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GENERALIZED REVERSE CAUCHY INEQUALITY AND APPLICATIONS TO OPERATOR MEANS

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Abstract. Let σ be an operator mean in the sense of Kubo-Ando and let ∇_α be a weighted arithmetic mean. If $\mathrm{Tr}(A\sigma B) \geqslant \mathrm{Tr}(A\nabla_\alpha B - \max\{\alpha, 1-\alpha\}|A-B|)$ holds for all positive semidefinite matrices A, B, then there exists $\beta \in [0,1]$ such that $\sigma = \nabla_\beta$.

1. Introduction

It is well-known as Young inequality that for $0 \leqslant \nu \leqslant 1$ and $a, b \geqslant 0$,

$$\nu a + (1-\nu)b \geqslant a^\nu b^{1-\nu} \geqslant \frac{a+b-|a-b|}{2}.$$

When $\nu = \frac{1}{2}$, the inequality $a^{\frac{1}{2}} b^{\frac{1}{2}} \geqslant \frac{a+b-|a-b|}{2}$ is called the reverse Cauchy inequality. A natural matrix form of the reverse Cauchy inequality could be written as

$$A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} \geqslant \frac{A+B}{2} - \frac{|A-B|}{2},$$

where A, B are positive semidefinite matrices. Furuichi, however, pointed out in [2] that the trace inequality $\mathrm{Tr}(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}) \geqslant \frac{1}{2}\mathrm{Tr}(A+B-|A-B|)$ is not true in general. Recall that $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ is the geometric mean and $A\nabla_\alpha B = (1-\alpha)A + \alpha B$ $(0 \leqslant \alpha \leqslant 1)$ is the α-weighted arithmetic mean of A and B, respectively.

Very recently, in [7], Hoa, Khue and the first author show that the inequality

$$A\sigma B \geqslant A\nabla B - \frac{1}{2}|A-B| \tag{1}$$

holds for any operator mean σ and positive semidefinite matrices A and B with $AB + BA \geqslant 0$. Later Hoa [6] shows that if σ is symmetric, that the inequality (1) holds for any positive semidefinite matrices A and B implies that $\sigma = \nabla$. (See Remark 15.)

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Motivatied by this, we study a characterization of operator means σ which satisfy a generalized reverse Cauchy inequality (1).

Our main result is the following: For $\alpha \in [0,1]$,

$$\begin{aligned} &\{\sigma \mid A\sigma B \geqslant A\nabla_\alpha B - \max\{\alpha, 1-\alpha\}|A-B| \text{ for all } A,B \geqslant 0\} \\ &= \{\sigma \mid \operatorname{Tr}(A\sigma B) \geqslant \operatorname{Tr}(A\nabla_\alpha B - \max\{\alpha, 1-\alpha\}|A-B|) \text{ for all } A,B \geqslant 0\} \\ &= \{\nabla_\beta \mid 0 \leqslant \beta \leqslant 1\}. \end{aligned}$$

We next introduce an operator inequality for an operator mean σ such that for all positive semidefinite matrices A and B,

$$\phi_1(A)\sigma\phi_1(B) \geqslant \phi_1(A\nabla_\nu B) - \phi_2(|A-B|), \qquad (\star)$$

where ϕ_1 and ϕ_2 are nonnegative continuous functions on $[0,\infty)$ with $\phi_1(0) = 0$ and $\phi_1(1) = 1$ and $\phi_2(0) = 0$ and $\lim_{t\downarrow 0} \phi_2(t)/t = 0$, and we show the characterization of σ to hold that $\sigma = \nabla_\nu$.

The paper is organized as follows. In section 2 we show our main theorem stated above. In section 3 some of equivalent conditions are provided for a given operator mean σ to be a weighted operator mean ∇_ν. In section 4 we investigate the generalized Cauchy inequality and we point out a counterexample for the inequality $(\star)$ when $\phi_1(t) = t^2$, $\phi_2(t) = \left(\frac{1}{2}t\right)^2$, and some positive definite matrices A and B with $AB + BA \geqslant 0$.

2. Weighted Cauchy reverse inequality

THEOREM 1. *Let σ be an operator mean in the sense of Kubo-Ando. Then the following are equivalent:*

1. *There exists $\beta \in [0,1]$ such that $\sigma = \nabla_\beta$;*
2. *$A\sigma B \geqslant A\nabla_\alpha B - \max\{\alpha, 1-\alpha\}|A-B|$ holds for all positive semi-definite matrices A,B;*
3. *$\operatorname{Tr}(A\sigma B) \geqslant \operatorname{Tr}(A\nabla_\alpha B - \max\{\alpha, 1-\alpha\}|A-B|)$ holds for all positive semi-definite matrices A,B.*

We need the following result in proving Theorem 1.

PROPOSITION 2. *Let $\alpha \in [0,1]$ and let $\alpha_0 = \max\{\alpha, 1-\alpha\}$. If for positive semi-definite matrices A,B, $A\nabla_\alpha B \geqslant \alpha_0|A-B|$, then*

$$A\sigma B \geqslant A\nabla_\alpha B - \alpha_0|A-B|$$

holds for any operator mean σ.

Proof. Note that

$$\begin{aligned} A &= (1-\alpha)A + \alpha B + \alpha(A-B) \\ &\geqslant A\nabla_\alpha B - \alpha|A-B| \\ &\geqslant A\nabla_\alpha B - \alpha_0|A-B| \end{aligned}$$

and

$$\begin{aligned} B &= (1-\alpha)A + \alpha B - (1-\alpha)(A-B) \\ &\geqslant A\nabla_\alpha B - (1-\alpha)|A-B| \\ &\geqslant A\nabla_\alpha B - \alpha_0|A-B|. \end{aligned}$$

Thus

$$\begin{aligned} A\sigma B &\geqslant (A\nabla_\alpha B - \alpha_0|A-B|)\sigma(A\nabla_\alpha B - \alpha_0|A-B|) \\ &= A\nabla_\alpha B - \alpha_0|A-B|. \quad \square \end{aligned}$$

COROLLARY 3. [7] *Let A and B be positive semi-definite matrices such that $AB+BA \geqslant 0$. Then*

$$A\sigma B \geqslant A\nabla B - \frac{1}{2}|A-B|$$

holds for all operator means σ.

Proof. In Proposition 2, take $\alpha = \frac{1}{2}$. Since $AB+BA \geqslant 0$, we have

$$\begin{aligned} \frac{A+B}{2} &= \frac{\left(A^2+B^2+(AB+BA)\right)^{\frac{1}{2}}}{2} \\ &\geqslant \frac{\left(A^2+B^2-(AB+BA)\right)^{\frac{1}{2}}}{2} \\ &= \frac{|A-B|}{2}. \quad \square \end{aligned}$$

LEMMA 4. *Let $\alpha \in [0,1]$ and $\alpha_0 = \max\{\alpha, 1-\alpha\}$, and let σ be an operator mean. If the following inequality*

$$\mathrm{Tr}(A\nabla_\alpha B - \alpha_0|A-B|) \leqslant \mathrm{Tr}(A\sigma B)$$

holds for every positive semi-definite matrices A and B, then $\sigma = \nabla_\beta$ for some $\beta \in [0,1]$.

Proof. Let P, Q be orthogonal projections on a Hilbert space H with $P \wedge Q = 0$, where $P \wedge Q$ is the orthogonal projection on $PH \cap QH$. From the assumption, we have

$$\mathrm{Tr}(P\nabla_\alpha Q - \alpha_0|P-Q|) \leqslant \mathrm{Tr}(P\sigma Q). \tag{2}$$

Furthermore, $P\ \sigma\ Q = aP + bQ$ [1], where $a = \inf_x 1\sigma x$, $b = \lim_{x\to\infty} \frac{1\sigma x}{x}$. Choose two orthogonal projections

$$P := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ Q := \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \quad \left(0 < \theta < \frac{\pi}{2}\right)$$

in the realization of the 2×2 matrix algebra in the set $B(H)$ of all bounded operators on H. Then $P\wedge Q = 0$,

$$P\ \nabla_\alpha\ Q = \left(\begin{bmatrix} (1-\alpha)+\alpha\cos^2\theta & \alpha\cos\theta\sin\theta \\ \alpha\cos\theta\sin\theta & \alpha\sin^2\theta \end{bmatrix}\right)$$

and

$$|P-Q| = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix}.$$

Letting $\theta \to 0^+$ from (2), we get

$$\lim_{\theta\to 0}\{\mathrm{Tr}(P\ \nabla_\alpha\ Q - \alpha_0|P-Q|)\} \leqslant \lim_{\theta\to 0}\mathrm{Tr}\left(\begin{bmatrix} a+b\cos^2\theta & b\cos\theta\sin\theta \\ b\cos\theta\sin\theta & b\sin^2\theta \end{bmatrix}\right)$$

or

$$\mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{Tr}\left(\begin{bmatrix} (1-\alpha)+\alpha & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \leqslant \mathrm{Tr}\left(\begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Then, we have

$$1 \leqslant a+b. \tag{3}$$

Furthermore, since $1\sigma t$ is an operator monotone function, there is a positive Radon measure μ on $[0,\infty]$ such that

$$1\sigma t = a + bt + \int_{(0,\infty)} \frac{(x+1)t}{x+t}\, d\mu(x).$$

Therefore,

$$1\sigma 1 = a + b + \int_{(0,\infty)} d\mu(t) = 1,$$

hence $\mu = 0$ by (3). We have then

$$1\sigma t = a + bt,\ \ 1 = a + b.$$

Thus $\sigma = \nabla_\beta$ for some $\beta \in [0,1]$. $\square$

Proof of Theorem 1. $1 \Rightarrow 2$: Put $\alpha_0 := \max\{\alpha, 1-\alpha\}$. From the proof of Proposition 2, we have

$$A \geqslant A\nabla_\alpha B - \alpha_0|A-B|$$

and

$$B \geqslant A\nabla_\alpha B - \alpha_0|A-B|.$$

Then

$$\begin{aligned} A\sigma B &= A\nabla_\beta B \\ &= (1-\beta)A+\beta B \\ &\geqslant (1-\beta)(A\nabla_\alpha B-\alpha_0|A-B|)+\beta(A\nabla_\alpha B-\alpha_0|A-B|) \\ &= A\nabla_\alpha B-\alpha_0|A-B|. \end{aligned}$$

$2\Rightarrow 3$: Immediate.

$3\Rightarrow 1$: From Lemm 4, the operator mean σ should be ∇_β for some $\beta\in[0,1]$. □

3. Characterization of weighted arithmetic means

PROPOSITION 5. *Let ϕ_1 be a nonnegative continuous function on $[0,\infty)$ such that $\phi_1(0)=0$, $\phi_1(1)=1$ and $\phi_1(t)$ has non-zero derivative at $t=1$. Let ϕ_2 be a nonnegative continuous function on $[0,\infty)$ such that $\phi_2(0)=0$ and $\lim_{t\downarrow 0}\phi_2(t)/t=0$. If σ is an operator mean satisfying*

$$\phi_1(A)\sigma\phi_1(B)\geqslant\phi_1(A\nabla_\lambda B)-\phi_2(|A-B|) \tag{4}$$

for $A,B\geqslant 0$, then $\sigma=\nabla_\lambda$.

LEMMA 6. *Let ϕ be a nonnegative continuous function on $[0,\infty)$ which has non-zero derivative at 1and let ψ be a continuous function on $[0,\infty)\times[0,\infty)$ such that $t\mapsto\psi(1,t)$ has derivative at $t=1$. Let σ be an operator mean satisfying*

$$\phi(x)\sigma\phi(y)\geqslant\psi(x,y)$$

for all scalars $x,y\geqslant 0$ and the equality hold when $(x,y)=(1,1)$. Then $\frac{d}{dt}(1\sigma t)\big|_{t=1}\frac{d\phi}{dt}\big|_{t=1}$ $=\frac{d}{dt}\psi(1,t)\big|_{t=1}$.

Proof. It is enough to assume $\phi(1)=\psi(1,1)=1$. By assumption, the inequality

$$\frac{1\sigma\phi(t)-1\sigma\phi(1)}{t-1}\geqslant\frac{\psi(1,t)-\psi(1,1)}{t-1}$$

holds for all $t>1$, which implies that

$$\lim_{t\downarrow 1}\frac{1\sigma\phi(t)-1\sigma\phi(1)}{t-1}=\frac{d}{dt}(1\sigma t)\Big|_{t=1}\frac{d\phi}{dt}\Big|_{t=1}\geqslant\frac{d}{dt}\psi(1,t)\big|_{t=1}.$$

We also obtain

$$\lim_{t\uparrow 1}\frac{1\sigma\phi(t)-1\sigma\phi(1)}{t-1}=\frac{d}{dt}(1\sigma t)\Big|_{t=1}\frac{d\phi}{dt}\Big|_{t=1}\leqslant\frac{d}{dt}\psi(1,t)\big|_{t=1}. \quad \square$$

Proof of Proposition 5. As in the proof of Lemma 4 we have

$$1\sigma t = a + bt, \quad 1 = a + b.$$

Moreover, for $A = xI_H$ and $B = yI_H$ $(x, y \geqslant 0)$, the inequality (4) in Proposition 5 becomes

$$\phi_1(x)\sigma\phi_1(y) \geqslant \phi_1(x\nabla_\lambda y) - \phi_2(|x-y|).$$

Put $\psi(x,y) = \phi_1(x\nabla_\lambda y) - \phi_2(|x-y|)$. Then the functions ϕ_1 and ψ satisfy the condtions in Lemma 6. Indeed, it follows from the following equation:

$$\begin{aligned}\frac{\psi(t,1)-1}{t-1} &= \frac{\phi_1(1\nabla_\lambda t) - \phi_2(|1-t|) - 1}{t-1} \\ &= \frac{\phi_1(1\nabla_\lambda t) - 1}{t-1} - \frac{\phi_2(|1-t|)}{t-1}.\end{aligned}$$

Thus

$$\frac{d}{dt}(1\sigma t)\Big|_{t=1}\frac{d\phi_1}{dt}\Big|_{t=1} = \frac{d}{dt}\psi(1,t)\Big|_{t=1} = \lambda\frac{d\phi_1}{dt}\Big|_{t=1},$$

which implies $\frac{d}{dt}(1\sigma t)\big|_{t=1} = \lambda$ and $\sigma = \nabla_\lambda$. □

The following are immediate.

COROLLARY 7. *Let $r > 1$, $\lambda \in [0,1]$, and let ϕ be a nonnegative continuous function on $[0,\infty)$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(t)$ has non-zero derivative at $t = 1$. Assume that $\phi'_+(0)$ exists. If*

$$\phi(A)\sigma\phi(B) \geqslant \phi(A\nabla_\lambda B) - \phi\left(|A-B|^r\right)$$

for $A, B \geqslant 0$, then $\sigma = \nabla_\lambda$.

COROLLARY 8. *Let $r > 0$, $\lambda \in [0,1]$ and ϕ be a nonnegative continuous function on $[0,\infty)$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(t)$ has non-zero derivative at $t = 1$. Assume $\phi'_+(0) = 0$. If*

$$\phi(A)\sigma\phi(B) \geqslant \phi(A\nabla_\lambda B) - \phi\left(r|A-B|\right)$$

for $A, B \geqslant 0$, then $\sigma = \nabla_\lambda$.

Proof. Put $\phi_1 = \phi$ and $\phi_2(t) = \phi(rt)$. The functions ϕ_1 and ϕ_2 satisfy the conditions in Proposition 5, since $\phi_2(0) = 0$ and $\lim_{t\downarrow 0}\phi_2(t)/t = r\phi'_+(0) = 0$. □

THEOREM 9. *Let $r > 1$, $\lambda \in [0,1]$, and σ be an operator mean. Suppose that ϕ is nonnegative operator convex with with $\phi(0) = 0$ and $\phi(1) = 1$, and $\phi'_+(0)$ exists. Then the following are equivalent:*

1. $\sigma = \nabla_\lambda$*;*

2. $\phi(A)\sigma\phi(B) \geqslant \phi(A\nabla_\lambda B) - \phi(|A-B|^r)$ *for* $A,B \geqslant 0$.

Proof. It follows from Corollary 7. □

From the above, we can see that if $r > 1$ and $\sigma \neq \nabla_\lambda$, then the inequality

$$A\sigma B \geqslant A\nabla_\lambda B - |A-B|^r$$

does not hold for some $A,B \geqslant 0$.

THEOREM 10. *Let* $r > 0$, $\lambda \in [0,1]$, *and* σ *be operator mean. Suppose that* ϕ *is nonnegative operator convex with with* $\phi(0) = 0$ *and* $\phi(1) = 1$. *Assume* $\phi'_+(0) = 0$. *Then the following are equivalent:*

1. $\sigma = \nabla_\lambda$;
2. $\phi(A)\sigma\phi(B) \geqslant \phi(A\nabla_\lambda B) - \phi(r|A-B|)$ *for* $A,B \geqslant 0$.

Proof. It follows from Corollary 8. □

We have the following theorem if $r = \frac{1}{2}$ in Theorem 10.

THEOREM 11. *Let* $\lambda \in [0,1]$ *and* σ *be operator mean. Suppose that* ϕ *is nonnegative operator convex with with* $\phi(0) = 0$, $\phi(1) = 1$, *and* $\phi'_+(0) = 0$. *Then the following are equivalent:*

1. $\sigma = \nabla_\lambda$;
2. $\phi(A)\sigma\phi(B) \geqslant \phi(A\nabla_\lambda B) - \phi\left(\frac{|A-B|}{2}\right)$ *for* $A,B \geqslant 0$.

4. Generalized Cauchy reverse inequality

From the proof of Proposition 5 we have the following observation.

PROPOSITION 12. *Let* $\lambda \in [0,1]$ *and* σ *be an operator mean. If*

$$A\sigma B \geqslant A\nabla_\lambda B - \frac{|A-B|}{2}$$

for every $A,B \geqslant 0$, *then there exists* $\beta \in [0,1]$ *such that* $\sigma = \nabla_\beta$.

Using this observation we have the following characterization.

PROPOSITION 13. *Let* $\lambda \in [0,1]$ *and* OM^1_+ *be the set of all operator means. Then we have*

$$\left\{\sigma \in OM^1_+ \mid A,B \geqslant 0 \to A\sigma B \geqslant A\nabla_\lambda B - \frac{|A-B|}{2}\right\}$$
$$= \left\{\nabla_\beta \mid \max\left\{\lambda - \frac{1}{2}, 0\right\} \leqslant \beta \leqslant \min\left\{\frac{1}{2} + \lambda, 1\right\}\right\}.$$

Proof. Let $\sigma \in \left\{\sigma \in OM_+^1 \mid A,B \geqslant 0 \to A\sigma B \geqslant A\nabla_\lambda B - \frac{|A-B|}{2}\right\}$. Then there exists $\beta \in [0,1]$ such that $\sigma = \nabla_\beta$ by Proposition 12.

For any $A,B \geqslant 0$,

$$\begin{aligned}&(1-\beta)A+\beta B \geqslant (1-\lambda)A+\lambda B - \frac{|A-B|}{2}\\ &\Leftrightarrow (\lambda-\beta)A+(\beta-\lambda)B \geqslant -\frac{|A-B|}{2}\\ &\Leftrightarrow (\lambda-\beta)(A-B) \geqslant -\frac{|A-B|}{2}.\end{aligned}$$

Set $A=a$ and $B=b$ for any positive number a,b. Then $(\lambda-\beta)(a-b) \geqslant -\frac{|a-b|}{2}$. If $a>b$, then $(\lambda-\beta)(a-b) \geqslant -\frac{a-b}{2}$, and $\beta \leqslant \lambda+\frac{1}{2}$. If $a<b$, then $(\lambda-\beta)(a-b) \geqslant \frac{a-b}{2}$, and $\lambda-\frac{1}{2} \leqslant \beta$. Hence $\lambda-\frac{1}{2} \leqslant \beta \leqslant \lambda+\frac{1}{2}$. This implies that $\sigma \in \left\{\nabla_\beta \mid \max\{\lambda-\frac{1}{2},0\} \leqslant \beta \leqslant \min\{\frac{1}{2}+\lambda,1\}\right\}$.

Conversely, when $\beta \in [0,1]$ and $\lambda-\frac{1}{2} \leqslant \beta \leqslant \lambda+\frac{1}{2}$, for any $A,B \geqslant 0$,

$$\begin{aligned}A\nabla_\beta B - A\nabla_\lambda B &= (\lambda-\beta)A+(\beta-\lambda)B\\ &= (\lambda-\beta)(A-B)\\ &\geqslant -\frac{1}{2}|A-B|,\end{aligned}$$

because that $(\beta-\lambda)(A-B) \leqslant |\beta-\lambda||A-B| \leqslant \frac{1}{2}|A-B|$. Hence we have the conclusion. $\square$

COROLLARY 14.

$$\left\{\sigma \in OM_+^1 \mid A,B \geqslant 0 \to A\sigma B \geqslant \frac{A+B}{2} - \frac{|A-B|}{2}\right\} = \left\{\nabla_\beta \mid 0 \leqslant \beta \leqslant 1\right\}.$$

REMARK 15.

1. When σ is symmetry, we know that

$$\left\{\sigma = \sigma' \in OM_+^1 \mid A,B \geqslant 0 \to A\sigma B \geqslant \frac{A+B}{2} - \frac{|A-B|}{2}\right\} = \{\nabla\}$$

   by [6].

2. When the correspondent operator monotone f_σ satisfies that $f_\sigma(0)=0$, $\sigma = \nabla$. Indeed, since $0 = f_\sigma(0) = \inf_x f_\sigma(x) = a$ in the condition of the proof of Proposition 5, we have $\beta = 1$, that is, $\sigma = \nabla_1$.

In the rest of this section we present the sufficient condition for the inequality (4) in Proposition 5 to hold.

DEFINITION 16. Let H and K be Hilbert spaces and $B(H)^+$ (resp. $B(K)^+$) be the set of all bounded positive linear operators on H (resp. on K). A map $\Phi : B(H)^+ \to B(H)^+$ is called a positive order preserving map if $\Phi(A) \geqslant \Phi(B)$ for $A, B \in B(H)^+$ with $A \geqslant B$.

THEOREM 17. *Let f be a positive operator monotone function on $[0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(1) = 1$ and let $\alpha \in [0,1]$ and $\alpha_0 = \max\{\alpha, 1-\alpha\}$. Let Φ be a positive order preserving and strong-topology continuous map on $B(H)$. Then for any positive invertible operators A and B in $B(H)$ with $AB + BA \geqslant 0$,*

$$\Phi(A)\sigma_f\Phi(B) \geqslant \Phi(A\nabla_\alpha B - \alpha_0|A-B|).$$

Proof. As in the proof of Proposition 2 we have

$$\Phi(A) \geqslant \Phi(A\nabla_\alpha B - \alpha_0|A-B|)$$

and

$$\Phi(B) \geqslant \Phi(A\nabla_\alpha B - \alpha_0|A-B|).$$

Thus

$$\begin{aligned}\Phi(A)\sigma\Phi(B) &\geqslant (\Phi(A\nabla_\alpha B - \alpha_0|A-B|))\sigma(\Phi(A\nabla_\alpha B - \alpha_0|A-B|))\\ &= \Phi(A\nabla_\alpha B - \alpha_0|A-B|). \quad \square\end{aligned}$$

REMARK 18.

1. It is obvious that a positive linear map on $B(H)$ is positive order preserving.
2. Let f be a strictly positive operator monotone function on $(0,\infty)$. Define Φ_f on $B(H)^+$ by $\Phi_f(A) = f(A)$. Then Φ_f is positive order preserving.

Related to Theorem 17, we recall Corollary 8 as followings.

PROPOSITION 19. *Let $r > 0$, $\lambda \in [0,1]$ and ϕ be a nonnegative continuous function on $[0,\infty)$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(t)$ has non-zero derivative at $t = 1$. Assume $\phi'_+(0) = 0$. If*

$$\phi(A)\sigma\phi(B) \geqslant \phi(A\nabla_\lambda B) - \phi(r|A-B|)$$

for $A, B \geqslant 0$, then $\sigma = \nabla_\lambda$.

Note that when $0 \leqslant r < \frac{1}{2}$ and $\phi(t) = t^2$, the inequality $\phi(\sqrt{ab}) \geqslant \phi(\frac{a+b}{2}) - \phi(r|a-b|)$ does not hold. Consider $a = 1$ and $b = 0$. Therefore, we may consider the case that $r = \frac{1}{2}$.

Compared with Theorem 17 and Proposition 19 the following is the natural question:

PROBLEM 20. Let f be an operator monotone with $f((0,\infty)) \subset (0,\infty)$ and $f(1) = 1$. For $\phi(t) = t^2$, is it true that for any $A, B \in B(H)^+$ with $AB + BA \geqslant 0$,

$$\phi(A)\sigma_f\phi(B) \geqslant \phi(A\nabla B) - \phi\left(\frac{1}{2}|A-B|\right)? \qquad (\star\star)$$

LEMMA 21.

1. *If $f(t) = \sqrt{t}$, the inequality $(\star\star)$ holds when $A = a$, $B = b$ for any $a, b \in (0,\infty)$.*
2. *If $f(t) = t^\alpha$ for $\alpha \in (0,1)\setminus\{\frac{1}{2}\}$, the inequality $(\star\star)$ does not hold for some nonnegative real number $A = a$, $B = b$.*

From Lemma 21, we may consider Problem 20 when $f(t) = \sqrt{t}$. However, we have a counterexample as follows.

PROPOSITION 22. *There are positive definite matrices A and B such that $AB + BA$ is nonnegative, and*

$$\mathrm{Tr}(\phi(A)\sharp\phi(B)) < \mathrm{Tr}\left(\phi(A\nabla B) - \phi\left(\frac{1}{2}|A-B|\right)\right).$$

Proof. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $AB + BA = \begin{pmatrix} 2 & 3 \\ 3 & 8 \end{pmatrix}$, which is positive definite. Then we have

$$\begin{aligned}
\phi(A)\sharp\phi(B) &= A(A^{-1}B^2A^{-1})^{\frac{1}{2}}A \\
&= A\left\{\frac{1}{4}\begin{pmatrix} 8 & 6 \\ 6 & 5 \end{pmatrix}\right\}^{\frac{1}{2}}A \\
&= \frac{1}{2}A\begin{pmatrix} a & b \\ b & c \end{pmatrix}A \\
&= \frac{1}{2}\begin{pmatrix} a & 2b \\ 2b & 4c \end{pmatrix},
\end{aligned}$$

where $a^2 + b^2 = 8$, $ab + bc = 6$, and $b^2 + c^2 = 5$. Hence, we have $a = \frac{10}{\sqrt{17}}$, $b = \frac{6}{\sqrt{17}}$, and $c = \frac{7}{\sqrt{17}}$. Then

$$\begin{aligned}
2 \times \mathrm{Tr}(\phi(A)\sharp\phi(B)) &= a + 4c \\
&= \frac{38}{\sqrt{17}} < 10.
\end{aligned}$$

On the contrary,

$$2 \times \mathrm{Tr}\left(\phi(A\nabla B) - \phi\left(\frac{1}{2}|A-B|\right)\right) = 2 \times \mathrm{Tr}\left(\frac{1}{2}(AB+BA)\right)$$
$$= \mathrm{Tr}\left(\begin{pmatrix} 2 & 3 \\ 3 & 8 \end{pmatrix}\right) = 10.$$

Hence,

$$\mathrm{Tr}(\phi(A)\sharp\phi(B)) < \mathrm{Tr}\left(\phi(A\nabla B) - \phi\left(\frac{1}{2}|A-B|\right)\right). \quad \square$$

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