1} R(k)e_{\ominus(-AQ)}(k + 1, 0)} \leq \frac{\frac{4}{3}(\sqrt{x^2(0) + 1} - 1)}{1 - \frac{2n}{3(n+2)}(\sqrt{x^2(0) + 1} - 1)} \leq \frac{4(n+2)}{n+6}(\sqrt{x^2(0) + 1} - 1); \quad x(0) \leq \sqrt{3}.$$

Here we choose $x(0) < \min \left\{ \sqrt{3}, \sqrt{\frac{\epsilon}{12} + \frac{\epsilon^2}{576}} \right\} \Rightarrow \sqrt{x^2(0) + 1} - 1 < \frac{\epsilon}{24}$, then we have

$$\begin{aligned} x(n) &\leq \omega^{-1}(\theta) = (\theta + 1)^2 - 1 \leq \left(\frac{4(n+2)}{n+6}(\sqrt{x^2(0) + 1} - 1) + 1 \right)^2 - 1 \\ &\leq \frac{8(n+2)(3n+10)}{(n+6)^2}(\sqrt{x^2(0) + 1} - 1) \leq 24(\sqrt{x^2(0) + 1} - 1) < \epsilon. \end{aligned}$$

This proves that the trivial solution of (3.4) is uniformly stable on \mathbb{N} .

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Boqun Ou
School of Mathematics and Statistics
Lingnan Normal College
Zhanjiang 524048, P. R. China
e-mail: boqunou@163.com