WEIGHTED ESTIMATES FOR MARCINKIEWICZ INTEGRALS WITH NON–SMOOTH KERNELS ON SPACES OF HOMOGENEOUS TYPE AND THEIR APPLICATIONS

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Abstract. Given a family of operators that act like approximations of the identity, we obtain weighted estimates for the Marcinkiewicz integrals with non-smooth kernels on spaces of homogeneous type. As applications of weighted estimates, we also establish the boundedness of these operators on the homogeneous Herz spaces over Euclidean spaces, and on the homogeneous weak Herz spaces for the endpoint case. We also study the boundedness of commutators of Marcinkiewicz integrals and BMO functions on various spaces.

1. Introduction

Let \((\mathcal{X}, d, \mu)\) be a space of homogeneous type, endowed with a metric distance \(d\) on \(\mathcal{X} \times \mathcal{X}\) satisfying

\[
d(x, z) \leq \kappa (d(x, y) + d(y, z))\]

for some fixed constant \(\kappa \geq 1\) and for all \(x, y, z \in \mathcal{X}\), \((1.1)\)

and a regular Borel measure \(\mu\) on \(\mathcal{X}\) such that the doubling property

\[
\mu(B(x; 2r)) \leq C\mu(B(x; r)) < \infty \]

holds for some fixed constant \(C \geq 1\), for all \(x \in \mathcal{X}\) and for all \(r > 0\), where \(B(x; r) = \{y \in \mathcal{X} : d(x, y) < r\}\). The above property implies that there exist some fixed constants \(C \geq 1, n > 0\) such that

\[
\mu(B(x; \lambda r)) \leq C\lambda^n \mu(B(x; r)), \]

uniformly for all \(\lambda \geq 1, x \in \mathcal{X}\), and \(r > 0\). The parameter \(n\) measures the “dimension” of the space \(\mathcal{X}\). There also exist constants \(C, N (C \geq 1, 0 \leq N \leq n)\) such that

\[
\mu(B(y; r)) \leq C \left(1 + \frac{d(x, y)}{r}\right)^N \mu(B(x; r)) \]


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uniformly for all \( x, y \in \mathcal{X} \) and all \( r > 0 \). The reader can find more information on this subject in [1, 2].

Let \( T \) be a bounded linear operator on \( L^2(\mathcal{X}) \) with an associated kernel \( K(x, y) \) in the sense that

\[
Tf(x) = \int_{\mathcal{X}} K(x, y) f(y) d\mu(y),
\]

where \( f \) is a continuous function with compact support, \( x \not\in \text{supp} f \); and \( K(x, y) \) is a measurable function defined on \( (\mathcal{X} \times \mathcal{X}) \setminus \Delta \) with \( \Delta = \{(x, x) : x \in \mathcal{X}\} \).

The authors in [3, 4] assumed that there exists a class of operators \( A_t \) (\( t > 0 \)) which can be represented by the kernels \( a_t(x, y) \) in the sense that

\[
A_t u(x) = \int_{\mathcal{X}} a_t(x, y) u(y) d\mu(y), \quad \text{for every function } u \in L^1(\mathcal{X}) \cap L^2(\mathcal{X}).
\]

Moreover, the kernels \( a_t(x, y) \) satisfy the following conditions

\[
|a_t(x, y)| \leq h_t(x, y), \quad \text{for all } x, y \in \mathcal{X},
\]

where \( h_t(x, y) = (\mu(B(x; t^{1/m})))^{-1} s((d(x, y))^{m} t^{-1}) \) for some positive constant \( m \).

Here \( s \) is a positive, bounded, decreasing function satisfying

\[
\lim_{r \to \infty} r^{n+\sigma} s(r^m) = 0,
\]

for some \( \sigma > N \), where \( n \) and \( N \) appear in (1.3) and (1.4) respectively.

**Remark 1.1.** The functions \( h_t \) above satisfy the following properties (see [4, 5]):

i) There exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \leq \int_{\mathcal{X}} h_t(x, y) d\mu(x) \leq C_2 \quad \text{uniformly in } t \text{ and } y.
\]

ii) There exists a positive constant \( C \) such that

\[
\int_{\mathcal{X}} h_t(x, y) |f(x)| d\mu(x) \leq C.\mathcal{M} f(y) \quad \text{and} \quad \int_{\mathcal{X}} h_t(x, y) |f(y)| d\mu(y) \leq C.\mathcal{M} f(x).
\]

Here \( \mathcal{M} f(x) \), the Hardy-Littlewood maximal function, is defined by

\[
\mathcal{M} f(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(B)} \int_{B} |f(y)| d\mu(y) \right\},
\]

where the supremum is taken over all balls \( B \) containing \( x \).

The class of operators \( A_t \) plays the role of approximation to the identity. The existence of such a class of operators \( A_t \) was verified in [3]. Now let \( A_t \) and \( B_t \) (\( t > 0 \)) be two classes of operators which satisfy (1.6)–(1.8). Denote by \( K(x, y) - K_t(x, y) \) the kernels of the operators \( (T - TB_t) \), and \( K(x, y) - K'(x, y) \) as the kernels of \( (T - A_t T) \) respectively. We state below a list of assumptions which lead to interesting results:
(i) $T$ is a bounded linear operator from $L^2(\mathcal{X})$ to $L^2(\mathcal{X})$; 
(ii) There exist positive constants $c_1$ and $C_A$ such that
\[ \int_{d(x,y) \geq c_1 t^{1/m}} |K(x,y) - K_t(x,y)| d\mu(x) \leq C_A, \text{ for all } y \in \mathcal{X}; \]
(iii) There exist positive constants $c_2$ and $C_A$ such that
\[ \int_{d(x,y) \geq c_2 t^{1/m}} |K(x,y) - K^t(x,y)| d\mu(y) \leq C_A, \text{ for all } x \in \mathcal{X}; \]
(iv) There exist positive constants $c_2$, $c_4$ and $\beta$ such that
\[ |K(x,y) - K^t(x,y)| \leq \frac{c_4}{\mu(B(x,d(x,y)))} \frac{t^{\beta/m}}{[d(x,y)]^{\beta}}, \text{ whenever } d(x,y) \geq c_2 t^{1/m}. \]

Using assumptions (i), (ii) and (iii), Duong and McIntosh [3] obtained the $L^p$-boundedness of the singular integral operator $T$. Afterward, Martell [16] extended their results to weighted spaces with weights $w \in A_p$, under hypotheses (i), (ii) and (iv). Based on a set of hypotheses that are almost similar to the assumptions (i), (ii) and (iv) above, the author in [12] recently obtained the $L^p$-boundedness of the Marcinkiewicz integral
\[ \nu(f)(x) = \left\{ \int_0^\infty \left( \int_{d(x,y) < \tau} K(x,y)f(y) d\mu(y) \right)^2 \frac{d\tau}{\tau^3} \right\}^{1/2}, \]  
and the commutator
\[ \nu_b(f)(x) = \left\{ \int_0^\infty \left( \int_{d(x,y) < \tau} \left( \prod_{i=1}^k (b_i(x) - b_i(y)) \right) K(x,y)f(y) d\mu(y) \right)^2 \frac{d\tau}{\tau^3} \right\}^{1/2}, \]  
where $b_i \in \text{BMO}(\mathcal{X}), 1 \leq i \leq k$. The reader may further view [4, 11, 12, 13, 14, 15, 19] for several interesting results about this topic.

The purpose of this paper is to extend the results of [12] to homogeneous Herz spaces $\dot{K}_q^{\alpha_p}(\mathbb{R}^n)$. In order to do so, we first obtain the weighted $L^p$ estimates for $\nu(f)$ and $\nu_b(f)$. We then use these estimates to further extend to homogeneous Herz spaces $K_q^{\alpha_p}(\mathbb{R}^n)$ and to homogeneous weak Herz spaces $W \dot{K}_q^{\alpha_p}(\mathbb{R}^n)$ for the endpoint case. The plan of this paper is as follows. Section 2 describes some background information such as definitions, notations and preliminary theorems. Our main theorems are given in sections 3–5. In the last section, we give an example about a classical kernel $K(x,y)$ that satisfies the hypotheses of our theorems. For the rest of this paper, the letter $C$ denotes a positive constant which may vary at each occurrence; however, it is independent of any essential variable.
2. Preliminaries

2.1. Approximation of the identity

Denote by $L^\infty_b(\mathcal{X})$ the set of all functions in $L^\infty(\mathcal{X})$ with bounded support. Note that $L^\infty_b(\mathcal{X})$ is dense in $L^p(\mathcal{X})$ for $p \in (0, \infty)$ (see [6, 7] for example). For $f \in L^\infty_b(\mathcal{X})$, define the linear operator $F$ by

$$F(f)(x, \tau) = \int_{d(x,y)<\tau} K(x,y) f(y) d\mu(y),$$

where $K(x,y)$ is a measurable function defined on $(\mathcal{X} \times \mathcal{X}) \setminus \Delta$ with $\Delta = \{(x,x) : x \in \mathcal{X}\}$. Define the Marcinkiewicz integral $v(f)$ by

$$v(f)(x) = \left\{ \int_0^\infty |F(f)(x, \tau)|^2 \frac{d\tau}{\tau^3} \right\}^{1/2}.$$

Denote $\mathcal{B}_1 = \mathbb{C}$ and $\mathcal{B}_2 = L^2 \left( \mathbb{R}^+; \frac{d\tau}{\tau^3} \right)$, where $\mathbb{R}^+ = (0, \infty)$.

Then $v(f)(x) = ||F(f)(x, \cdot)||_{\mathcal{B}_2}$, so that $||v(f)||_{L^p(\mathcal{X})} = ||||F(f)||_{L^p(\mathcal{X})} = ||||F(f)||_{L^p(\mathcal{X}; \mathcal{B}_2)}$. Note also that $L^p(\mathcal{X}; \mathcal{B}_1) = L^p(\mathcal{X})$.

In the sequel, we assume the existence of two classes of operators $A_t$ and $B_t$ ($t > 0$), both of which can be represented by kernels $a_t(x,y)$ and $b_t(x,y)$ respectively in the sense that

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y) u(y) d\mu(y),$$

for every function $u \in L^1(\mathcal{X}) \cap L^r(\mathcal{X})$ and for some $r > 1$ and similar definition for $B_t$. Moreover, both kernels $a_t(x,y)$ and $b_t(x,y)$ are assumed to satisfy inequalities (1.6)–(1.8). Let $K_t(x,y)$ and $K^t(x,y)$ represent the kernels of the operators $FB_t$ and $A_tF$ ($t > 0$) respectively. We may assume that both $FB_t$ and $A_tF$ have integral forms

$$\left(FB_t\right)(f)(x, \tau) = \int_{d(x,y)<\tau} K_t(x,y) f(y) d\mu(y)$$

and

$$\left(A_tF\right)(f)(x, \tau) = \int_{d(x,y)<\tau} K^t(x,y) f(y) d\mu(y).$$

To see this, consider the kernel $b_t$ of the operator $B_t$ defined in [3] by

$$b_t(y,z) = \chi_{B(z; \tau^1/m)}(y) [\mu(B(z; \tau^{1/m})]^{-1}.$$

Now, let $B_{t,\tau}$ be the operator whose kernel $b_t, \tau(y,z)$ is defined by $b_t, \tau(y,z) = \chi_{B(z; \tau)}(y) b_t(y,z)$. Then $|b_t, \tau(y,z)| \leq |b_t(y,z)| \leq h_t(y,z)$ for all $t, \tau > 0$ and $y, z \in \mathcal{X}$. Moreover,

$$||\left(FB_t, \tau\right)(f)(x, \cdot)||_{\mathcal{B}_2} = \left|\left| \int_{d(x,z)<2\kappa \tau} K_t(x,z) f(z) d\mu(z) \right|\right|_{\mathcal{B}_2} = 2\kappa \left|\left| \int_{d(x,z)<\tau} K_t(x,z) f(z) d\mu(z) \right|\right|_{\mathcal{B}_2},$$
where $\kappa$ appears in (1.1). Similarly, if we let $A_{t,\tau}$ be the operator whose kernel $a_t,\tau(x, y)$ is given by

$$a_t,\tau(x, y) = \chi_{B(x, \tau)}(y) a_t(x, y) = \chi_{B(x, t^{1/m})}(y) \chi_{B(x, t^{1/m})}(y) [\mu(B(x, t^{1/m})]^{-1},$$

then the above equation also holds for $\|(A_{t,\tau}F)(x, \cdot)\|_{\mathcal{A}_2}$ (with $K_t(x, z)$ instead of $K_t(x, z)$). Therefore, for simplicity, we will assume that (2.3) holds true; and we will work with the operators $A_t$ and $B_t$ for the rest of this paper.

For a ball $B \subset \mathcal{X}$, denote the radius of the ball $B$ by $r_B$. Let $t_B = r_B^m$, where $m$ appears in (1.7)–(1.8). For $g \in L^\infty_b(\mathcal{X}; \mathcal{A}_2)$, we define the sharp maximal function $M_A^g(||g||_{\mathcal{A}_2})$ by

$$M_A^g(||g||_{\mathcal{A}_2})(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(B)} \int_B \|(g - A_B g)(y, \cdot)\|_{\mathcal{A}_2} d\mu(y) \right\}. \quad (2.4)$$

We now state the following assumptions for our theorems:

(a) $\nu$ is a bounded operator from $L^r(\mathcal{X})$ to $L^r(\mathcal{X})$ with the bound $C_r$ for some $r > 1$;

(b) there exist positive constants $c_1$ and $C_A$ such that

$$\int_{d(x, y) > c_1 t^{1/m}} \frac{|K(x, y) - K_t(x, y)|}{d(x, y)} d\mu(x) \leq C_A, \text{ for all } y \in \mathcal{X};$$

(c) there exist positive constants $c_1$ and $c_3$ such that

$$\frac{|K(x, y) - K_t(x, y)|}{d(x, y)} \leq \frac{c_3}{\mu(B(x; d(x, y))) (d(x, y))^\beta} \frac{t^{\beta/m}}{(d(x, y))^\beta}$$

whenever $d(x, y) \geq c_1 t^{1/m}$;

(c) there exist positive constants $c_2$ and $c_4$ such that

$$\frac{|K(x, y) - K'(x, y)|}{d(x, y)} \leq \frac{c_4}{\mu(B(x; d(x, y))) (d(x, y))^\beta} \frac{t^{\beta/m}}{(d(x, y))^\beta}$$

whenever $d(x, y) \geq c_2 t^{1/m}$.

**Remark 2.1.** Note that hypothesis (b) implies hypothesis (b). Assumption (b) will be used to prove weak type $(1, 1)$ inequality for the Marcinkiewicz integral.

### 2.2. Muckenhoupt weights

For $p \in (1, \infty)$, let $p'$ be the dual exponent of $p$. That is, $\frac{1}{p} + \frac{1}{p'} = 1$. A weight $w$ is said to belong to the Muckenhoupt class $\mathcal{A}_p(\mathcal{X})$, $1 < p < \infty$, if there exists a positive constant $C$ such that

$$\left( \frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right) \left( \frac{1}{\mu(B)} \int_B w^{-p'/p}(x) d\mu(x) \right)^{p/p'} \leq C < \infty,$$
for all balls \( B \subset \mathcal{X} \). The smallest constant \( C \) for which the above inequality holds is the \( A_p \) bound of \( w \). For a ball \( B \subset \mathcal{X} \) and any weight \( w \), let \( w(B) = \int_B w(x) d\mu(x) \).

The class \( A_1(\mathcal{X}) \) consists of non-negative functions \( w \) such that \[
\frac{w(B)}{\mu(B)} \leq C \text{ess inf}_{x \in B} w(x)
\]
for all balls \( B \subset \mathcal{X} \). A weight \( w \) belongs to \( A_\infty(\mathcal{X}) \) if there exist positive constants \( C_w \) and \( \delta_w \) (\( 0 < \delta_w < 1 \)) such that, for any ball \( B \subset \mathcal{X} \) and any measurable subset \( E \subset B \), \[
\frac{w(E)}{w(B)} \leq C_w \left( \frac{\mu(E)}{\mu(B)} \right)^{\delta_w}.
\]

If \( w \in A_p(\mathcal{X}) \) (\( 1 \leq p < \infty \)), we denote \( L^p(\mathcal{X}, w) \) to be the space of all measurable functions \( f \) on \( \mathcal{X} \) whose norms are finite:

\[
\|f\|_{L^p(\mathcal{X}, w)} = \left\{ \int_{\mathcal{X}} \|f(x)\|^p w(x) d\mu(x) \right\}^{1/p} < \infty.
\]

For more information on topics of weights, the reader may view [17, 18].

### 2.3. Preliminary lemmas and theorems

The following lemmas and theorems are necessary for the proof of our theorems.

**Lemma 2.2.** [2] Let \( f \in L^1(\mathcal{X}) \) with bounded support and \( \alpha > \|f\|_{L^1(\mathcal{X})} (\mu(\mathcal{X}))^{-1} \). Then there exist positive constants \( C, M_1 \) depending only on \( \mathcal{X} \) and a sequence of metric balls \( \{B_i\}_i \equiv \{B(x_i, r_i)\}_i \) such that

\[
\Omega_\alpha := \{ x \in \mathcal{X} : \mathcal{M}(f)(x) > C_\alpha \alpha \} = \bigcup_i B_i,
\]

and

a) \( f(x) = g(x) + b(x) \), where

\[
g(x) = f(x) \chi_{\alpha_\alpha}(x) + \sum_i \left( \frac{1}{\mu(B_i)} \int_{B_i} f(y) \eta_i(y) d\mu(y) \right) \chi_{B_i}(x),
\]

\[
\eta_i(x) = \frac{\chi_{B_i}(x)}{\sum_j \chi_{B_j}(x)}, \quad b(x) = \sum_i b_i(x),
\]

\[
b_i(x) = f(x) \eta_i(x) - \left( \frac{1}{\mu(B_i)} \int_{B_i} f(y) \eta_i(y) d\mu(y) \right) \chi_{B_i}(x)
\]

for all \( x \in \mathcal{X} \);

b) \( |g(x)| \leq C \alpha \) for almost all \( x \in \mathcal{X} \);

c) \( \|g\|_{L^1(\mathcal{X})} \leq C \|f\|_{L^1(\mathcal{X})} \).
d) for all \( i \in \mathbb{N} \), \( \text{supp}(b_i) \subset B_i \) and \( \sum_i \mu(B_i) \leq C \alpha^{-1} \|f\|_{L^1(X)} \);

e) for all \( i \in \mathbb{N} \), \( \int_X b_i(x) d\mu(x) = 0 \);

f) for all \( i \in \mathbb{N} \), \( \frac{1}{\mu(B_i)} \int_{B_i} |b_i(x)| d\mu(x) \leq C \alpha \), and \( \sum_i \|b_i\|_{L^1(X)} \leq C \|f\|_{L^1(X)} \);

g) every point of \( X \) belongs to no more than \( M_1 \) balls of \( \{B_i\}_i \).

**Lemma 2.3.** [12] Assume that hypotheses (a) and (c) (as stated in section 2.1) hold true. Then there exists a positive constant \( c_9 \) such that, for every \( f \in L^\infty_b(X) \) and any \( r > 1 \),

\[
M_9^r(\|F(f)\|_{\mathcal{B}_2})(x) \leq c_9 M_r f(x),
\]

where the definitions of \( \|F(f)(x, \cdot)\|_{\mathcal{B}_2} \) and \( M_9^r \) were given by equations (2.1) and (2.4) respectively; and

\[
M_r f(x) := \sup_{\mathcal{B} \ni x} \left\{ \frac{1}{\mu(B)} \int_B |f(y)|^r d\mu(y) \right\}^{1/r} = \{\mathcal{M}(|f|^r)(x)\}^{1/r}.
\]

**Theorem 2.4.** [12] Assume that hypotheses (a), (b) and (c) (as stated in section 2.1) hold true. Then the operator \( v \) has well-defined extensions on \( L^p(X) \) for \( 1 \leq p < \infty \). Moreover, there exist positive constants \( C_X \) and \( C \) (where \( C \) depends on \( p \), \( C_X \), \( C_A \) and \( C_r \)) such that

\[
\|v(f)\|_{L^1(X)} \leq C_X (C_A + C_r) \|f\|_{L^1(X)},
\]

and

\[
\|v(f)\|_{L^p(X)} \leq C \|f\|_{L^p(X)} \text{ for } 1 < p < \infty.
\]

**Theorem 2.5.** [13, 16] Let \( p \in (0, \infty), w \in \mathcal{A}_\infty \) and \( g \in L^1_b(X; \mathcal{B}_2) \). Assume that \( \mathcal{M}(g(\cdot)) \in L^p(X, w) \). Then

(i) \( \|\mathcal{M}(g(\cdot))\|_{L^p(X, w)} \leq C \|M^r_A(g(\cdot))\|_{L^p(X, w)} \) if \( X \) is unbounded.

(ii) \( \|\mathcal{M}(g(\cdot))\|_{L^p(X, w)} \leq C \|M^r_A(g(\cdot))\|_{L^p(X, w)} + C \|g\|_{L^1(X; \mathcal{B}_2)} \) if \( X \) is bounded.

3. Weighted estimates for Marcinkiewicz integrals

We need this lemma below in order to obtain weighted estimates of Marcinkiewicz integrals.

**Lemma 3.1.** Let \( w \in \mathcal{A}_p \ (1 < p < \infty) \) and \( f \in L^\infty_b(X^-) \), where \( L^\infty_b(X^-) \) denotes the set of all functions \( f \in L^\infty_b(X^-) \) that have bounded support. Assume that hypotheses (a), (b) and (c) hold true. Then \( v(f) \in L^p(X^-, w) \).
Proof. We will employ some ideas in [16] to prove this lemma. Recall that \( V(f)(x) = ||F(f)(x, \cdot)||_{\mathcal{B}_2} \). We may assume that \( \text{supp}(f) \subset B(x_0; R) \) for some \( x_0 \in \mathcal{X}^c \) and some \( R > 0 \). By Reverse’s Hölder’s inequality, there exists \( \varepsilon > 0 \), depending on \( p \) and the \( \mathcal{A}_p \) bound for \( w \), such that

\[
\left( \frac{1}{\mu(B)} \int_B w(x)^{1+\varepsilon} \, d\mu(x) \right)^{1+\varepsilon} \leq \frac{C}{\mu(B)} \int_B w(x) \, d\mu(x) \text{ for every ball } B \subset \mathcal{X}.
\]

An application of Hölder’s inequality (with \( q = 1 + \varepsilon \)) yields

\[
\int_{B(x_0; 2R)} ||F(f)(x, \cdot)||_{\mathcal{B}_2}^p \, w(x) \, d\mu(x) \leq ||||F(f)||_{\mathcal{B}_2}||_{L^p(1+\varepsilon)'} \left( \int_{B(x_0; 2R)} w(x)^{1+\varepsilon} \, d\mu(x) \right)^{1+\varepsilon}
\]

\[
\leq C \frac{w(B(x_0; 2R))}{\mu(B(x_0; 2R))^{1+\varepsilon}} ||V(f)||_{L^p(1+\varepsilon)'}^p
\]

\[
\leq C \frac{w(B(x_0; 2R))}{\mu(B(x_0; 2R))^{1+\varepsilon}} ||f||_{L^p(1+\varepsilon)'}^p,
\]

where the last inequality follows from Theorem 2.4. Let \( t_k = (2^{k-1}R)^m \). We write

\[
\int_{B(x_0; 2R)^c} ||F(f)(x, \cdot)||_{\mathcal{B}_2}^p \, w(x) \, d\mu(x)
\]

\[
= \sum_{k=1}^{\infty} \int_{d(x, x_0) \geq 2^k R} ||F(f)(x, \cdot)||_{\mathcal{B}_2}^p \, w(x) \, d\mu(x)
\]

\[
\leq C_p \sum_{k=1}^{\infty} \int_{d(x, x_0) \geq 2^k R} ||(F(f) - A_{k} F(f))(x, \cdot)||_{\mathcal{B}_2}^p \, w(x) \, d\mu(x)
\]

\[
+ C_p \sum_{k=1}^{\infty} \int_{d(x, x_0) \geq 2^k R} ||A_{k} F(f))(x, \cdot)||_{\mathcal{B}_2}^p \, w(x) \, d\mu(x)
\]

\[
= J_1 + J_2,
\]

where \( d(x, x_0) \geq 2^k R \) means \( \{ x \in \mathcal{X}^c : 2^k R \leq d(x, x_0) < 2^{k+1} R \} \). By Minkowski’s inequality,

\[
||F(f) - A_{k} F(f))(x, \cdot)||_{\mathcal{B}_2} \leq \left\{ \int_0^\infty \left| \int_{d(x, z) < \tau} (K(x, z) - K^k(x, z)) \, f(z) \, d\mu(z) \right|^2 \frac{d\tau}{\tau^3} \right\}^{1/2}
\]

\[
\leq \int_{d(x, z) \geq t_{k}^{1/m}} \frac{|K(x, z) - K^k(x, z)|}{d(x, z)} f(z) \, d\mu(z)
\]

\[
\leq C \int_{d(x, z) \geq t_{k}^{1/m}} \frac{t_{k}^{\beta/m}}{d(x, z)^{\beta}} \frac{f(z)}{\mu(B(x; d(x, z)))} \, d\mu(z)
\]

\[
\leq C \sum_{j=0}^{\infty} \frac{2^{-\beta j}}{\mu(B(x; 2^{j+1} t_{k}^{1/m})))} \int_{d(x, z) \geq 2^{j+1/m}} |f(z)| \, d\mu(z)
\]

\[
\leq C \mathcal{M} f(x),
\]
where the second and third inequalities follow from hypothesis (e) and (1.3) respectively. Therefore,

\[
J_1 \leq C \sum_{k=1}^{\infty} \int_{d(x,x_0) \geq 2^k R} [\mathcal{M} f(x)]^p w(x) \, d\mu(x) \leq C \int_{\mathcal{X}} [\mathcal{M} f(x)]^p w(x) \, d\mu(x) \\
\leq C \int_{\mathcal{X}} |f(x)|^p w(x) \, d\mu(x) \leq C \|f\|^p_{L^p(\mathcal{X})} \, w(B(x_0 : R)) < \infty. \tag{3.3}
\]

Note that \( w \in \mathcal{A}_p \Rightarrow w \in \mathcal{A}_{p/s} \) for some \( s > 1 \) with \( p/s > 1 \). By Minkowski’s inequality, Hölder’s inequality, inequality (1.7), Remark 1.1-i and by Theorem 2.4, we obtain

\[
\|A_k F(f)(x, \cdot)\|_{\mathcal{B}_2} = \left( \int_0^\infty \left( \int_{\mathcal{X}} |a_{k}(x, y)| F(f)(y, t) \, d\mu(y) \right)^2 \frac{dt}{t^3} \right)^{1/2} \\
\leq \int_{\mathcal{X}} |a_{k}(x, y)| \left( \int_0^\infty |F(f)(y, t)|^2 \frac{dt}{t^3} \right)^{1/2} \, d\mu(y) \\
\leq \int_{\mathcal{X}} h_{k}(x, y) \|F(f)(y, \cdot)\|_{\mathcal{B}_2} \, d\mu(y) \\
\leq \|F(f)\|_{L^p(\mathcal{X} ; \mathcal{B}_2)} \left( \int_{\mathcal{X}} (h_{k}(x, y))^{s'-1} h_{k}(x, y) \, d\mu(y) \right)^{1/s'} \\
\leq \|v(f)\|_{L^s(\mathcal{X})} \frac{C}{\mu(B(x; t_k^{1/m}))^{1/s}} \left( \int_{\mathcal{X}} h_{k}(x, y) \, d\mu(y) \right)^{1/s'} \\
\leq C \|f\|_{L^s(\mathcal{X})} \|\mu(B(x; t_k^{1/m}))\|^{-1/s}.
\]

From Remark 5.6 [16], we see that

\[
J_2 \leq C \|f\|^p_{L^s(\mathcal{X})} \sum_{k=1}^{\infty} \int_{d(x,x_0) \geq 2^k R} \frac{w(x)}{[\mu(B(x; t_k^{1/m}))]^{p/s}} \, d\mu(x) \leq C \|f\|^p_{L^s(\mathcal{X})}. \tag{3.4}
\]

It follows from (3.1)–(3.4) that \( F(f) \in L^p(\mathcal{X}, \mu ; \mathcal{B}_2) \), or equivalently, \( v(f) \in L^p(\mathcal{X}, w) \).

Lemma 3.1 is proved. \( \square \)

**Theorem 3.2.** Let \( w \in \mathcal{A}_p \), \( 1 < p < \infty \). Assume that hypotheses (a), (b) and (c) hold true. Then there exists a positive constant \( C_{p,w} \), depending on \( p \) and \( w \), such that

\[
\|v(f)\|_{L^p(\mathcal{X}, w)} \leq C_{p,w} \|f\|_{L^p(\mathcal{X}, w)}.
\]

**Proof.** It suffices to prove the theorem for \( f \in L^p_b(\mathcal{X}) \), since this space is densely contained in \( L^p(\mathcal{X}, w) \) for \( 1 \leq p < \infty \). Recall that \( w \in \mathcal{A}_p \Rightarrow w \in \mathcal{A}_{p/s} \) for some \( s > 1 \) with \( p/s > 1 \). We first consider the case that \( \mathcal{X} \) is unbounded. By Lemma 3.1, \( v(f) = \|F(f)(\cdot)\|_{\mathcal{B}_2} \in L^p(\mathcal{X}, w) \), and thus \( \mathcal{M}(\|F(f)(\cdot)\|_{\mathcal{B}_2}) \in L^p(\mathcal{X}, w) \). We then
have
\[
\int_{\mathcal{X}} |v(f)(x)|^p w(x) d\mu(x) = \int_{\mathcal{X}} |F(f)(x, \cdot)|^p \mathcal{B}_2 w(x) d\mu(x)
\]
\[
\leq \int_{\mathcal{X}} [\mathcal{M}(||F(f)(\cdot)||_{\mathcal{B}_2}(x))]^p w(x) d\mu(x)
\]
\[
\leq C \int_{\mathcal{X}} [M_A^p(||F(f)(\cdot)||_{\mathcal{B}_2}(x))]^p w(x) d\mu(x)
\]
\[
\leq C \int_{\mathcal{X}} [M_s f(x)]^p w(x) d\mu(x)
\]
\[
\leq C \int_{\mathcal{X}} [f(x)]^p w(x) d\mu(x),
\]
where the second and third inequalities follow from Theorem 2.5 and Lemma 2.3 respectively.

We now consider the case that $\mathcal{X}$ is bounded. Recall from Theorem 2.4 that $v$ is a bounded operator on $L^q(\mathcal{X})$ for $q > 1$. Since $\mathcal{X}$ is bounded, $v$ is also bounded on $L^1(\mathcal{X})$. By Theorem 2.5 and by Lemma 2.3, we have
\[
||v(f)||_{L^p(\mathcal{X},w)} = |||F(f)||_{\mathcal{B}_2}|_{L^p(\mathcal{X},w)} \leq ||\mathcal{M}(||F(f)||_{\mathcal{B}_2})||_{L^p(\mathcal{X},w)}
\]
\[
\leq C ||M_A^p(\mathcal{M}(||F(f)||_{\mathcal{B}_2}))||_{L^p(\mathcal{X},w)} + C ||||F(f)||_{\mathcal{B}_2}||_{L^1(\mathcal{X})}
\]
\[
\leq C ||M_s f||_{L^p(\mathcal{X},w)} + C ||v(f)||_{L^1(\mathcal{X})}
\]
\[
\leq C ||f||_{L^p(\mathcal{X},w)} + C ||v(f)||_{L^1(\mathcal{X})} \leq C ||f||_{L^p(\mathcal{X},w)}.
\]
The last inequality is a consequence of successive applications of Hölder’s inequality:
\[
||v(f)||_{L^1(\mathcal{X})} \leq ||v(f)||_{L^q(\mathcal{X})} [\mu(\mathcal{X})]^{1/q} \leq C ||f||_{L^p(\mathcal{X})} [\mu(\mathcal{X})]^{1/q}
\]
\[
\leq C ||f||_{L^p(\mathcal{X},w)} [\mu(\mathcal{X})]^{1/q} \left( \int_{\mathcal{X}} [w(x)]^{-r'/r} d\mu(x) \right)^{1/(sr')}
\]
\[
\leq C ||f||_{L^p(\mathcal{X},w)}.
\]
The proof of this theorem is finished. \(\square\)

**Theorem 3.3.** Let $w \in \mathcal{A}_1$. Assume that hypotheses (a), (b) and (c) hold true. Then there exists a positive constant $C_w$, depending on $w$, such that
\[
||v(f)||_{L^{1,w}(\mathcal{X},w)} \leq C_w ||f||_{L^1(\mathcal{X},w)}.
\]

**Proof.** It is enough to prove the theorem for $f \in L^p_w(\mathcal{X})$. Observe that hypothesis (b) implies hypothesis (b). Thus we may conclude from Theorem 3.2 that if $w \in \mathcal{A}_1 \subset \mathcal{A}_r$ for some fixed $r > 1$, then there exists a positive constant $C_{r,w}$ such that
\[
||v(f)||_{L^r(\mathcal{X},w)} \leq C_{r,w} ||f||_{L^r(\mathcal{X},w)}.
\]
Now suppose that $\lambda > C_{r, w} ||f||_{L^1(\mathcal{X}, w)}(\mu(\mathcal{X}))^{-1}$. We apply Lemma 2.2 to $f \in L_b^\infty(\mathcal{X})$ at the height $\alpha = C_{r, w}^{-1} \lambda$ to obtain a sequence of metric balls $\{B_i\}_i \equiv \{B(x_i, r_i)\}_i$ such that $\Omega_\lambda := \{x \in \mathcal{X} : \mathcal{M} f(x) > C_{\mathcal{X}} C_{r, w}^{-1} \lambda\} = \bigcup_i B_i$, and $f(x) = g(x) + \sum b_i(x)$. Let $\tilde{B}_i = B(x_i; r_i) := B(x_i; (1 + c_1) r_i)$, where $c_1$ appears in hypothesis (b). We have

$$w(\{x \in \mathcal{X} : |v(f)(x)| > \lambda\}) \leq w \left( \left\{ x \in \mathcal{X} : \|F(g)(x, \cdot)||_{\mathcal{P}_2} > \frac{\lambda}{2} \right\} \right)$$

$$+ w \left( \left\{ x \in \mathcal{X} : \|F(b)(x, \cdot)||_{\mathcal{P}_2} > \frac{\lambda}{2} \right\} \right)$$

$$\equiv Z_1 + Z_2.$$

The $L'$-boundedness of $v$ together with properties (b) and (c) of Lemma 2.2 imply that

$$Z_1 \leq \left( \frac{\lambda}{2} \right)^{-r} \int_{\mathcal{X}} ||F(g)(x, \cdot)||_{\mathcal{P}_2} w(x) d\mu(x) = \left( \frac{\lambda}{2} \right)^{-r} \int_{\mathcal{X}} |v(g)(x)|^r w(x) d\mu(x)$$

$$\leq C_{r, w} \left( \frac{\lambda}{2} \right)^{-r} \int_{\mathcal{X}} |g(x)|^r w(x) d\mu(x) \leq C_{r, w} \int_{\mathcal{X}} |g(x)| w(x) d\mu(x)$$

$$\leq C_{r, w} \left( \frac{\lambda}{2} \right) \left\{ ||f||_{L^1(\mathcal{X}, w)} + \sum_i \left( \int_{B_i} |f(y)||w(y)| d\mu(y) \right) (w(B_i))^{-1} w(x) \chi_{B_i}(x) d\mu(x) \right\}$$

$$\leq C_{r, w} \left( \frac{\lambda}{2} \right) \left\{ ||f||_{L^1(\mathcal{X}, w)} + \sum_i \int_{B_i} |f(y)||w(y)| d\mu(y) \right\}$$

$$\leq C_{r, w} \left( \frac{\lambda}{2} \right) \left\{ ||f||_{L^1(\mathcal{X}, w)} + M_1 \int_{\bigcup B_i} |f(y)||w(y)| d\mu(y) \right\}$$

$$\leq C_{r, w} \left( \frac{\lambda}{2} \right) ||f||_{L^1(\mathcal{X}, w)},$$

where $M_1$ appears in part (g) of Lemma 2.2. Observe that

$$Z_2 \leq Z_3 + Z_4 + Z_5,$$

where $Z_3 = w(\bigcup_i \tilde{B}_i)$, $Z_4 = w(\{x \notin \bigcup_i \tilde{B}_i : \sum ||(F - FB_{\tilde{B}_i})(b_i)(x, \cdot)||_{\mathcal{P}_2} > \lambda / 4\})$, and $Z_5 = w(\{x \in \mathcal{X} : ||\sum_i FB_{\tilde{B}_i} b_i(x, \cdot)||_{\mathcal{P}_2} > \lambda / 4\})$.

We choose $t_i = r_i^m$, where $m$ is the constant appearing in (1.7). It follows from the doubling property of $w$, properties (d) and (g) of Lemma 2.2, and weak type (1, 1) of the Hardy-Littlewood maximal operator that

$$Z_3 = w(\bigcup_i \tilde{B}_i) \leq \sum_i w(\tilde{B}_i) \leq C \sum_i w(B_i) \leq C M_1 w(\bigcup_i B_i) = C M_1 w(\Omega_\lambda)$$

$$= C M_1 w(\{x \in \mathcal{X} : \mathcal{M} f(x) > C_{\mathcal{X}} C_{r, w}^{-1} \lambda\}) \leq C_{r, w} \left( \frac{\lambda}{2} \right) ||f||_{L^1(\mathcal{X}, w)},$$

where $C_{r, w}$ is the constant appearing in part (g) of Lemma 2.2.
We now estimate $Z_4$. If $y \in B_i$ and $x \notin \bigcup_i \tilde{B}_i$, then $d(x, y) \geq c_1 r_i = c_1 t_i^{1/m}$. Note that

$$||(F - FB_t)(b_i)(x, \cdot)||_{\mathcal{B}_2} = \left\{ \int_0^\infty \left[ \int_{d(x, y) < \tau} (K(x, y) - K_t(x, y)) b_i(y) d\mu(y) \right]^2 \frac{d\tau}{\tau^3} \right\}^{1/2} \leq \int_{B_i} |K(x, y) - K_t(x, y)| (d(x, y))^{-1} |b_i(y)| d\mu(y).$$

Thus we have

$$Z_4 \leq \frac{4}{\lambda} \int_{(U, B_i)} \sum_i \left\{ \left[ \sum_{j=0}^\infty 2^{-\beta j} \int_{d(x, y) \geq 2^{-j} c_1 t_i^{1/m}} \frac{w(x)}{\mu(B(x; d(x, y)))} d\mu(x) \right] |b_i(y)| d\mu(y) \right\} \leq \frac{C}{\lambda} \sum_i \int_{B_i} w(y) b_i(y) d\mu(y) \leq \frac{C}{\lambda} \sum_i \int_{B_i} |b_i(y)| w(y) d\mu(y) \leq \frac{C}{\lambda} \sum_i \int_{B_i} |f(y)| w(y) d\mu(y) + \sum_i \int_{B_i} \left( \frac{1}{\mu(B_i)} \int_{B_i} w(y) d\mu(y) \right) |f(z)| d\mu(z) \leq \frac{C}{\lambda} \left\{ M_1 \int_{\mathcal{B}_i} |f(y)| w(y) d\mu(y) + \sum_i \int_{B_i} |f(z)| w(z) d\mu(z) \right\} \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{B}, w)}.$$

By assumption (b) and inequalities (1.3)–(1.4), we obtain

$$Z_4 \leq \frac{C}{\lambda} \sum_i \int_{B_i} \left\{ \int_{d(x, y) \geq c_1 t_i^{1/m}} |K(x, y) - K_t(x, y)| (d(x, y))^{-1} w(x) d\mu(x) \right\} |b_i(y)| d\mu(y) \leq \frac{C}{\lambda} \sum_i \int_{B_i} |f(y)| w(y) d\mu(y) + \sum_i \int_{B_i} \left( \frac{1}{\mu(B_i)} \int_{B_i} w(y) d\mu(y) \right) |f(z)| d\mu(z) \leq \frac{C}{\lambda} \left\{ M_1 \int_{\mathcal{B}_i} |f(y)| w(y) d\mu(y) + \sum_i \int_{B_i} |f(z)| w(z) d\mu(z) \right\} \leq \frac{C}{\lambda} \|f\|_{L^1(\mathcal{B}, w)}.$$

It remains to estimate $Z_5$. It follows from inequality (1.6), Lemma 2.2-f and Proposition 2.5 [5] (or see Lemma 1 [3]) that

$$|B_t b_i(x)| \leq \int_{\mathcal{B}_i} h_t(x, y) |b_i(y)| d\mu(y) \leq \sup_{y \in B_i} h_t(x, y) \int_{B_i} |b_i(y)| d\mu(y) \leq CC_{r, w}^{-1} \lambda_\mu (B_i) \sup_{y \in B_i} h_t(x, y) \leq CC_{r, w}^{-1} \lambda_\mu (B_i) \inf_{y \in B_i} h_{\theta t_i}(x, y) \leq CC_{r, w}^{-1} \lambda_\mu \int_{\mathcal{B}_i} h_{\theta t_i}(x, y) \chi_{B_i}(y) d\mu(y),$$

where $\theta \geq 1$ is a positive constant (see Lemma 1 [3]). Recall that $w \in \mathcal{A}_1 \subset \mathcal{A}_r \Rightarrow w^{1-\rho} \in \mathcal{A}_r$. Now take a function $u \geq 0$, $u \in L^r(\mathcal{B}, w)$ with $||u||_{L^r(\mathcal{B}, w)} \leq 1$. Then the

...
function \( v = uw \in L^{r'}(\mathcal{X}, w^{1-r'}) \) and \( \|v\|_{L^{r'}(\mathcal{X}, w^{1-r'})} = \|u\|_{L^{r'}(\mathcal{X}, w)} \leq 1 \). By inequality (3.5) and Remark 1.1-ii, we obtain

\[
\left| \int_{\mathcal{X}} \sum_i |B_i b_i(x)| u(x) w(x) \, d\mu(x) \right|
\]

\( \leq C C_{r,w}^{-1} \lambda \sum_i \int_{\mathcal{X}} \left( \int_{\mathcal{X}} h_{B_i}(x,y) u(x) w(x) \, d\mu(x) \right) \chi_{B_i}(y) \, d\mu(y) \)

\( \leq C C_{r,w}^{-1} \lambda \sum_i \int_{\mathcal{X}} \mathcal{M}(uw)(y) \chi_{B_i}(y) \, d\mu(y) \)

\( \leq C M_1 C_{r,w}^{-1} \lambda \int_{\cup B_i} \mathcal{M}(uw)(y) \, d\mu(y) \)

\( \leq C M_1 C_{r,w}^{-1} \lambda \left\{ \int_{\mathcal{X}} |\mathcal{M}(uw)(y)|^{r'} w(y)^{1-r'} \, d\mu(y) \right\}^{1/r'} \{w(\cup B_i)\}^{1/r} \)

\( \leq C C_{r,w}^{-1} \lambda \|u\|_{L^{r'}(\mathcal{X}, w)} w(O^\lambda)^{1/r} \)

\( \leq C C_{r,w}^{-1} \lambda \left\{ w \{x \in \mathcal{X} : \mathcal{M} f(x) > C_{r,w}^{-1} \lambda \} \right\}^{1/r} \)

\( \leq C \left\{ \frac{\lambda}{C_{r,w}} \right\}^{1/r'} \|f\|_{L^1(\mathcal{X}, w)}^{1/r}. \)

Therefore,

\[
\left\| \sum_i |B_i b_i| \right\|_{L^{r'}(\mathcal{X}, w)} = \sup_{\|u\|_{L^{r'}(\mathcal{X}, w)} \leq 1} \left\{ \left| \int_{\mathcal{X}} \sum_i |B_i b_i(x)| u(x) w(x) \, d\mu(x) \right| \right\}
\]

\( \leq C \left\{ \frac{\lambda}{C_{r,w}} \right\}^{1/r'} \|f\|_{L^1(\mathcal{X}, w)}^{1/r}. \)

Since \( w \in \mathcal{A}_1 \subset \mathcal{A}_r \), it follows from the \( L^r - \) boundedness of \( v \) and from the estimate above that

\[
Z_5 \leq \left( \frac{4}{\lambda} \right)^r \int_{\mathcal{X}} \|F(\sum_i B_i b_i)(x,\cdot)\|^r_{L^2} w(x) \, d\mu(x)
\]

\( \leq C \left( \frac{C_{r,w}}{\lambda} \right)^r \int_{\mathcal{X}} \left| \sum_i B_i b_i(x) \right|^r w(x) \, d\mu(x) \)

\( \leq C \left( \frac{C_{r,w}}{\lambda} \right)^r \int_{\mathcal{X}} \left( \sum_i |B_i b_i(x)| \right)^r w(x) \, d\mu(x) \)

\( \leq C \frac{C_{r,w}}{\lambda} \|f\|_{L^1(\mathcal{X}, w)}. \)

Summing all of the estimates of \( Z_1, \ldots, Z_5 \), we obtain for \( \lambda > C_{r,w} \|f\|_{L^1(\mathcal{X})} (\mu(\mathcal{X}))^{-1} \),

\[ w(\{x \in \mathcal{X} : v(f)(x) > \lambda \}) \leq \frac{C_{r'}}{\lambda} (C_A + C_{r,w}) \|f\|_{L^1(\mathcal{X}, w)}. \]
If \( \mathcal{X} \) is unbounded, then the above inequality holds for all \( \lambda > 0 \); and thus we are done. Now suppose that \( \mathcal{X} \) is bounded. Then for \( 0 < \lambda \leq C_{r,w}||f||L^1(\mathcal{X}) (\mu(\mathcal{X}))^{-1} \), we have
\[
w(\{x \in \mathcal{X} : v(f)(x) > \lambda \}) \leq \frac{C_{r,w}}{\lambda} \frac{w(\mathcal{X})}{\mu(\mathcal{X})} ||f||L^1(\mathcal{X}) \leq \frac{C_{r,w}}{\lambda} ||f||L^1(\mathcal{X},w).
\]
The proof of this theorem is complete. \( \square \)

4. Commutators of BMO functions and Marcinkiewicz integrals

Given a fixed positive integer \( k \), let \( \vec{b} = (b_1, b_2, \ldots, b_k) \), where each \( b_i \in \text{BMO}(\mathcal{X}) \) \((1 \leq i \leq k)\). For each \( b_i \) \((1 \leq i \leq k)\), denote by \( ||b_i||_\sigma \) the BMO norm of \( b_i \); and let \( ||\vec{b}||_\sigma = \prod_{i=1}^k ||b_i||_\sigma \).

Let \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_k) = ((b_1)_B, \ldots, (b_k)_B) \), where \( (b_i)_B = \frac{1}{\mu(B)} \int_B b_i(x) d\mu(x), 1 \leq i \leq k \). Let \( C_j^k \) \((1 \leq j \leq k)\) stand for the family of all finite subsets \( \sigma = \{\sigma(1), \ldots, \sigma(j)\} \) of different elements of \( \{1, \ldots, k\} \). For any \( \sigma \in C_j^k \), we denote the complement sequence of \( \sigma \) by \( \sigma' = \{1, \ldots, k\} \setminus \sigma \). Let \( \vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)}) \) and let the product \( b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)} \). Similarly, denote \( (\vec{b} - \vec{\lambda})_\sigma = (b_{\sigma(1)} - \lambda_{\sigma(1)}, \ldots, b_{\sigma(j)} - \lambda_{\sigma(j)}) \) and \( (b - \lambda)_\sigma = (b_{\sigma(1)} - \lambda_{\sigma(1)}, \ldots, b_{\sigma(j)} - \lambda_{\sigma(j)}) \).

For \( f \in L^p_\vec{b}(\mathcal{X}) \), define the linear operator \( F_{\vec{b}}(f) \) by
\[
F_{\vec{b}}(f)(x, \tau) = \int_{d(x,y) < \tau} \left\{ \prod_{i=1}^k (b_i(x) - b_i(y)) \right\} K(x,y)f(y) d\mu(y),
\]
where \( K(x,y) \) is a measurable function defined on \( (\mathcal{X} \times \mathcal{X}) \setminus \Delta \) with \( \Delta = \{(x,x) : x \in \mathcal{X}\} \).

Define the commutator of Marcinkiewicz integral and BMO functions, \( v_{\vec{b}}(f) \), by
\[
v_{\vec{b}}(f)(x) = \left\{ \int_0^\infty |F_{\vec{b}}(f)(x, \tau)|^2 \frac{d\tau}{\tau^3} \right\}^{1/2}.
\]
We have the following theorem.

**Theorem 4.1.** Let \( \mathcal{X} \) have infinite measure. Let \( v_{\vec{b}}(f) \) be the commutator of Marcinkiewicz integral and BMO functions with non-smooth kernel satisfying hypotheses (a), (b) and (c). Let \( w \in A_p \), \( 1 < p < \infty \). Then there exists a positive constant \( C \) (depending on \( p \) and \( w \)) such that, for all \( f \in L^p(\mathcal{X}) \),
\[
||v_{\vec{b}}(f)||_{LP(\mathcal{X},w)} \leq C ||\vec{b}||_\sigma ||f||_{LP(\mathcal{X},w)}.
\]

**Proof.** In order to prove this theorem, we need the following lemma.
LEMMA 4.2. [12] Let $t_B = r_B^m$, where $r_B$ is the radius of the ball $B$, and $m$ appears in (1.7)–(1.8). Given any $r > 1$, there exists a positive constant $C$ (depending on $r$) such that

$$M^2_A (||F_B (f)||_{\mathcal{B}_2} (x)) = \sup_{\mathcal{B} \ni x} \left\{ \frac{1}{\mu (B)} \int_B ||F_B (f) (y, \cdot) - A_{t_B} F_B (f) (y, \cdot)||_{\mathcal{B}_2} d\mu (y) \right\}$$

$$\leq C ||\vec{b}||_* \{ |M_r f (x) + M_r (||F (f)||_{\mathcal{B}_2}) (x)\} + C \sum_{i=1}^{k-1} \sum_{\sigma \in C_i} c_{k,i} \left( \prod_{l=1}^{i} ||b_{\sigma (l)}||_* \right) M_r (||F_{\vec{b}_{\sigma^l}} (f)||_{\mathcal{B}_2}) (x).$$

Given $w \in \mathcal{A}_p$, $1 < p < \infty$, there exists $\varepsilon > 0$ such that $w \in \mathcal{A}_{p-\varepsilon}$, where $p - \varepsilon > 1$. We choose $r > 1$ such that $1 < r < \frac{p}{p - \varepsilon} < p$. Note that $w \in \mathcal{A}_{p-\varepsilon} \implies w \in \mathcal{A}_{p/r}$ since $p - \varepsilon < \frac{p}{r}$.

By Theorem 2.5, Lemma 4.2, Theorem 3.2 and by induction argument, we may conclude that

$$||v_B (f)||_{L^p (\mathcal{X}, w)} \leq ||\mathcal{M} (v_B f)||_{L^p (\mathcal{X}, w)} = ||\mathcal{M} (||F_B (f)||_{\mathcal{B}_2})||_{L^p (\mathcal{X}, w)}$$

$$\leq C ||M^2_A (||F_B (f)||_{\mathcal{B}_2})||_{L^p (\mathcal{X}, w)}$$

$$\leq C ||\vec{b}||_* \left\{ ||M_r f||_{L^p (\mathcal{X}, w)} + ||M_r (||F (f)||_{\mathcal{B}_2})||_{L^p (\mathcal{X}, w)} \right\} + C \sum_{i=1}^{k-1} \sum_{\sigma \in C_i} c_{k,i} \left( \prod_{l=1}^{i} ||b_{\sigma (l)}||_* \right) ||M_r (||F_{\vec{b}_{\sigma^l}} (f)||_{\mathcal{B}_2})||_{L^p (\mathcal{X}, w)}$$

$$\leq C ||\vec{b}||_* ||f||_{L^p (\mathcal{X}, w)} + C \sum_{i=1}^{k-1} \sum_{\sigma \in C_i} c_{k,i} \left( \prod_{l=1}^{i} ||b_{\sigma (l)}||_* \right) ||v_{\vec{b}_{\sigma^l}} (f)||_{L^p (\mathcal{X}, w)}$$

$$\leq C ||\vec{b}||_* ||f||_{L^p (\mathcal{X}, w)}. \quad \Box$$

5. Marcinkiewicz integrals on homogeneous Herz spaces

5.1. Homogeneous Herz spaces

For $k \in \mathbb{Z}$, let $A_k = \left\{ x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k \right\}$. Denote the characteristic function $\chi_{A_k}$ on the set $A_k$ by $\chi_k$. For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, the homogeneous Herz spaces $K^\alpha_{q,p} (\mathbb{R}^n)$ are defined by

$$K^\alpha_{q,p} (\mathbb{R}^n) = \left\{ f \in L^q_{\text{loc}} (\mathbb{R}^n \setminus \{0\}) : ||f||_{K^\alpha_{q,p} (\mathbb{R}^n)} := \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} ||f \chi_k||_{L^q (\mathbb{R}^n)}^p \right)^{1/p} < \infty \right\},$$

with usual modifications when $p = \infty$ or $q = \infty$. Observe that $K^\alpha_{q/(p-q)} (\mathbb{R}^n) = L^q (\mathbb{R}^n, |x|^{\alpha})$, and $K^0_{q,p} (\mathbb{R}^n) = L^q (\mathbb{R}^n)$.
Denote \( m_k(\lambda, f) := |\{ x \in \mathbb{R}^n : |f(x)| > \lambda \}| \), where \( \lambda > 0 \). For \( \alpha \in \mathbb{R} \), \( 0 < p \leq \infty \) and \( 0 < q < \infty \), the homogeneous weak Herz spaces \( W^\alpha_{q,p}(\mathbb{R}^n) \) consist of all measurable functions \( f \) on \( \mathbb{R}^n \) whose norms are finite:

\[
\|f\|_{W^\alpha_{q,p}(\mathbb{R}^n)} := \sup_{\lambda > 0} \left\{ \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} (m_k(\lambda, f))^{p/q} \right)^{1/p} \right\} < \infty \quad (0 < p < \infty),
\]

and

\[
\|f\|_{W^\alpha_{q,\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \left\{ \lambda \sup_{k \in \mathbb{Z}} \left( 2^{k\alpha} (m_k(\lambda, f))^{1/q} \right) \right\} < \infty.
\]

Note that \( W^\alpha_{q,q}(\mathbb{R}^n) = L^{q,\infty}(\mathbb{R}^n, |x|^\alpha) \) and \( W^0_{q,q}(\mathbb{R}^n) = L^{q,\infty}(\mathbb{R}^n) \), where \( L^{q,\infty}(\mathbb{R}^n, w) \) is the space of all measurable functions \( f \) such that

\[
\sup_{\lambda > 0} \left\{ \lambda (w(x \in \mathbb{R}^n : |f(x)| > \lambda))^{1/q} \right\} < \infty.
\]

For topics related to Herz spaces, the reader may view [8, 9, 10] among many other good references.

In order to obtain the boundedness of Marcinkiewicz integrals and their commutators on homogeneous Herz spaces, we need the following theorems.

**Theorem 5.1.** [9] If a sublinear operator \( T \) is bounded on \( L^q(\mathbb{R}^n, |x|^\beta) \) for some \( q \in (1, \infty) \) and for all \( \beta \in (\beta_1, \beta_2) \), where \( \beta_1, \beta_2 \in \mathbb{R} \), then \( T \) is also bounded on \( K^\alpha_{q,p}(\mathbb{R}^n) \) for all \( \alpha \in (\beta_1/q, \beta_2/q) \) and all \( p \in (0, \infty] \).

**Theorem 5.2.** Suppose that a sublinear operator \( T \) is bounded from \( L^1(\mathbb{R}^n, |x|^\beta) \) to \( L^{1,\infty}(\mathbb{R}^n, |x|^\beta) \) for all \( \beta \in (\beta_1, \beta_2) \), where \( \beta_1, \beta_2 \in \mathbb{R} \). Then \( T \) is bounded from \( K^\alpha_{1,p}(\mathbb{R}^n) \) to \( \dot{W}^\alpha_{1,p}(\mathbb{R}^n) \) for all \( \alpha \in (\beta_1, \beta_2) \) and all \( p \in (0, \infty] \). That is, there exists a positive constant \( C \), depending on \( p \) and \( \alpha \), such that

\[
\|TF\|_{\dot{W}^\alpha_{1,p}(\mathbb{R}^n)} \leq C \|f\|_{K^\alpha_{1,p}(\mathbb{R}^n)} \quad \text{for all } \alpha \in (\beta_1, \beta_2) \text{ and all } p \in (0, \infty].
\]

**Proof.** We apply some ideas in [9] to prove this theorem. Note that

\[
\|TF\|_{\dot{W}^\alpha_{1,p}(\mathbb{R}^n)} = \sup_{\lambda > 0} \left\{ \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} m_k(\lambda, Tf)^p \right)^{1/p} \right\}, \quad 0 < p < \infty, \tag{5.1}
\]

and

\[
\|TF\|_{W^\alpha_{1,\infty}(\mathbb{R}^n)} = \sup_{\lambda > 0} \left\{ \sup_{k \in \mathbb{Z}} \left( 2^{k\alpha} m_k(\lambda, Tf) \right) \right\}, \tag{5.2}
\]

where \( m_k(\lambda, Tf) = |\{ x \in A_k : |Tf(x)| > \lambda \}| \) and \( A_k = \{ x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k \} \), \( k \in \mathbb{Z} \).
For $j \in \mathbb{Z}$, denote $f_j = f \chi_j \equiv f \chi_{A_j}$. Pick $\alpha_1, \alpha_2 \in (\beta_1, \beta_2)$ such that $\beta_1 < \alpha_1 < \alpha < \alpha_2 < \beta_2$. We first consider the case $0 < p < \infty$. By hypothesis, we have

$$J_\lambda := \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha_p} m_k(\lambda, Tf)^p \right)^{1/p}$$

$$\leq C_p \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha_p} \left\{ x \in A_k : \left| T \left( \sum_{j=-\infty}^{k} f_j \right)(x) \right| > \frac{\lambda}{2} \right\} \right)^{1/p}$$

$$+ C_{p} \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha_p} \left\{ x \in A_k : \left| T \left( \sum_{j=k+1}^{\infty} f_j \right)(x) \right| > \frac{\lambda}{2} \right\} \right)^{1/p}$$

$$\leq C \left\{ \sum_{k \in \mathbb{Z}} 2^{k(\alpha_1-\alpha_2)p} \left( \sum_{j=-\infty}^{k} ||f_j||_{L^1(\mathbb{R}^n, |x|^{\alpha_2})} \right)^p \right\}^{1/p}$$

$$+ C \left\{ \sum_{k \in \mathbb{Z}} 2^{k(\alpha_1-\alpha_3)p} \left( \sum_{j=k+1}^{\infty} ||f_j||_{L^1(\mathbb{R}^n, |x|^{\alpha_1})} \right)^p \right\}^{1/p}$$

$$\equiv J_3 + J_4.$$  

(5.3)
\[
= C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{k = -\infty}^{2^j} 2^{k(\alpha - \alpha_1)p} \right) 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} \\
\leq C \left\{ \sum_{j \in \mathbb{Z}} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} = C \|f\|_{K^{\alpha,p}_1(\mathbb{R}^n)}.
\]

Hence,
\[
J_\lambda \leq J_3 + J_4 \leq C \|f\|_{K^{\alpha,p}_1(\mathbb{R}^n)}.
\]

If \(1 < p < \infty\), then we apply Hölder’s inequality to obtain
\[
J_\lambda \leq C \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{j = -\infty}^{k} 2^{(k-j)(\alpha - \alpha_2)p/2} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right) \left( \sum_{j = -\infty}^{k} 2^{(k-j)(\alpha - \alpha_2)p'/2} \right)^{p'/p} \right\}^{1/p} \\
+ C \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{j = k+1}^{\infty} 2^{(k-j)(\alpha - \alpha_2)p/2} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right) \left( \sum_{j = k+1}^{\infty} 2^{(k-j)(\alpha - \alpha_2)p'/2} \right)^{p'/p} \right\}^{1/p} \\
= C \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{j = -\infty}^{k} 2^{(k-j)(\alpha - \alpha_2)p/2} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right) \left( \sum_{j = -\infty}^{k} 2^{j(\alpha - \alpha_2)p'/2} \right)^{p'/p} \right\}^{1/p} \\
+ C \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{j = k+1}^{\infty} 2^{(k-j)(\alpha - \alpha_2)p/2} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right) \left( \sum_{j = k+1}^{\infty} 2^{j(\alpha - \alpha_2)p'/2} \right)^{p'/p} \right\}^{1/p} \\
\leq C \left\{ \sum_{k \in \mathbb{Z}} \sum_{j = -\infty}^{k} 2^{(k-j)(\alpha - \alpha_2)p/2} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} \\
+ C \left\{ \sum_{k \in \mathbb{Z}} \sum_{j = k+1}^{\infty} 2^{(k-j)(\alpha - \alpha_1)p/2} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} \\
\leq C \left\{ \sum_{j \in \mathbb{Z}} 2^j \|f_j\|^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} = C \|f\|_{K^{\alpha,p}_1(\mathbb{R}^n)},
\]

where the last inequality follows from similar calculations as in the case \(0 < p < 1\).

Therefore, for \(0 < p < \infty\),
\[
\|TF\|_{W^{K^{\alpha,p}_1(\mathbb{R}^n)}} = \sup_{\lambda > 0} \{J_\lambda\} \leq C \|f\|_{K^{\alpha,p}_1(\mathbb{R}^n)}.
\]

When \(p = \infty\), we have
\[
J_\lambda \leq J_3 + J_4 \\
\leq C \sum_{j = -\infty}^{k} 2^{(k-j)(\alpha - \alpha_1)p} 2^j \|f_j\|_{L^1(\mathbb{R}^n)} + C \sum_{j = k+1}^{\infty} 2^{(k-j)(\alpha - \alpha_1)p} 2^j \|f_j\|_{L^1(\mathbb{R}^n)}
\]
\begin{align*}
&\leq C \sup_{j \in \mathbb{Z}} \left\{ 2^{j \alpha} \|f_j\|_{L^1(\mathbb{R}^n)} \right\} \left( \sum_{j=-\infty}^{k} 2^{(k-j)(\alpha-\alpha_2)} \right) \\
&+ C \sup_{j \in \mathbb{Z}} \left\{ 2^{j \alpha} \|f_j\|_{L^1(\mathbb{R}^n)} \right\} \left( \sum_{j=k+1}^{\infty} 2^{(k-j)(\alpha-\alpha_1)} \right) \\
&= C \sup_{j \in \mathbb{Z}} \left\{ 2^{j \alpha} \|f_j\|_{L^1(\mathbb{R}^n)} \right\} \left( \sum_{j=0}^{\infty} 2^{j(\alpha-\alpha_2)} \right) \\
&+ C \sup_{j \in \mathbb{Z}} \left\{ 2^{j \alpha} \|f_j\|_{L^1(\mathbb{R}^n)} \right\} \left( \sum_{j=-\infty}^{0} 2^{j(\alpha-\alpha_1)} \right) \\
&\leq C \sup_{j \in \mathbb{Z}} \left\{ 2^{j \alpha} \|f_j\|_{L^1(\mathbb{R}^n)} \right\} = C \|f\|_{K_1^{\alpha,\infty}(\mathbb{R}^n)}. 
\end{align*}

Hence,

\[ \|TF\|_{W^{\alpha,\infty}(\mathbb{R}^n)} \leq \sup_{\lambda > 0} \left\{ \sup_{k \in \mathbb{Z}} \{J_3 + J_4\} \right\} \leq C \|f\|_{K_1^{\alpha,\infty}(\mathbb{R}^n)}. \]

The proof of this theorem is finished. \( \square \)

**Theorem 5.3.** Assume that hypotheses (a), (b) and (c) hold true. Then there exist positive constants \( C_1 \) and \( C_2 \) (depending on \( p \), \( q \) and \( \alpha \)) such that

\[ \|\nu(f)\|_{K_1^{\alpha,p}(\mathbb{R}^n)} \leq C_1 \|f\|_{K_1^{\alpha,p}(\mathbb{R}^n)}, \quad (5.4) \]

and

\[ \|\nu_b(f)\|_{K_1^{\alpha,p}(\mathbb{R}^n)} \leq C_2 \|\vec{b}\|_* \|f\|_{K_1^{\alpha,p}(\mathbb{R}^n)}, \quad (5.5) \]

for \( 1 < q < \infty \), \( 0 < p \leq \infty \), and \( -\frac{n}{q} < \alpha < \frac{n}{q'} \), where \( \frac{1}{q} + \frac{1}{q'} = 1 \).

If hypotheses (a), (\( \vec{b} \)) and (c) hold true, then there exists a positive constant \( C_3 \) (depending on \( p \) and \( \alpha \)) such that

\[ \|\nu(f)\|_{W^{\alpha,p}(\mathbb{R}^n)} \leq C_3 \|f\|_{K_1^{\alpha,p}(\mathbb{R}^n)}, \quad (5.6) \]

for all \( \alpha \in (-n, 0) \) and all \( p \in (0, \infty] \).

**Proof.** Recall that \( w(x) = |x|^a \in \mathcal{A}_p \) if and only if \( -n < a < n(p-1) \) for \( 1 < p < \infty \); and \( w(x) = |x|^a \in \mathcal{A}_1 \) if and only if \( -n < a \leq 0 \) (see [6]). Inequality (5.4) follows from Theorem 3.2 and Theorem 5.1. By Theorem 4.1 and Theorem 5.1, we obtain inequality (5.5). Finally, inequality (5.6) is a consequence of Theorem 3.3 and Theorem 5.2. \( \square \)
5.2. Example

We will need the following propositions for the example below.

**Proposition 5.4.** Suppose that there exist constants $\delta_1 > 1$, $C_S > 0$ and $\beta > 0$ such that

$$
\frac{|K(x, y) - K(x, z)|}{d(x, y)} \leq \frac{C_S}{\mu(B(x; d(x, y)))} \left\{ \frac{d(y, z)}{d(x, y)} \right\}^\beta \text{ whenever } d(x, y) \geq \delta_1 d(y, z).
$$

Then there exist positive constants $C_6$ and $\delta_2$ such that

$$
\frac{|K(x, y) - K_t(x, y)|}{d(x, y)} \leq \frac{C_6}{\mu(B(x; d(x, y)))} \left( \frac{t}{d(x, y)} \right)^{\beta/m} \text{ whenever } d(x, y) \geq \delta_2 t^{1/m}.
$$

**Proposition 5.5.** Suppose that there exist constants $\delta_3 > 1$, $C_7 > 0$ and $\beta > 0$ such that

$$
\frac{|K(x, y) - K(z, y)|}{d(x, y)} \leq \frac{C_7}{\mu(B(x; d(x, y)))} \left\{ \frac{d(x, z)}{d(x, y)} \right\}^\beta \text{ whenever } d(x, y) \geq \delta_3 d(x, z).
$$

Then there exist positive constants $C_8$ and $\delta_4$ such that

$$
\frac{|K(x, y) - K^t(x, y)|}{d(x, y)} \leq \frac{C_8}{\mu(B(x; d(x, y)))} \left( \frac{t}{d(x, y)} \right)^{\beta/m} \text{ whenever } d(x, y) \geq \delta_4 t^{1/m}.
$$

We omit the proofs of these propositions since they are essentially similar to the proof of Proposition 2 in [3].

Now let $\mathcal{X} = \mathbb{R}^n$, $n \geq 2$. Let $\Omega$ be homogeneous of degree zero, $\Omega(x) = \Omega(\frac{x}{|x|})$ for every nonzero $x \in \mathbb{R}^n$, and satisfy the cancellation condition

$$
\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.
$$

Assume further that $\Omega$ satisfies the $\beta$-Hölder’s continuity:

There exist constants $C > 0$ and $\beta \in (0, 1]$ such that for all nonzero $x, y \in \mathbb{R}^n$,

$$
|\Omega(x) - \Omega(y)| \leq C \frac{|x|}{|y|}^\beta.
$$

Let $K$ be the convolution kernel defined by $K(x) = \frac{\Omega(x)}{|x|^{n-1}}$, $n \geq 2$. Let

$$
F_\Omega(f)(x, t) = \int_{|x-y| \leq t} K(x-y)f(y)dy,
$$

and

$$
F_{\Omega, \beta}(f)(x, t) = \int_{|x-y| \leq t} \left( \prod_{i=1}^k (b_i(x) - b_i(y)) \right) K(x-y)f(y)dy,
$$

whenever $d(x, y) \geq \delta_1 d(y, z)$.
where each $b_i \in \text{BMO}(\mathbb{R}^n)$, $1 \leq i \leq k$. Recall that the Marcinkiewicz integral and its commutator are given by
\[

v_\Omega(f)(x) = \left( \int_0^\infty |F_\Omega(f)(x,t)|^2 \frac{dt}{t^3} \right)^{1/2} \quad \text{and} \quad v_{\Omega,b}(f)(x) = \left( \int_0^\infty |F_{\Omega,b}(f)(x,t)|^2 \frac{dt}{t^3} \right)^{1/2}.

\]

Since $K$ is a convolution kernel satisfying the cancellation condition, it can be shown via Fourier transform that $v_\Omega(f)$ is bounded on $L^2(\mathbb{R}^n)$. Thus hypothesis (a) is satisfied. Moreover, it is straightforward to obtain inequalities (5.7)–(5.8) from inequality (5.9). By propositions 5.4–5.5, hypotheses (b) and (c) are fulfilled. Consequently, we may infer from Theorem 5.3 that there exist positive constants $C_1$, $C_2$ and $C_3$ such that
\[

\|v_\Omega(f)\|_{K^{\alpha,p}_q(\mathbb{R}^n)} \leq C_1 \|f\|_{K^{\alpha,p}_q(\mathbb{R}^n)},
\]
\[

\|v_{\Omega,b}(f)\|_{K^{\alpha,p}_q(\mathbb{R}^n)} \leq C_2 \|b\|_\infty \|f\|_{K^{\alpha,p}_q(\mathbb{R}^n)},
\]

for $1 < q < \infty$, $0 < p \leq \infty$, and $-\frac{n}{q} < \alpha < \frac{n}{q'}$, where $\frac{1}{q} + \frac{1}{q'} = 1$, and
\[

\|v_\Omega(f)\|_{WK^{\alpha,p}_1(\mathbb{R}^n)} \leq C_3 \|f\|_{K^{\alpha,p}_1(\mathbb{R}^n)}
\]

for all $\alpha \in (-n, 0)$ and all $p \in (0, \infty]$.

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