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WEIGHTED ESTIMATES FOR MARCINKIEWICZ INTEGRALS WITH NON–SMOOTH KERNELS ON SPACES OF HOMOGENEOUS TYPE AND THEIR APPLICATIONS

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Abstract. Given a family of operators that act like approximations of the identity, we obtain weighted estimates for the Marcinkiewicz integrals with non-smooth kernels on spaces of homogeneous type. As applications of weighted estimates, we also establish the boundedness of these operators on the homogeneous Herz spaces over Euclidean spaces, and on the homogeneous weak Herz spaces for the endpoint case. We also study the boundedness of commutators of Marcinkiewicz integrals and BMO functions on various spaces.

1. Introduction

Let $(\mathscr{X}, d, \mu)$ be a space of homogeneous type, endowed with a metric distance d on $\mathscr{X} \times \mathscr{X}$ satisfying

$$d(x,z) \leqslant \kappa(d(x,y)+d(y,z)) \text{ for some fixed constant } \kappa \geqslant 1 \text{ and for all } x,y,z \in \mathscr{X}, \tag{1.1}$$

and a regular Borel measure μ on $\mathscr{X}$ such that the doubling property

$$\mu(B(x;2r)) \leqslant C\mu(B(x;r)) < \infty \tag{1.2}$$

holds for some fixed constant $C \geqslant 1$, for all $x \in \mathscr{X}$ and for all $r > 0$, where $B(x;r) = \{y \in \mathscr{X} : d(x,y) < r\}$. The above property implies that there exist some fixed constants $C \geqslant 1$, $n > 0$ such that

$$\mu(B(x;\lambda r)) \leqslant C\lambda^n \mu(B(x;r)), \tag{1.3}$$

uniformly for all $\lambda \geqslant 1$, $x \in \mathscr{X}$, and $r > 0$. The parameter n measures the "dimension" of the space $\mathscr{X}$. There also exist constants C, N $(C \geqslant 1,\ 0 \leqslant N \leqslant n)$ such that

$$\mu(B(y;r)) \leqslant C\left(1+\frac{d(x,y)}{r}\right)^N \mu(B(x;r)) \tag{1.4}$$

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uniformly for all $x, y \in \mathscr{X}$ and all $r > 0$. The reader can find more information on this subject in [1, 2].

Let T be a bounded linear operator on $L^2(\mathscr{X})$ with an associated kernel $K(x, y)$ in the sense that

$$Tf(x) = \int_{\mathscr{X}} K(x, y) f(y)\, d\mu(y), \tag{1.5}$$

where f is a continuous function with compact support, $x \notin \operatorname{supp} f$; and $K(x, y)$ is a measurable function defined on $(\mathscr{X} \times \mathscr{X}) \setminus \Delta$ with $\Delta = \{(x, x) : x \in \mathscr{X}\}$.

The authors in [3, 4] assumed that there exists a class of operators A_t $(t > 0)$ which can be represented by the kernels $a_t(x, y)$ in the sense that

$$A_t u(x) = \int_{\mathscr{X}} a_t(x, y) u(y)\, d\mu(y), \text{ for every function } u \in L^1(\mathscr{X}) \cap L^2(\mathscr{X}).$$

Moreover, the kernels $a_t(x, y)$ satisfy the following conditions

$$|a_t(x, y)| \leqslant h_t(x, y), \text{ for all } x, y \in \mathscr{X}, \tag{1.6}$$

$$\text{where } h_t(x, y) = (\mu(B(x; t^{1/m})))^{-1} s((d(x, y))^m t^{-1}) \text{ for some positive constant } m. \tag{1.7}$$

Here s is a positive, bounded, decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\sigma} s(r^m) = 0, \tag{1.8}$$

for some $\sigma > N$, where n and N appear in (1.3) and (1.4) respectively.

REMARK 1.1. The functions h_t above satisfy the following properties (see [4, 5]):
i) There exist positive constants C_1 and C_2 such that

$$C_1 \leqslant \int_{\mathscr{X}} h_t(x, y)\, d\mu(x) \leqslant C_2 \text{ uniformly in } t \text{ and } y.$$

ii) There exists a positive constant C such that

$$\int_{\mathscr{X}} h_t(x, y) |f(x)|\, d\mu(x) \leqslant C \mathscr{M} f(y) \text{ and } \int_{\mathscr{X}} h_t(x, y) |f(y)|\, d\mu(y) \leqslant C \mathscr{M} f(x).$$

Here $\mathscr{M} f(x)$, the Hardy-Littlewood maximal function, is defined by

$$\mathscr{M} f(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y) \right\},$$

where the supremum is taken over all balls B containing x.

The class of operators A_t plays the role of approximation to the identity. The existence of such a class of operators A_t was verified in [3]. Now let A_t and B_t $(t > 0)$ be two classes of operators which satisfy (1.6)–(1.8). Denote by $K(x, y) - K_t(x, y)$ the kernels of the operators $(T - TB_t)$, and $K(x, y) - K^t(x, y)$ as the kernels of $(T - A_t T)$ respectively. We state below a list of assumptions which lead to interesting results:

(i) T is a bounded linear operator from $L^2(\mathscr{X})$ to $L^2(\mathscr{X})$;

(ii) There exist positive constants c_1 and C_A such that

$$\int_{d(x,y)\geqslant c_1 t^{1/m}} |K(x,y)-K_t(x,y)|\,d\mu(x) \leqslant C_A, \text{ for all } y\in\mathscr{X};$$

(iii) There exist positive constants c_2 and C_A such that

$$\int_{d(x,y)\geqslant c_2 t^{1/m}} |K(x,y)-K^t(x,y)|\,d\mu(y) \leqslant C_A, \text{ for all } x\in\mathscr{X};$$

(iv) There exist positive constants c_2, c_4 and β such that

$$|K(x,y)-K^t(x,y)| \leqslant \frac{c_4}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{[d(x,y)]^{\beta}}, \text{ whenever } d(x,y)\geqslant c_2 t^{1/m}.$$

Using assumptions (i), (ii) and (iii), Duong and McIntosh [3] obtained the L^p-boundedness of the singular integral operator T. Afterward, Martell [16] extended their results to weighted spaces with weights $w\in A_p$, under hypotheses (i), (ii) and (iv). Based on a set of hypotheses that are almost similar to the assumptions (i), (ii) and (iv) above, the author in [12] recently obtained the L^p-boundedness of the Marcinkiewicz integral

$$\nu(f)(x)=\left\{\int_0^\infty\left|\int_{d(x,y)<\tau}K(x,y)f(y)\,d\mu(y)\right|^2\frac{d\tau}{\tau^3}\right\}^{1/2}, \tag{1.9}$$

and the commutator

$$\nu_{\vec{b}}(f)(x)=\left\{\int_0^\infty\left|\int_{d(x,y)<\tau}\left(\prod_{i=1}^k(b_i(x)-b_i(y))\right)K(x,y)f(y)\,d\mu(y)\right|^2\frac{d\tau}{\tau^3}\right\}^{1/2}, \tag{1.10}$$

where $b_i\in\mathrm{BMO}(\mathscr{X})$, $1\leqslant i\leqslant k$. The reader may further view [4, 11, 12, 13, 14, 15, 19] for several interesting results about this topic.

The purpose of this paper is to extend the results of [12] to homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. In order to do so, we first obtain the weighted L^p estimates for $\nu(f)$ and $\nu_{\vec{b}}(f)$. We then use these estimates to further extend to homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and to homogeneous weak Herz spaces $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ for the endpoint case. The plan of this paper is as follows. Section 2 describes some background information such as definitions, notations and preliminary theorems. Our main theorems are given in sections 3–5. In the last section, we give an example about a classical kernel $K(x,y)$ that satisfies the hypotheses of our theorems. For the rest of this paper, the letter C denotes a positive constant which may vary at each occurrence; however, it is independent of any essential variable.

2. Preliminaries

2.1. Approximation of the identity

Denote by $L_b^\infty(\mathscr{X})$ the set of all functions in $L^\infty(\mathscr{X})$ with bounded support. Note that $L_b^\infty(\mathscr{X})$ is dense in $L^p(\mathscr{X})$ for $p \in (0,\infty)$ (see [6, 7] for example). For $f \in L_b^\infty(\mathscr{X})$, define the linear operator F by

$$F(f)(x,\tau) = \int_{d(x,y)<\tau} K(x,y) f(y)\, d\mu(y), \tag{2.1}$$

where $K(x,y)$ is a measurable function defined on $(\mathscr{X}\times\mathscr{X})\setminus\Delta$ with $\Delta = \{(x,x) : x \in \mathscr{X}\}$. Define the Marcinkiewicz integral $\nu(f)$ by

$$\nu(f)(x) = \left\{ \int_0^\infty |F(f)(x,\tau)|^2 \frac{d\tau}{\tau^3} \right\}^{1/2}. \tag{2.2}$$

Denote $\mathscr{B}_1 = \mathbb{C}$ and $\mathscr{B}_2 = L^2\left(\mathbb{R}^+; \dfrac{d\tau}{\tau^3}\right)$, where $\mathbb{R}^+ = (0,\infty)$.
Then $\nu(f)(x) = ||F(f)(x,\cdot)||_{\mathscr{B}_2}$, so that $||\nu(f)||_{L^p(\mathscr{X})} = \big|\big| \|F(f)\|_{\mathscr{B}_2} \big|\big|_{L^p(\mathscr{X})} := ||F(f)||_{L^p(\mathscr{X};\mathscr{B}_2)}$.
Note also that $L^p(\mathscr{X};\mathscr{B}_1) = L^p(\mathscr{X})$.

In the sequel, we assume the existence of two classes of operators A_t and B_t $(t>0)$, both of which can be represented by kernels $a_t(x,y)$ and $b_t(x,y)$ respectively in the sense that

$$A_t u(x) = \int_{\mathscr{X}} a_t(x,y) u(y)\, d\mu(y),$$

for every function $u \in L^1(\mathscr{X}) \cap L^r(\mathscr{X})$ and for some $r>1$ and similar definition for B_t. Moreover, both kernels $a_t(x,y)$ and $b_t(x,y)$ are assumed to satisfy inequalities (1.6)–(1.8). Let $K_t(x,y)$ and $K^t(x,y)$ represent the kernels of the operators FB_t and A_tF $(t>0)$ respectively. We may assume that both FB_t and A_tF have integral forms

$$(FB_t)(f)(x,\tau) = \int_{d(x,y)<\tau} K_t(x,y) f(y)\, d\mu(y)$$

and

$$(A_tF)(f)(x,\tau) = \int_{d(x,y)<\tau} K^t(x,y) f(y)\, d\mu(y). \tag{2.3}$$

To see this, consider the kernel b_t of the operator B_t defined in [3] by

$$b_t(y,z) = \chi_{B(z;t^{1/m})}(y) [\mu(B(z;t^{1/m}))]^{-1}.$$

Now, let $B_{t,\tau}$ be the operator whose kernel $b_{t,\tau}(y,z)$ is defined by $b_{t,\tau}(y,z) = \chi_{B(z;\tau)}(y)\, b_t(y,z)$.
Then $|b_{t,\tau}(y,z)| \leqslant |b_t(y,z)| \leqslant h_t(y,z)$ for all $t,\ \tau > 0$ and $y, z \in \mathscr{X}$. Moreover,

$$\begin{aligned} \|(FB_{t,\tau})(f)(x,\cdot)\|_{\mathscr{B}_2} &= \left|\left| \int_{d(x,z)<2\kappa\tau} K_t(x,z) f(z)\, d\mu(z) \right|\right|_{\mathscr{B}_2} \\ &= 2\kappa \left|\left| \int_{d(x,z)<\tau} K_t(x,z) f(z)\, d\mu(z) \right|\right|_{\mathscr{B}_2}, \end{aligned}$$

where κ appears in (1.1). Similarly, if we let $A_{t,\tau}$ be the operator whose kernel $a_{t,\tau}(x,y)$ is given by

$$a_{t,\tau}(x,y) = \chi_{B(x;\tau)}(y)\,a_t(x,y) = \chi_{B(x;\tau)}(y)\,\chi_{B(x;t^{1/m})}(y)[\mu(B(x;t^{1/m}))]^{-1},$$

then the above equation also holds for $\|(A_{t,\tau}F)(f)(x,\cdot)\|_{\mathscr{B}_2}$ (with $K^t(x,z)$ instead of $K_t(x,z)$). Therefore, for simplicity, we will assume that (2.3) holds true; and we will work with the operators A_t and B_t for the rest of this paper.

For a ball $B \subset \mathscr{X}$, denote the radius of the ball B by r_B. Let $t_B = r_B^m$, where m appears in (1.7)–(1.8). For $g \in L_b^\infty(\mathscr{X};\mathscr{B}_2)$, we define the sharp maximal function $M_A^\sharp(||g||_{\mathscr{B}_2})$ by

$$M_A^\sharp(||g||_{\mathscr{B}_2})(x) = \sup_{B\ni x}\left\{\frac{1}{\mu(B)}\int_B ||(g - A_{t_B}g)(y,\cdot)||_{\mathscr{B}_2}\,d\mu(y)\right\}. \tag{2.4}$$

We now state the following assumptions for our theorems:

(a) ν is a bounded operator from $L^r(\mathscr{X})$ to $L^r(\mathscr{X})$ with the bound C_r for some $r > 1$;

(b) there exist positive constants c_1 and C_A such that

$$\int_{d(x,y)\geqslant c_1 t^{1/m}} \frac{|K(x,y) - K_t(x,y)|}{d(x,y)}\,d\mu(x) \leqslant C_A, \text{ for all } y \in \mathscr{X};$$

($\tilde{\mathbf{b}}$) there exist positive constants c_1 and c_3 such that

$$\frac{|K(x,y) - K_t(x,y)|}{d(x,y)} \leqslant \frac{c_3}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^\beta} \text{ whenever } d(x,y) \geqslant c_1 t^{1/m};$$

(c) there exist positive constants c_2 and c_4 such that

$$\frac{|K(x,y) - K^t(x,y)|}{d(x,y)} \leqslant \frac{c_4}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^\beta} \text{ whenever } d(x,y) \geqslant c_2 t^{1/m}.$$

REMARK 2.1. Note that hypothesis **($\tilde{\mathbf{b}}$)** implies hypothesis **(b)**. Assumption **($\tilde{\mathbf{b}}$)** will be used to prove weak type $(1,1)$ inequality for the Marcinkiewicz integral.

2.2. Muckenhoupt weights

For $p \in (1,\infty)$, let p' be the dual exponent of p. That is, $\frac{1}{p} + \frac{1}{p'} = 1$. A weight w is said to belong to the Muckenhoupt class $\mathscr{A}_p(\mathscr{X})$, $1 < p < \infty$, if there exists a positive constant C such that

$$\left(\frac{1}{\mu(B)}\int_B w(x)\,d\mu(x)\right)\left(\frac{1}{\mu(B)}\int_B w^{-p'/p}(x)\,d\mu(x)\right)^{p/p'} \leqslant C < \infty,$$

for all balls $B \subset \mathscr{X}$. The smallest constant C for which the above inequality holds is the $\mathscr{A}_p$ bound of w. For a ball $B \subset \mathscr{X}$ and any weight w, let $w(B) = \int_B w(x)\,d\mu(x)$. The class $\mathscr{A}_1(\mathscr{X})$ consists of non-negative functions w such that

$$\frac{w(B)}{\mu(B)} \leqslant C \operatorname{ess\,inf}_{x\in B} w(x)$$

for all balls $B \subset \mathscr{X}$. A weight w belongs to $\mathscr{A}_\infty(\mathscr{X})$ if there exist positive constants C_w and δ_w $(0 < \delta_w < 1)$ such that, for any ball $B \subset \mathscr{X}$ and any measurable subset $E \subset B$,

$$\frac{w(E)}{w(B)} \leqslant C_w \left(\frac{\mu(E)}{\mu(B)}\right)^{\delta_w}. \tag{2.5}$$

If $w \in \mathscr{A}_p(\mathscr{X})$ $(1 \leqslant p < \infty)$, we denote $L^p(\mathscr{X}, w)$ to be the space of all measurable functions f on $\mathscr{X}$ whose norms are finite:

$$\|f\|_{L^p(\mathscr{X},w)} := \left\{\int_{\mathscr{X}} ||f(x)||^p\, w(x)\,d\mu(x)\right\}^{1/p} < \infty.$$

For more information on topics of weights, the reader may view [17, 18].

2.3. Preliminary lemmas and theorems

The following lemmas and theorems are necessary for the proof of our theorems.

LEMMA 2.2. [2] *Let $f \in L^1(\mathscr{X})$ with bounded support and $\alpha > \|f\|_{L^1(\mathscr{X})}(\mu(\mathscr{X}))^{-1}$. Then there exist positive constants C, M_1 depending only on $\mathscr{X}$ and a sequence of metric balls $\{B_i\}_i \equiv \{B(x_i, r_i)\}_i$ such that*

$$\Omega_\alpha := \{x \in \mathscr{X} : \mathscr{M}(f)(x) > C_{\mathscr{X}}\alpha\} = \bigcup_i B_i,$$

and

a) $f(x) = g(x) + b(x)$, where

$$g(x) = f(x)\chi_{\Omega_\alpha^c}(x) + \sum_i \left(\frac{1}{\mu(B_i)}\int_{B_i} f(y)\eta_i(y)d\mu(y)\right)\chi_{B_i}(x),$$

$$\eta_i(x) = \frac{\chi_{B_i}(x)}{\sum_j \chi_{B_j}(x)}, \quad b(x) = \sum_i b_i(x),$$

$$b_i(x) = f(x)\eta_i(x) - \left(\frac{1}{\mu(B_i)}\int_{B_i} f(y)\eta_i(y)d\mu(y)\right)\chi_{B_i}(x)$$

for all $x \in \mathscr{X}$;

b) $|g(x)| \leqslant C\alpha$ for almost all $x \in \mathscr{X}$;

c) $||g||_{L^1(\mathscr{X})} \leqslant C||f||_{L^1(\mathscr{X})}$;

d) for all $i \in \mathbb{N}$, *supp*$(b_i) \subset B_i$ *and* $\sum_i \mu(B_i) \leqslant C\alpha^{-1}||f||_{L^1(\mathscr{X})}$;

e) for all $i \in \mathbb{N}$, $\int_{\mathscr{X}} b_i(x)d\mu(x) = 0$;

f) for all $i \in \mathbb{N}$, $\frac{1}{\mu(B_i)}\int_{B_i} |b_i(x)|d\mu(x) \leqslant C\alpha$, *and* $\sum_i ||b_i||_{L^1(\mathscr{X})} \leqslant C||f||_{L^1(\mathscr{X})}$;

g) every point of $\mathscr{X}$ *belongs to no more than* M_1 *balls of* $\{B_i\}_i$.

LEMMA 2.3. [12] *Assume that hypotheses **(a)** and **(c)** (as stated in section 2.1) hold true. Then there exists a positive constant* c_9 *such that, for every* $f \in L_b^\infty(\mathscr{X})$ *and any* $r > 1$,

$$M_A^\sharp(||F(f)||_{\mathscr{B}_2})(x) \leqslant c_9 M_r f(x),$$

where the definitions of $||F(f)(x,\cdot)||_{\mathscr{B}_2}$ *and* $M_A^\sharp$ *were given by equations* (2.1) *and* (2.4) *respectively; and*

$$M_r f(x) := \sup_{B \ni x} \left\{ \frac{1}{\mu(B)} \int_B |f(y)|^r \, d\mu(y) \right\}^{1/r} = \{\mathscr{M}(|f|^r)(x)\}^{1/r}.$$

THEOREM 2.4. [12] *Assume that hypotheses **(a)**, **(b)** and **(c)** (as stated in section 2.1) hold true. Then the operator* ν *has well-defined extensions on* $L^p(\mathscr{X})$ *for* $1 \leqslant p < \infty$. *Moreover, there exist positive constants* $C_{\mathscr{X}}$ *and* C *(where* C *depends on* p, $C_{\mathscr{X}}$, C_A *and* C_r*) such that*

$$||\nu(f)||_{L^{1,\infty}(\mathscr{X})} \leqslant C_{\mathscr{X}}(C_A + C_r)\,||f||_{L^1(\mathscr{X})},$$

and

$$||\nu(f)||_{L^p(\mathscr{X})} \leqslant C||f||_{L^p(\mathscr{X})} \text{ for } 1 < p < \infty.$$

THEOREM 2.5. [13, 16] *Let* $p \in (0,\infty)$, $w \in \mathscr{A}_\infty$ *and* $g \in L_b^1(\mathscr{X};\mathscr{B}_2)$. *Assume that* $\mathscr{M}(||g(\cdot)||_{\mathscr{B}_2}) \in L^p(\mathscr{X},w)$. *Then*

(i) $\|\mathscr{M}(||g(\cdot)||_{\mathscr{B}_2})\|_{L^p(\mathscr{X},w)} \leqslant C\|M_A^\sharp(||g(\cdot)||_{\mathscr{B}_2})\|_{L^p(\mathscr{X},w)}$ *if* $\mathscr{X}$ *is unbounded.*

(ii) $\|\mathscr{M}(||g(\cdot)||_{\mathscr{B}_2})\|_{L^p(\mathscr{X},w)} \leqslant C\|M_A^\sharp(||g(\cdot)||_{\mathscr{B}_2})\|_{L^p(\mathscr{X},w)} + C||g||_{L^1(\mathscr{X};\mathscr{B}_2)}$ *if* $\mathscr{X}$ *is bounded.*

3. Weighted estimates for Marcinkiewicz integrals

We need this lemma below in order to obtain weighted estimates of Marcinkiewicz integrals.

LEMMA 3.1. *Let* $w \in \mathscr{A}_p$ $(1 < p < \infty)$ *and* $f \in L_b^\infty(\mathscr{X})$, *where* $L_b^\infty(\mathscr{X})$ *denotes the set of all functions* $f \in L^\infty(\mathscr{X})$ *that have bounded support. Assume that hypotheses **(a)**, **(b)** and **(c)** hold true. Then* $\nu(f) \in L^p(\mathscr{X},w)$.

Proof. We will employ some ideas in [16] to prove this lemma. Recall that $\nu(f)(x) = ||F(f)(x,\cdot)||_{\mathscr{B}_2}$. We may assume that $\mathrm{supp}(f) \subset B(x_o;R)$ for some $x_o \in \mathscr{X}$ and some $R > 0$. By Reverse's Hölder's inequality, there exists $\varepsilon > 0$, depending on p and the $\mathscr{A}_p$ bound for w, such that

$$\left(\frac{1}{\mu(B)}\int_B w(x)^{1+\varepsilon}\,d\mu(x)\right)^{\frac{1}{1+\varepsilon}} \leqslant \frac{C}{\mu(B)}\int_B w(x)\,d\mu(x) \text{ for every ball } B \subset \mathscr{X}.$$

An application of Hölder's inequality (with $q = 1+\varepsilon$) yields

$$\begin{aligned}
&\int_{B(x_o;2R)} ||F(f)(x,\cdot)||^p_{\mathscr{B}_2}\, w(x)\,d\mu(x)\\
&\leqslant \big|\big|\,||F(f)||_{\mathscr{B}_2}\big|\big|^p_{L^{p(1+\varepsilon)'}} \left(\int_{B(x_o;2R)} w(x)^{1+\varepsilon}\,d\mu(x)\right)^{\frac{1}{1+\varepsilon}}\\
&\leqslant C\frac{w(B(x_o;2R))}{\mu(B(x_o;2R))^{\frac{\varepsilon}{1+\varepsilon}}}||\nu(f)||^p_{L^{p(1+\varepsilon)'}}\\
&\leqslant C\frac{w(B(x_o;2R))}{\mu(B(x_o;2R))^{\frac{\varepsilon}{1+\varepsilon}}}||f||^p_{L^{p(1+\varepsilon)'}}, \qquad (3.1)
\end{aligned}$$

where the last inequality follows from Theorem 2.4. Let $t_k = (2^{k-1}R)^m$. We write

$$\begin{aligned}
&\int_{B(x_o;2R)^c} ||F(f)(x,\cdot)||^p_{\mathscr{B}_2}\, w(x)\,d\mu(x)\\
&= \sum_{k=1}^{\infty}\int_{d(x,x_o)\cong 2^kR} ||F(f)(x,\cdot)||^p_{\mathscr{B}_2}\, w(x)\,d\mu(x)\\
&\leqslant C_p\sum_{k=1}^{\infty}\int_{d(x,x_o)\cong 2^kR} ||(F(f)-A_{t_k}F(f))(x,\cdot)||^p_{\mathscr{B}_2}\, w(x)\,d\mu(x)\\
&\quad + C_p\sum_{k=1}^{\infty}\int_{d(x,x_o)\cong 2^kR} ||A_{t_k}F(f))(x,\cdot)||^p_{\mathscr{B}_2}\, w(x)\,d\mu(x)\\
&\equiv J_1 + J_2, \qquad (3.2)
\end{aligned}$$

where $d(x,x_o) \cong 2^kR$ means $\{x \in \mathscr{X} : 2^kR \leqslant d(x,x_o) < 2^{k+1}R\}$. By Minkowski's inequality,

$$\begin{aligned}
||(F(f)-A_{t_k}F(f))(x,\cdot)||_{\mathscr{B}_2} &= \left\{\int_0^\infty \left|\int_{d(x,z)<\tau}\big(K(x,z)-K^{t_k}(x,z)\big)\,f(z)d\mu(z)\right|^2 \frac{d\tau}{\tau^3}\right\}^{1/2}\\
&\leqslant \int_{d(x,z)\geqslant t_k^{1/m}} \frac{|K(x,z)-K^{t_k}(x,z)|}{d(x,z)} f(z)\,d\mu(z)\\
&\leqslant C\int_{d(x,z)\geqslant t_k^{1/m}} \frac{t_k^{\beta/m}}{d(x,z)^\beta}\,\frac{f(z)}{\mu(B(x;d(x,z)))}d\mu(z)\\
&\leqslant C\sum_{j=0}^{\infty}\frac{2^{-\beta j}}{\mu(B(x;2^{j+1}t_k^{1/m}))}\int_{d(x,z)\cong 2^jt_k^{1/m}} |f(z)|\,d\mu(z)\\
&\leqslant C\mathscr{M}f(x),
\end{aligned}$$

where the second and third inequalities follow from hypothesis **(c)** and (1.3) respectively. Therefore,

$$
\begin{aligned}
J_1 &\leqslant C\sum_{k=1}^{\infty}\int_{d(x,x_o)\cong 2^kR}[\mathscr{M}f(x)]^p w(x)\,d\mu(x) \leqslant C\int_{\mathscr{X}}[\mathscr{M}f(x)]^p w(x)\,d\mu(x)\\
&\leqslant C\int_{\mathscr{X}}|f(x)|^p w(x)\,d\mu(x) \leqslant C||f||^p_{L^\infty(\mathscr{X})}w(B(x_o:R))<\infty. \qquad (3.3)
\end{aligned}
$$

Note that $w\in\mathscr{A}_p \Rightarrow w\in\mathscr{A}_{p/s}$ for some $s>1$ with $p/s>1$. By Minkowski's inequality, Hölder's inequality, inequality (1.7), Remark 1.1-i and by Theorem 2.4, we obtain

$$
\begin{aligned}
||A_{t_k}F(f))(x,\cdot)||_{\mathscr{B}_2} &= \left(\int_0^\infty\left|\int_{\mathscr{X}}a_{t_k}(x,y)F(f)(y,t)\,d\mu(y)\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&\leqslant \int_{\mathscr{X}}|a_{t_k}(x,y)|\left(\int_0^\infty|F(f)(y,t)|^2\frac{dt}{t^3}\right)^{1/2}d\mu(y)\\
&\leqslant \int_{\mathscr{X}}h_{t_k}(x,y)\,||F(f)(y,\cdot)||_{\mathscr{B}_2}d\mu(y)\\
&\leqslant ||F(f)||_{L^s(\mathscr{X};\mathscr{B}_2)}\left(\int_{\mathscr{X}}(h_{t_k}(x,y))^{s'-1}h_{t_k}(x,y)\,d\mu(y)\right)^{1/s'}\\
&\leqslant ||\nu(f)||_{L^s(\mathscr{X})}\frac{C}{\mu(B(x;t_k^{1/m}))^{\frac{s'-1}{s'}}}\left(\int_{\mathscr{X}}h_{t_k}(x,y)\,d\mu(y)\right)^{1/s'}\\
&\leqslant C||f||_{L^s(\mathscr{X})}[\mu(B(x;t_k^{1/m}))]^{-1/s}.
\end{aligned}
$$

From Remark 5.6 [16], we see that

$$
J_2 \leqslant C||f||^p_{L^s(\mathscr{X})}\sum_{k=1}^{\infty}\int_{d(x,x_o)\cong 2^kR}\frac{w(x)}{[\mu(B(x;t_k^{1/m}))]^{p/s}}d\mu(x) \leqslant C||f||^p_{L^s(\mathscr{X})}. \qquad (3.4)
$$

It follows from (3.1)–(3.4) that $F(f)\in L^p(\mathscr{X},w;\mathscr{B}_2)$, or equivalently, $\nu(f)\in L^p(\mathscr{X},w)$.
Lemma 3.1 is proved. $\square$

THEOREM 3.2. *Let $w\in\mathscr{A}_p$, $1<p<\infty$. Assume that hypotheses **(a)**, **(b)** and **(c)** hold true. Then there exists a positive constant $C_{p,w}$, depending on p and w, such that*

$$
||\nu(f)||_{L^p(\mathscr{X},w)} \leqslant C_{p,w}||f||_{L^p(\mathscr{X},w)}.
$$

Proof. It suffices to prove the theorem for $f\in L_b^\infty(\mathscr{X})$, since this space is densely contained in $L^p(\mathscr{X},w)$ for $1\leqslant p<\infty$. Recall that $w\in\mathscr{A}_p\Rightarrow w\in\mathscr{A}_{p/s}$ for some $s>1$ with $p/s>1$. We first consider the case that $\mathscr{X}$ is unbounded. By Lemma 3.1, $\nu(f)=||F(f)(\cdot)||_{\mathscr{B}_2}\in L^p(\mathscr{X},w)$, and thus $\mathscr{M}(||F(f)(\cdot)||_{\mathscr{B}_2})\in L^p(\mathscr{X},w)$. We then have

$$\begin{aligned}\int_{\mathscr{X}} |\nu(f)(x)|^p w(x)\,d\mu(x) &= \int_{\mathscr{X}} ||F(f)(x,\cdot)||_{\mathscr{B}_2}^p w(x)\,d\mu(x)\\ &\leqslant \int_{\mathscr{X}} [\mathscr{M}(||F(f)(\cdot)||_{\mathscr{B}_2})(x)]^p w(x)\,d\mu(x)\\ &\leqslant C\int_{\mathscr{X}} [M_A^\sharp(||F(f)(\cdot)||_{\mathscr{B}_2})(x)]^p w(x)\,d\mu(x)\\ &\leqslant C\int_{\mathscr{X}} [M_s f(x)]^p w(x)\,d\mu(x)\\ &\leqslant C\int_{\mathscr{X}} |f(x)|^p w(x)\,d\mu(x),\end{aligned}$$

where the second and third inequalities follow from Theorem 2.5 and Lemma 2.3 respectively.

We now consider the case that $\mathscr{X}$ is bounded. Recall from Theorem 2.4 that ν is a bounded operator on $L^q(\mathscr{X})$ for $q>1$. Since $\mathscr{X}$ is bounded, ν is also bounded on $L^1(\mathscr{X})$. By Theorem 2.5 and by Lemma 2.3, we have

$$\begin{aligned}||\nu(f)||_{L^p(\mathscr{X},w)} &= \big|\big|\,||F(f)||_{\mathscr{B}_2}\big|\big|_{L^p(\mathscr{X},w)} \leqslant \big|\big|\mathscr{M}(||F(f)||_{\mathscr{B}_2})\big|\big|_{L^p(\mathscr{X},w)}\\ &\leqslant C\left|\left|M_A^\sharp(||F(f)||_{\mathscr{B}_2})\right|\right|_{L^p(\mathscr{X},w)} + C\big|\big|\,||F(f)||_{\mathscr{B}_2}\big|\big|_{L^1(\mathscr{X})}\\ &\leqslant C||M_s f||_{L^p(\mathscr{X},w)} + C||\nu(f)||_{L^1(\mathscr{X})}\\ &\leqslant C||f||_{L^p(\mathscr{X},w)} + C||\nu(f)||_{L^1(\mathscr{X})} \leqslant C||f||_{L^p(\mathscr{X},w)}.\end{aligned}$$

The last inequality is a consequence of successive applications of Hölder's inequality:

$$\begin{aligned}||\nu(f)||_{L^1(\mathscr{X})} &\leqslant ||\nu(f)||_{L^s(\mathscr{X})}[\mu(\mathscr{X})]^{1/s'} \leqslant C||f||_{L^s(\mathscr{X})}[\mu(\mathscr{X})]^{1/s'}\\ &\leqslant C||f||_{L^p(\mathscr{X},w)}[\mu(\mathscr{X})]^{1/s'}\left(\int_{\mathscr{X}}\{w(x)\}^{-r'/r}\,d\mu(x)\right)^{1/(sr')} \quad (r=p/s)\\ &\leqslant C||f||_{L^p(\mathscr{X},w)}.\end{aligned}$$

The proof of this theorem is finished. □

THEOREM 3.3. *Let $w\in\mathscr{A}_1$. Assume that hypotheses* **(a)**, **($\tilde{\mathbf{b}}$)** *and* **(c)** *hold true. Then there exists a positive constant C_w, depending on w, such that*

$$||\nu(f)||_{L^{1,\infty}(\mathscr{X},w)} \leqslant C_w||f||_{L^1(\mathscr{X},w)}.$$

Proof. It is enough to prove the theorem for $f\in L_b^\infty(\mathscr{X})$. Observe that hypothesis **($\tilde{\mathbf{b}}$)** implies hypothesis **(b)**. Thus we may conclude from Theorem 3.2 that if $w\in\mathscr{A}_1\subset\mathscr{A}_r$ for some fixed $r>1$, then there exists a positive constant $C_{r,w}$ such that

$$||\nu(f)||_{L^r(\mathscr{X},w)} \leqslant C_{r,w}||f||_{L^r(\mathscr{X},w)}.$$

Now suppose that $\lambda > C_{r,w}||f||_{L^1(\mathscr{X})}(\mu(\mathscr{X}))^{-1}$. We apply Lemma 2.2 to $f \in L_b^\infty(\mathscr{X})$ at the height $\alpha = C_{r,w}^{-1}\lambda$ to obtain a sequence of metric balls $\{B_i\}_i \equiv \{B(x_i, r_i)\}_i$ such that $\Omega_\lambda := \{x \in \mathscr{X} : \mathscr{M} f(x) > C_{\mathscr{X}} C_{r,w}^{-1}\lambda\} = \bigcup_i B_i$, and $f(x) = g(x) + \sum_i b_i(x)$.

Let $\tilde{B}_i = B(x_i; \tilde{r}_i) := B(x_i; (1+c_1)r_i)$, where c_1 appears in hypothesis **($\tilde{\mathbf{b}}$)**. We have

$$
\begin{aligned}
w(\{x \in \mathscr{X} : |\nu(f)(x)| > \lambda\}) &\leqslant w\left(\left\{x \in \mathscr{X} : ||F(g)(x,\cdot)||_{\mathscr{B}_2} > \frac{\lambda}{2}\right\}\right) \\
&\quad + w\left(\left\{x \in \mathscr{X} : ||F(b)(x,\cdot)||_{\mathscr{B}_2} > \frac{\lambda}{2}\right\}\right) \\
&\equiv Z_1 + Z_2.
\end{aligned}
$$

The L^r-boundedness of ν together with properties (b) and (c) of Lemma 2.2 imply that

$$
\begin{aligned}
Z_1 &\leqslant \left(\frac{\lambda}{2}\right)^{-r} \int_{\mathscr{X}} ||F(g)(x,\cdot)||_{\mathscr{B}_2}^r w(x)\, d\mu(x) = \left(\frac{\lambda}{2}\right)^{-r} \int_{\mathscr{X}} |\nu(g)(x)|^r w(x)\, d\mu(x) \\
&\leqslant C_{r,w}^r \left(\frac{\lambda}{2}\right)^{-r} \int_{\mathscr{X}} |(g)(x)|^r w(x)\, d\mu(x) \leqslant C\frac{C_{r,w}}{\lambda} \int_{\mathscr{X}} |g(x)|\, w(x)\, d\mu(x) \\
&\leqslant C\frac{C_{r,w}}{\lambda} \left\{ ||f||_{L^1(\mathscr{X},w)} + \int_{\mathscr{X}} \sum_i \left(\int_{B_i} |f(y)| w(y) d\mu(y) \right) (w(B_i))^{-1} w(x) \chi_{B_i}(x)\, d\mu(x) \right\} \\
&\leqslant C\frac{C_{r,w}}{\lambda} \left\{ ||f||_{L^1(\mathscr{X},w)} + \sum_i \int_{B_i} |f(y)|\, w(y)\, d\mu(y) \right\} \\
&\leqslant C\frac{C_{r,w}}{\lambda} \left\{ ||f||_{L^1(\mathscr{X},w)} + M_1 \int_{\cup_i B_i} |f(y)|\, w(y)\, d\mu(y) \right\} \\
&\leqslant C\frac{C_{r,w}}{\lambda} ||f||_{L^1(\mathscr{X},w)},
\end{aligned}
$$

where M_1 appears in part (g) of Lemma 2.2. Observe that

$$
Z_2 \leqslant Z_3 + Z_4 + Z_5,
$$

where $Z_3 = w\left(\bigcup_i \tilde{B}_i\right)$, $Z_4 = w\left(\left\{x \notin \bigcup_i \tilde{B}_i : \sum_i ||(F - FB_{t_i})(b_i)(x,\cdot)||_{\mathscr{B}_2} > \lambda/4\right\}\right)$, and $Z_5 = w\left(\left\{x \in \mathscr{X} : ||\sum_i FB_{t_i} b_i(x,\cdot)||_{\mathscr{B}_2} > \lambda/4\right\}\right)$.

We choose $t_i = r_i^m$, where m is the constant appearing in (1.7). It follows from the doubling property of w, properties (d) and (g) of Lemma 2.2, and weak type $(1,1)$ of the Hardy-Littlewood maximal operator that

$$
\begin{aligned}
Z_3 = w\left(\cup_i \tilde{B}_i\right) &\leqslant \sum_i w(\tilde{B}_i) \leqslant C \sum_i w(B_i) \leqslant C M_1\, w(\cup_i B_i) = C M_1\, w(\Omega_\lambda) \\
&= C M_1\, w\left(\left\{x \in \mathscr{X} : \mathscr{M} f(x) > C_{\mathscr{X}} C_{r,w}^{-1} \lambda\right\}\right) \leqslant C\frac{C_{r,w}}{\lambda} ||f||_{L^1(\mathscr{X},w)}.
\end{aligned}
$$

We now estimate Z_4. If $y \in B_i$ and $x \notin \bigcup_i \tilde{B}_i$, then $d(x,y) \geqslant c_1 r_i = c_1 t_i^{1/m}$. Note that

$$\begin{aligned}||(F - FB_{t_i})(b_i)(x,\cdot)||_{\mathscr{B}_2} &= \left\{\int_0^\infty \left|\int_{d(x,y)<\tau} (K(x,y) - K_{t_i}(x,y))\, b_i(y)\, d\mu(y)\right|^2 \frac{d\tau}{\tau^3}\right\}^{1/2} \\ &\leqslant \int_{B_i} |K(x,y) - K_{t_i}(x,y)|(d(x,y))^{-1}|b_i(y)|\, d\mu(y).\end{aligned}$$

Thus we have

$$\begin{aligned}Z_4 &\leqslant \frac{4}{\lambda}\int_{(\bigcup_i \tilde{B}_i)^c} \sum_i ||(F - FB_{t_i})(b_i)(x,\cdot)||_{\mathscr{B}_2}\, w(x)\, d\mu(x) \\ &\leqslant \frac{4}{\lambda}\int_{(\bigcup_i \tilde{B}_i)^c} \left(\sum_i \int_{B_i} |K(x,y) - K_{t_i}(x,y)|(d(x,y))^{-1}|b_i(y)|\, d\mu(y)\right) w(x)\, d\mu(x) \\ &= \frac{4}{\lambda}\sum_i \int_{B_i} \left\{\int_{d(x,y)\geqslant c_1 t_i^{1/m}} |K(x,y) - K_{t_i}(x,y)|(d(x,y))^{-1} w(x)\, d\mu(x)\right\} |b_i(y)|\, d\mu(y).\end{aligned}$$

By assumption $(\tilde{\mathbf{b}})$ and inequalities (1.3)–(1.4), we obtain

$$\begin{aligned}Z_4 &\leqslant \frac{C}{\lambda}\sum_i \int_{B_i}\left\{\sum_{j=0}^\infty 2^{-\beta j}\int_{d(x,y)\cong 2^j c_1 t_i^{1/m}} \frac{w(x)}{\mu(B(x;d(x,y)))}\, d\mu(x)\right\} |b_i(y)|\, d\mu(y) \\ &\leqslant \frac{C}{\lambda}\sum_i \int_{B_i}\left\{\sum_{j=0}^\infty \frac{2^{-\beta j}}{\mu(B(y;2^{j+1}c_1 t_i^{1/m}))}\int_{B(y;2^{j+1}c_1 t_i^{1/m})} w(x)\, d\mu(x)\right\} |b_i(y)|\, d\mu(y) \\ &\leqslant \frac{C}{\lambda}\sum_i \int_{B_i} \mathscr{M}w(y)\, |b_i(y)|\, d\mu(y) \leqslant \frac{C}{\lambda}\sum_i \int_{B_i} |b_i(y)|\, w(y)\, d\mu(y) \\ &\leqslant \frac{C}{\lambda}\left\{\sum_i \int_{B_i} |f(y)|\, w(y)\, d\mu(y) + \sum_i \int_{B_i}\left(\frac{1}{\mu(B_i)}\int_{B_i} w(y)\, d\mu(y)\right)|f(z)|\, d\mu(z)\right\} \\ &\leqslant \frac{C}{\lambda}\left\{M_1\int_{\cup B_i} |f(y)|\, w(y)\, d\mu(y) + \sum_i \int_{B_i} |f(z)|\, w(z)\, d\mu(z)\right\} \leqslant \frac{C}{\lambda}||f||_{L^1(\mathscr{X},w)}.\end{aligned}$$

It remains to estimate Z_5. It follows from inequality (1.6), Lemma 2.2-f and Proposition 2.5 [5] (or see Lemma 1 [3]) that

$$\begin{aligned}|B_{t_i} b_i(x)| &\leqslant \int_{\mathscr{X}} h_{t_i}(x,y)|b_i(y)|\, d\mu(y) \leqslant \sup_{y\in B_i} h_{t_i}(x,y)\int_{B_i} |b_i(y)|\, d\mu(y) \\ &\leqslant CC_{r,w}^{-1}\lambda\, \mu(B_i)\sup_{y\in B_i} h_{t_i}(x,y) \leqslant CC_{r,w}^{-1}\lambda\, \mu(B_i)\inf_{y\in B_i} h_{\theta t_i}(x,y) \\ &\leqslant CC_{r,w}^{-1}\lambda \int_{\mathscr{X}} h_{\theta t_i}(x,y)\chi_{B_i}(y)\, d\mu(y), \qquad (3.5)\end{aligned}$$

where $\theta \geqslant 1$ is a positive constant (see Lemma 1 [3]). Recall that $w \in \mathscr{A}_1 \subset \mathscr{A}_r \Rightarrow w^{1-r'} \in \mathscr{A}_{r'}$. Now take a function $u \geqslant 0$, $u \in L^{r'}(\mathscr{X},w)$ with $||u||_{L^{r'}(\mathscr{X},w)} \leqslant 1$. Then the function $v = uw \in L^{r'}(\mathscr{X}, w^{1-r'})$ and $||v||_{L^{r'}(\mathscr{X}, w^{1-r'})} = ||u||_{L^{r'}(\mathscr{X}, w)} \leqslant 1$. By inequality (3.5) and Remark 1.1-ii, we obtain

$$
\begin{aligned}
&\left| \int_{\mathscr{X}} \sum_i |B_{t_i} b_i(x)|\, u(x)\, w(x)\, d\mu(x) \right| \qquad (3.6)\\
&\leqslant C C_{r,w}^{-1} \lambda \sum_i \int_{\mathscr{X}} \left(\int_{\mathscr{X}} h_{\theta t_i}(x,y) u(x)\, w(x)\, d\mu(x) \right) \chi_{B_i}(y)\, d\mu(y)\\
&\leqslant C C_{r,w}^{-1} \lambda \sum_i \int_{\mathscr{X}} \mathscr{M}(uw)(y)\, \chi_{B_i}(y)\, d\mu(y)\\
&\leqslant C M_1 C_{r,w}^{-1} \lambda \int_{\cup_i B_i} \mathscr{M}(uw)(y)\, d\mu(y)\\
&\leqslant C M_1 C_{r,w}^{-1} \lambda \left\{ \int_{\mathscr{X}} |\mathscr{M}(uw)(y)|^{r'} w(y)^{1-r'}\, d\mu(y) \right\}^{1/r'} \{w(\cup_i B_i)\}^{1/r}\\
&\leqslant C C_{r,w}^{-1} \lambda\, ||u||_{L^{r'}(\mathscr{X}, w)} w(\Omega_\lambda)^{1/r}\\
&\leqslant C C_{r,w}^{-1} \lambda\, \left\{ w\left(\{x \in \mathscr{X} : \mathscr{M} f(x) > C_{\mathscr{X}} C_{r,w}^{-1} \lambda\}\right) \right\}^{1/r}\\
&\leqslant C \left\{ \frac{\lambda}{C_{r,w}} \right\}^{1/r'} ||f||_{L^1(\mathscr{X}, w)}^{1/r}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\| \sum_i |B_{t_i} b_i| \right\|_{L^r(\mathscr{X}, w)} &= \sup_{||u||_{L^{r'}(\mathscr{X}, w)} \leqslant 1} \left\{ \left| \int_{\mathscr{X}} \sum_i |B_{t_i} b_i(x)|\, u(x)\, w(x)\, d\mu(x) \right| \right\}\\
&\leqslant C \left\{ \frac{\lambda}{C_{r,w}} \right\}^{1/r'} ||f||_{L^1(\mathscr{X}, w)}^{1/r}.
\end{aligned}
$$

Since $w \in \mathscr{A}_1 \subset \mathscr{A}_r$, it follows from the L^r- boundedness of ν and from the estimate above that

$$
\begin{aligned}
Z_5 &\leqslant \left(\frac{4}{\lambda} \right)^r \int_{\mathscr{X}} \|F(\sum_i B_{t_i} b_i)(x, \cdot)\|_{\mathscr{B}_2}^r\, w(x)\, d\mu(x)\\
&\leqslant C \left(\frac{C_{r,w}}{\lambda} \right)^r \int_{\mathscr{X}} \left| \sum_i B_{t_i} b_i(x) \right|^r w(x)\, d\mu(x)\\
&\leqslant C \left(\frac{C_{r,w}}{\lambda} \right)^r \int_{\mathscr{X}} \left(\sum_i |B_{t_i} b_i(x)| \right)^r w(x)\, d\mu(x)\\
&\leqslant C \frac{C_{r,w}}{\lambda} ||f||_{L^1(\mathscr{X}, w)}.
\end{aligned}
$$

Summing all of the estimates of $Z_1, \ldots, Z_5$, we obtain for $\lambda > C_{r,w} ||f||_{L^1(\mathscr{X})} (\mu(\mathscr{X}))^{-1}$,

$$
w(\{x \in \mathscr{X} : \nu(f)(x) > \lambda\}) \leqslant \frac{C_{\mathscr{X}}}{\lambda} (C_A + C_{r,w}) \|f\|_{L^1(\mathscr{X}, w)}.
$$

If $\mathscr{X}$ is unbounded, then the above inequality holds for all $\lambda > 0$; and thus we are done. Now suppose that $\mathscr{X}$ is bounded. Then for $0 < \lambda \leqslant C_{r,w}||f||_{L^1(\mathscr{X})}(\mu(\mathscr{X}))^{-1}$, we have

$$w(\{x \in \mathscr{X} : \nu(f)(x) > \lambda\}) \leqslant w(\mathscr{X}) \leqslant \frac{C_{r,w}}{\lambda}\frac{w(\mathscr{X})}{\mu(\mathscr{X})}||f||_{L^1(\mathscr{X})} \leqslant \frac{C_{r,w}}{\lambda}||f||_{L^1(\mathscr{X},w)}.$$

The proof of this theorem is complete. □

4. Commutators of BMO functions and Marcinkiewicz integrals

Given a fixed positive integer k, let $\vec{b} = (b_1, b_2, \ldots, b_k)$, where each $b_i \in \mathrm{BMO}(\mathscr{X})$ $(1 \leqslant i \leqslant k)$. For each b_i $(1 \leqslant i \leqslant k)$, denote by $||b_i||_*$ the BMO norm of b_i; and let $||\vec{b}||_* = \prod_{i=1}^k ||b_i||_*$.

Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_k) = ((b_1)_B, \ldots, (b_k)_B)$, where $(b_i)_B = \dfrac{1}{\mu(B)}\displaystyle\int_B b_i(x)\,d\mu(x)$, $1 \leqslant i \leqslant k$. Let C_j^k $(1 \leqslant j \leqslant k)$ stand for the family of all finite subsets $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$ of j different elements of $\{1, \ldots, k\}$. For any $\sigma \in C_j^k$, we denote the complement sequence of σ by $\sigma' = \{1, \ldots, k\} \setminus \sigma$. Let $\vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)})$ and let the product $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$. Similarly, denote $(\vec{b} - \vec{\lambda})_\sigma = (b_{\sigma(1)} - \lambda_{\sigma(1)}, \ldots, b_{\sigma(j)} - \lambda_{\sigma(j)})$ and $(b - \lambda)_\sigma = (b_{\sigma(1)} - \lambda_{\sigma(1)}) \cdots (b_{\sigma(j)} - \lambda_{\sigma(j)})$.

For $f \in L_b^\infty(\mathscr{X})$, define the linear operator $F_{\vec{b}}(f)$ by

$$F_{\vec{b}}(f)(x, \tau) = \int_{d(x,y)<\tau} \left\{\prod_{i=1}^k (b_i(x) - b_i(y))\right\} K(x, y) f(y)\, d\mu(y),$$

where $K(x, y)$ is a measurable function defined on $(\mathscr{X} \times \mathscr{X}) \setminus \Delta$ with $\Delta = \{(x, x) : x \in \mathscr{X}\}$.

Define the commutator of Marcinkiewicz integral and BMO functions, $\nu_{\vec{b}}(f)$, by

$$\nu_{\vec{b}}(f)(x) = \left\{\int_0^\infty |F_{\vec{b}}(f)(x, \tau)|^2 \frac{d\tau}{\tau^3}\right\}^{1/2}.$$

We have the following theorem.

THEOREM 4.1. *Let $\mathscr{X}$ have infinite measure. Let $\nu_{\vec{b}}(f)$ be the commutator of Marcinkiewicz integral and BMO functions with non-smooth kernel satisfying hypotheses* **(a)**, **(b)** *and* **(c)**. *Let $w \in \mathscr{A}_p$, $1 < p < \infty$. Then there exists a positive constant C (depending on p and w) such that, for all $f \in L^p(\mathscr{X})$,*

$$||\nu_{\vec{b}}(f)||_{L^p(\mathscr{X},w)} \leqslant C||\vec{b}||_* ||f||_{L^p(\mathscr{X},w)}.$$

Proof. In order to prove this theorem, we need the following lemma.

LEMMA 4.2. [12] *Let $t_B = r_B^m$, where r_B is the radius of the ball B, and m appears in* (1.7)–(1.8). *Given any $r>1$, there exists a positive constant C (depending on r) such that*

$$\begin{aligned}
M_A^{\sharp}(||F_{\vec{b}}(f)||_{\mathscr{B}_2})(x) &= \sup_{B\ni x}\left\{\frac{1}{\mu(B)}\int_B ||F_{\vec{b}}(f)(y,\cdot)-A_{t_B}F_{\vec{b}}(f)(y,\cdot)||_{\mathscr{B}_2}\,d\mu(y)\right\}\\
&\leqslant C||\vec{b}||_*\left\{M_rf(x)+M_r(||F(f)||_{\mathscr{B}_2})(x)\right\}\\
&\quad +C\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\left(\prod_{l=1}^{i}||b_{\sigma(l)}||_*\right)M_r(||F_{\vec{b}_{\sigma'}}(f)||_{\mathscr{B}_2})(x).
\end{aligned}$$

Given $w\in\mathscr{A}_p$, $1<p<\infty$, there exists $\varepsilon>0$ such that $w\in\mathscr{A}_{p-\varepsilon}$, where $p-\varepsilon>1$. We choose $r>1$ such that $1<r<\dfrac{p}{p-\varepsilon}<p$. Note that $w\in\mathscr{A}_{p-\varepsilon}\Rightarrow w\in\mathscr{A}_{p/r}$ since $p-\varepsilon<\dfrac{p}{r}$.

By Theorem 2.5, Lemma 4.2, Theorem 3.2 and by induction argument, we may conclude that

$$\begin{aligned}
||\nu_{\vec{b}}(f)||_{L^p(\mathscr{X},w)} &\leqslant ||\mathscr{M}(\nu_{\vec{b}}f)||_{L^p(\mathscr{X},w)} = ||\mathscr{M}(||F_{\vec{b}}(f)||_{\mathscr{B}_2})||_{L^p(\mathscr{X},w)}\\
&\leqslant C||M_A^{\sharp}(||F_{\vec{b}}(f)||_{\mathscr{B}_2})||_{L^p(\mathscr{X},w)}\\
&\leqslant C||\vec{b}||_*\left\{||M_rf||_{L^p(\mathscr{X},w)}+||M_r(||F(f)||_{\mathscr{B}_2})||_{L^p(\mathscr{X},w)}\right\}\\
&\quad +C\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\left(\prod_{l=1}^{i}||b_{\sigma(l)}||_*\right)||M_r(||F_{\vec{b}_{\sigma'}}(f)||_{\mathscr{B}_2})||_{L^p(\mathscr{X},w)}\\
&\leqslant C||\vec{b}||_*||f||_{L^p(\mathscr{X},w)}+C\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\left(\prod_{l=1}^{i}||b_{\sigma(l)}||_*\right)||\nu_{\vec{b}_{\sigma'}}(f)||_{L^p(\mathscr{X},w)}\\
&\leqslant C||\vec{b}||_*\,||f||_{L^p(\mathscr{X},w)}. \qquad \square
\end{aligned}$$

5. Marcinkiewicz integrals on homogeneous Herz spaces

5.1. Homogeneous Herz spaces

For $k\in\mathbb{Z}$, let $A_k=\left\{x\in\mathbb{R}^n : 2^{k-1}<|x|\leqslant 2^k\right\}$. Denote the characteristic function χ_{A_k} on the set A_k by χ_k. For $\alpha\in\mathbb{R}$, $0<p,q\leqslant\infty$, the homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ are defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n)=\left\{f\in L_{\mathrm{loc}}^q(\mathbb{R}^n\setminus\{0\}) :\ ||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} := \left(\sum_{k\in\mathbb{Z}}2^{k\alpha p}||f\chi_k||_{L^q(\mathbb{R}^n)}^p\right)^{1/p}<\infty\right\},$$

with usual modifications when $p=\infty$ or $q=\infty$. Observe that $\dot{K}_q^{\alpha/q,q}(\mathbb{R}^n)=L^q(\mathbb{R}^n,|x|^{\alpha})$, and $\dot{K}_q^{0,q}(\mathbb{R}^n)=L^q(\mathbb{R}^n)$.

Denote $m_k(\lambda, f) := |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|$, where $\lambda > 0$. For $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$ and $0 < q < \infty$, the homogeneous weak Herz spaces $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ consist of all measurable functions f on $\mathbb{R}^n$ whose norms are finite:

$$||f||_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} := \sup_{\lambda>0}\left\{\lambda\left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\,(m_k(\lambda, f))^{p/q}\right)^{1/p}\right\} < \infty \ (0 < p < \infty),$$

and

$$||f||_{W\dot{K}_q^{\alpha,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0}\left\{\lambda \sup_{k\in\mathbb{Z}}\left\{2^{k\alpha}\,(m_k(\lambda, f))^{1/q}\right\}\right\} < \infty.$$

Note that $W\dot{K}_q^{\alpha/q,q}(\mathbb{R}^n) = L^{q,\infty}(\mathbb{R}^n, |x|^\alpha)$ and $W\dot{K}_q^{0,q}(\mathbb{R}^n) = L^{q,\infty}(\mathbb{R}^n)$, where $L^{q,\infty}(\mathbb{R}^n, w)$ is the space of all measurable functions f such that

$$\sup_{\lambda>0}\left\{\lambda\,(w(x \in \mathbb{R}^n : |f(x)| > \lambda))^{1/q}\right\} < \infty.$$

For topics related to Herz spaces, the reader may view [8, 9, 10] among many other good references.

In order to obtain the boundedness of Marcinkiewicz integrals and their commutators on homogeneous Herz spaces, we need the following theorems.

THEOREM 5.1. [9] *If a sublinear operator T is bounded on $L^q(\mathbb{R}^n, |x|^\beta)$ for some $q \in (1, \infty)$ and for all $\beta \in (\beta_1, \beta_2)$, where $\beta_1, \beta_2 \in \mathbb{R}$, then T is also bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ for all $\alpha \in (\beta_1/q, \beta_2/q)$ and all $p \in (0, \infty]$.*

THEOREM 5.2. *Suppose that a sublinear operator T is bounded from $L^1(\mathbb{R}^n, |x|^\beta)$ to $L^{1,\infty}(\mathbb{R}^n, |x|^\beta)$ for all $\beta \in (\beta_1, \beta_2)$, where $\beta_1, \beta_2 \in \mathbb{R}$. Then T is bounded from $\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ to $W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)$ for all $\alpha \in (\beta_1, \beta_2)$ and all $p \in (0, \infty]$. That is, there exists a positive constant C, depending on p and α, such that*

$$||TF||_{W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)} \leqslant C||f||_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)} \text{ for all } \alpha \in (\beta_1, \beta_2) \text{ and all } p \in (0, \infty].$$

Proof. We apply some ideas in [9] to prove this theorem. Note that

$$||TF||_{W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0}\left\{\lambda\left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} m_k(\lambda, Tf)^p\right)^{1/p}\right\},\ 0 < p < \infty, \tag{5.1}$$

and

$$||TF||_{W\dot{K}_1^{\alpha,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0}\left\{\sup_{k\in\mathbb{Z}}\left\{2^{k\alpha} m_k(\lambda, Tf)\right\}\right\}, \tag{5.2}$$

where $m_k(\lambda, Tf) = |\{x \in A_k : |Tf(x)| > \lambda\}|$ and $A_k = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \leqslant 2^k\}$, $k \in \mathbb{Z}$.

For $j \in \mathbb{Z}$, denote $f_j = f\chi_j \equiv f\chi_{A_j}$. Pick $\alpha_1, \alpha_2 \in (\beta_1, \beta_2)$ such that $\beta_1 < \alpha_1 < \alpha < \alpha_2 < \beta_2$. We first consider the case $0 < p < \infty$. By hypothesis, we have

$$\begin{aligned}
J_\lambda &:= \lambda \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} m_k(\lambda, Tf)^p \right)^{1/p} \\
&\leqslant C_p \lambda \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} \left| \left\{ x \in A_k : \left| T\left(\sum_{j=-\infty}^{k} f_j \right)(x) \right| > \frac{\lambda}{2} \right\} \right|^p \right)^{1/p} \\
&\quad + C_p \lambda \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} \left| \left\{ x \in A_k : \left| T\left(\sum_{j=k+1}^{\infty} f_j \right)(x) \right| > \frac{\lambda}{2} \right\} \right|^p \right)^{1/p} \\
&\leqslant C \left\{ \sum_{k\in\mathbb{Z}} 2^{k(\alpha-\alpha_2)p} \left(\sum_{j=-\infty}^{k} ||f_j||_{L^1(\mathbb{R}^n, |x|^{\alpha_2})} \right)^p \right\}^{1/p} \\
&\quad + C \left\{ \sum_{k\in\mathbb{Z}} 2^{k(\alpha-\alpha_1)p} \left(\sum_{j=k+1}^{\infty} ||f_j||_{L^1(\mathbb{R}^n, |x|^{\alpha_1})} \right)^p \right\}^{1/p} \\
&\equiv J_3 + J_4.
\end{aligned} \tag{5.3}$$

If $0 < p \leqslant 1$, then

$$\begin{aligned}
J_3 &\leqslant C \left\{ \sum_{k\in\mathbb{Z}} \sum_{j=-\infty}^{k} 2^{(k-j)(\alpha-\alpha_2)p} 2^{j\alpha p} ||f_j||^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} \\
&= C \left\{ \sum_{j\in\mathbb{Z}} \left(\sum_{k=j}^{\infty} 2^{(k-j)(\alpha-\alpha_2)p} \right) 2^{j\alpha p} ||f_j||^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} \\
&= C \left\{ \sum_{j\in\mathbb{Z}} \left(\sum_{k=0}^{\infty} 2^{k(\alpha-\alpha_2)p} \right) 2^{j\alpha p} ||f_j||^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} \\
&\leqslant C \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha p} ||f_j||^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} = C\, ||f||_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_4 &\leqslant C \left\{ \sum_{k\in\mathbb{Z}} \sum_{j=k+1}^{\infty} 2^{(k-j)(\alpha-\alpha_1)p} 2^{j\alpha p} ||f_j||^p_{L^1(\mathbb{R}^n)} \right\}^{1/p} \\
&= C \left\{ \sum_{j\in\mathbb{Z}} \left(\sum_{k=-\infty}^{j-1} 2^{(k-j)(\alpha-\alpha_1)p} \right) 2^{j\alpha p} ||f_j||^p_{L^1(\mathbb{R}^n)} \right\}^{1/p}
\end{aligned}$$

$$= C\left\{\sum_{j\in\mathbb{Z}}\left(\sum_{k=-\infty}^{-1}2^{k(\alpha-\alpha_1)p}\right)2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right\}^{1/p}$$

$$\leqslant C\left\{\sum_{j\in\mathbb{Z}}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right\}^{1/p} = C\,||f||_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}.$$

Hence,

$$J_\lambda \leqslant J_3+J_4 \leqslant C\,||f||_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}.$$

If $1<p<\infty$, then we apply Hölder's inequality to obtain

$$\begin{aligned}
J_\lambda \leqslant\ & C\left\{\sum_{k\in\mathbb{Z}}\left(\sum_{j=-\infty}^{k}2^{(k-j)(\alpha-\alpha_2)p/2}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right)\left(\sum_{j=-\infty}^{k}2^{(k-j)(\alpha-\alpha_2)p'/2}\right)^{p/p'}\right\}^{1/p}\\
&+C\left\{\sum_{k\in\mathbb{Z}}\left(\sum_{j=k+1}^{\infty}2^{(k-j)(\alpha-\alpha_1)p/2}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right)\left(\sum_{j=k+1}^{\infty}2^{(k-j)(\alpha-\alpha_1)p'/2}\right)^{p/p'}\right\}^{1/p}\\
=\ & C\left\{\sum_{k\in\mathbb{Z}}\left(\sum_{j=-\infty}^{k}2^{(k-j)(\alpha-\alpha_2)p/2}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right)\left(\sum_{j=0}^{\infty}2^{j(\alpha-\alpha_2)p'/2}\right)^{p/p'}\right\}^{1/p}\\
&+C\left\{\sum_{k\in\mathbb{Z}}\left(\sum_{j=k+1}^{\infty}2^{(k-j)(\alpha-\alpha_1)p/2}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right)\left(\sum_{j=-\infty}^{-1}2^{j(\alpha-\alpha_1)p'/2}\right)^{p/p'}\right\}^{1/p}\\
\leqslant\ & C\left\{\sum_{k\in\mathbb{Z}}\sum_{j=-\infty}^{k}2^{(k-j)(\alpha-\alpha_2)p/2}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right\}^{1/p}\\
&+C\left\{\sum_{k\in\mathbb{Z}}\sum_{j=k+1}^{\infty}2^{(k-j)(\alpha-\alpha_1)p/2}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right\}^{1/p}\\
\leqslant\ & C\left\{\sum_{j\in\mathbb{Z}}2^{j\alpha p}||f_j||^p_{L^1(\mathbb{R}^n)}\right\}^{1/p} = C||f||_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)},
\end{aligned}$$

where the last inequality follows from similar calculations as in the case $0<p<1$. Therefore, for $0<p<\infty$,

$$||TF||_{W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0}\{J_\lambda\} \leqslant C\,||f||_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}.$$

When $p=\infty$, we have

$$\begin{aligned}
J_\lambda &\leqslant J_3+J_4\\
&\leqslant C\sum_{j=-\infty}^{k}2^{(k-j)(\alpha-\alpha_2)}2^{j\alpha}||f_j||_{L^1(\mathbb{R}^n)} + C\sum_{j=k+1}^{\infty}2^{(k-j)(\alpha-\alpha_1)}2^{j\alpha}||f_j||_{L^1(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C \sup_{j\in\mathbb{Z}}\left\{2^{j\alpha}\|f_j\|_{L^1(\mathbb{R}^n)}\right\}\left(\sum_{j=-\infty}^{k} 2^{(k-j)(\alpha-\alpha_2)}\right)\\
&\quad + C \sup_{j\in\mathbb{Z}}\left\{2^{j\alpha}\|f_j\|_{L^1(\mathbb{R}^n)}\right\}\left(\sum_{j=k+1}^{\infty} 2^{(k-j)(\alpha-\alpha_1)}\right)\\
&= C \sup_{j\in\mathbb{Z}}\left\{2^{j\alpha}\|f_j\|_{L^1(\mathbb{R}^n)}\right\}\left(\sum_{j=0}^{\infty} 2^{j(\alpha-\alpha_2)}\right)\\
&\quad + C \sup_{j\in\mathbb{Z}}\left\{2^{j\alpha}\|f_j\|_{L^1(\mathbb{R}^n)}\right\}\left(\sum_{j=-\infty}^{-1} 2^{j(\alpha-\alpha_1)}\right)\\
&\leqslant C \sup_{j\in\mathbb{Z}}\left\{2^{j\alpha}\|f_j\|_{L^1(\mathbb{R}^n)}\right\} = C\|f\|_{\dot K_1^{\alpha,\infty}(\mathbb{R}^n)}.
\end{aligned}
$$

Hence,

$$
\|TF\|_{W\dot K_1^{\alpha,\infty}(\mathbb{R}^n)} \leqslant \sup_{\lambda>0}\left\{\sup_{k\in\mathbb{Z}}\{J_3+J_4\}\right\} \leqslant C\|f\|_{\dot K_1^{\alpha,\infty}(\mathbb{R}^n)}.
$$

The proof of this theorem is finished. □

THEOREM 5.3. *Assume that hypotheses **(a)**, **(b)** and **(c)** hold true. Then there exist positive constants C_1 and C_2 (depending on p, q and α) such that*

$$
\|\nu(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leqslant C_1\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}, \tag{5.4}
$$

and

$$
\|\nu_{\vec b}(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leqslant C_2\|\vec b\|_*\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}, \tag{5.5}
$$

for $1<q<\infty$, $0<p\leqslant\infty$, and $-\frac{n}{q}<\alpha<\frac{n}{q'}$, where $\frac{1}{q}+\frac{1}{q'}=1$.

*If hypotheses **(a)**, **($\tilde{b}$)** and **(c)** hold true, then there exists a positive constant C_3 (depending on p and α) such that*

$$
\|\nu(f)\|_{W\dot K_1^{\alpha,p}(\mathbb{R}^n)} \leqslant C_3\|f\|_{\dot K_1^{\alpha,p}(\mathbb{R}^n)} \tag{5.6}
$$

for all $\alpha\in(-n,0)$ and all $p\in(0,\infty]$.

Proof. Recall that $w(x)=|x|^a\in\mathscr{A}_p$ if and only if $-n<a<n(p-1)$ for $1<p<\infty$; and $w(x)=|x|^a\in\mathscr{A}_1$ if and only if $-n<a\leqslant 0$ (see [6]). Inequality (5.4) follows from Theorem 3.2 and Theorem 5.1. By Theorem 4.1 and Theorem 5.1, we obtain inequality (5.5). Finally, inequality (5.6) is a consequence of Theorem 3.3 and Theorem 5.2. □

5.2. Example

We will need the following propositions for the example below.

PROPOSITION 5.4. *Suppose that there exist constants* $\delta_1 > 1,\ C_5 > 0$ *and* $\beta > 0$ *such that*

$$\frac{|K(x,y)-K(x,z)|}{d(x,y)} \leqslant \frac{C_5}{\mu(B(x;d(x,y)))}\left\{\frac{d(y,z)}{d(x,y)}\right\}^{\beta} \text{ whenever } d(x,y) \geqslant \delta_1 d(y,z). \tag{5.7}$$

Then there exist positive constants C_6 *and* δ_2 *such that*

$$\frac{|K(x,y)-K_t(x,y)|}{d(x,y)} \leqslant \frac{C_6}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^{\beta}} \text{ whenever } d(x,y) \geqslant \delta_2 t^{1/m}.$$

PROPOSITION 5.5. *Suppose that there exist constants* $\delta_3 > 1,\ C_7 > 0$ *and* $\beta > 0$ *such that*

$$\frac{|K(x,y)-K(z,y)|}{d(x,y)} \leqslant \frac{C_7}{\mu(B(x;d(x,y)))}\left\{\frac{d(x,z)}{d(x,y)}\right\}^{\beta} \text{ whenever } d(x,y) \geqslant \delta_3 d(x,z). \tag{5.8}$$

Then there exist positive constants C_8 *and* δ_4 *such that*

$$\frac{|K(x,y)-K^t(x,y)|}{d(x,y)} \leqslant \frac{C_8}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^{\beta}} \text{ whenever } d(x,y) \geqslant \delta_4 t^{1/m}.$$

We omit the proofs of these propositions since they are essentially similar to the proof of Proposition 2 in [3].

Now let $\mathscr{X} = \mathbb{R}^n$, $n \geqslant 2$. Let Ω be homogeneous of degree zero, $\Omega(x) = \Omega(\frac{x}{|x|})$ for every nonzero $x \in \mathbb{R}^n$, and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

Assume further that Ω satisfies the β-Hölder's continuity:

There exist constants $C > 0$ and $\beta \in (0,1]$ such that for all nonzero $x, y \in \mathbb{R}^n$,

$$|\Omega(x)-\Omega(y)| \leqslant C\left|\frac{x}{|x|}-\frac{y}{|y|}\right|^{\beta}. \tag{5.9}$$

Let K be the convolution kernel defined by $K(x) = \dfrac{\Omega(x)}{|x|^{n-1}}$, $n \geqslant 2$. Let

$$F_{\Omega}(f)(x,t) = \int_{|x-y|<t} K(x-y)\, f(y)\, dy,$$

and

$$F_{\Omega,\vec{b}}(f)(x,t) = \int_{|x-y|<t}\left(\prod_{i=1}^{k}(b_i(x)-b_i(y))\right) K(x-y)\, f(y)\, dy,$$

where each $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $1 \leqslant i \leqslant k$. Recall that the Marcinkiewicz integral and its commutator are given by

$$\nu_\Omega(f)(x) = \left(\int_0^\infty |F_\Omega(f)(x,t)|^2 \frac{dt}{t^3} \right)^{1/2} \text{ and } \nu_{\Omega,\vec{b}}(f)(x) = \left(\int_0^\infty |F_{\Omega,\vec{b}}(f)(x,t)|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Since K is a convolution kernel satisfying the cancellation condition, it can be shown via Fourier transform that $\nu_\Omega(f)$ is bounded on $L^2(\mathbb{R}^n)$. Thus hypothesis **(a)** is satisfied. Moreover, it is straightforward to obtain inequalities (5.7)–(5.8) from inequality (5.9). By propositions 5.4–5.5, hypotheses **($\tilde{\mathbf{b}}$)** and **(c)** are fulfilled. Consequently, we may infer from Theorem 5.3 that there exist positive constants C_1, C_2 and C_3 such that

$$||\nu_\Omega(f)||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leqslant C_1 \, ||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},$$

$$||\nu_{\Omega,\vec{b}}(f)||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leqslant C_2 \, ||\vec{b}||_* \, ||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},$$

for $1 < q < \infty$, $0 < p \leqslant \infty$, and $-\dfrac{n}{q} < \alpha < \dfrac{n}{q'}$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, and

$$||\nu_\Omega(f)||_{W\dot{K}_1^{\alpha,p}(\mathbb{R}^n)} \leqslant C_3 \, ||f||_{\dot{K}_1^{\alpha,p}(\mathbb{R}^n)}$$

for all $\alpha \in (-n, 0)$ and all $p \in (0, \infty]$.

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