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ON THE COMPLETE CONVERGENCE OF WEIGHTED SUMS FOR WIDELY ORTHANT DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, some results of complete convergence of weighted sums for widely orthant dependent (WOD, in short) random variables are established. The results obtained in the paper generalize and improve some corresponding ones for extended negatively dependent (END, in short) random variables and WOD random variables.

1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants and $S_n = \sum_{i=1}^{n} a_{ni}X_i,\ S_0 = 0$. The concept of negatively associated (NA) random variables was introduced by Alam and Saxena [1] and was carefully studied by Joag-Dev and Proschan [2]. A finite family of random variables $\{X_i, 1 \leqslant i \leqslant n\}$ is said to be NA if for every pair of disjoint $A,\ B \subset \{1, 2, \ldots, n\}$, $cov(f(X_i, i \in A), f(X_j, j \in B)) \leqslant 0$, whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. The properties for weighted sums $S_n = \sum_{i=1}^{n} a_{ni}X_i$ have been studied by many authors, we can refer to Wang et al. [3], Wu and Jiang [4], Gan [5], and so forth.

The concept of widely orthant dependent (WOD, in short) random variables was introduced by Wang et al. [6]. In various cases of dominating coefficients, the WOD structure contains many other dependence structures. Wang et al. [6] give some examples showing that WOD random variables contain negatively dependent random variables, positively dependent random variables, and some other classes of dependent random variables. In this paper, complete convergence for WOD random variables was obtained.

Firstly, let us recall the definitions of WOD random variables, complete convergence and stochastic dominance.

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DEFINITION 1.1. [6] For the random variables $\{X_n, n \geqslant 1\}$, if there exists a finite positive sequence $\{g_U(n), n \geqslant 1\}$ satisfying for each $n \geqslant 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leqslant i \leqslant n$,

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i), \tag{1.1}$$

then we say that the random variables $\{X_n, n \geqslant 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite positive sequence $\{g_L(n), n \geqslant 1\}$ satisfying for each $n \geqslant 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leqslant i \leqslant n$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \cdots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i), \tag{1.2}$$

then we say that the $\{X_n, n \geqslant 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{X_n, n \geqslant 1\}$ are WOD random variables, and $g_U(n)$, $g_L(n)$, $n \geqslant 1$, are called dominating coefficients.

It is easily seen that $g_U(n) \geqslant 1$, $g_L(n) \geqslant 1$. If both (1.1) and (1.2) hold for $g_L(n) = g_U(n) = M \geqslant 1$ for any $n \geqslant 1$, then $\{X_n, n \geqslant 1\}$ are extended negatively dependent (END, in short) random variables. If both (1.1) and (1.2) hold for $g_L(n) = g_U(n) = 1$ for any $n \geqslant 1$, then $\{X_n, n \geqslant 1\}$ are called negatively orthant dependent (NOD, in short) random variables. It is well known that NA random variables are NOD random variables. For more details about NOD sequence, we can refer to Shen et al. [13], Wu and Jiang [14], Sung [15], and so on. Hu [7] pointed out that negatively superadditive dependent (NSD, in short) random variables are NOD. For the details about the concept and the probability limit theory of NSD sequence, one can refer to Shen et al. [9], Wang and Chen [17], Wang et al. [18], and so forth. Hence, the class of WOD random variables include independent sequence, NA sequence, NSD sequence, NOD sequence and END sequence as special cases. So, it is interesting and necessary to study the convergence properties of WOD random variables.

Many literatures have discussed the probability limiting behavior of WOD random variables and a lots of applications have been obtained. For example, Wang et al. [8] investigated the complete convergence for WOD random variables, Shen et al. [19] and Chen et al. [20] established some probability inequalities for WOD random variables, Wang and Hu [21] investigated the consistency of the nearest neighbor estimator of the density function based on WOD samples, and so on.

Throughout this paper, let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables. $I(A)$ is the indicator of the set A. C denotes a positive constant which may be different in various places. $a_n = O(b_n)$ stands for $a_n \leqslant C b_n$. Let $\psi(x) = lm(x)$ or $\psi(x) = 1$, $lm(x) = \ln \max\{x, 1\}$, and $g(n) = \max\{g_L(n), g_U(n)\}$.

DEFINITION 1.2. A sequence of random variables $\{X_n, n \geqslant 1\}$ converges completely to a constant θ if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty. \tag{1.3}$$

From the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely as $n \to \infty$. Therefore the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables.

DEFINITION 1.3. A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be stochastically dominated by random variable X if there exists a positive constant C such that

$$P(|X_n| \geqslant x) \leqslant CP(|X| \geqslant x) \tag{1.4}$$

for all $x \geqslant 0$ and all $n \geqslant 1$.

2. Lemmas

This section will give some lemmas, which are useful and necessary to prove main results.

LEMMA 2.1. [6] *Let $\{X_n, n \geqslant 1\}$ be WLOD (WUOD) with dominating coefficients $g_L(n)$, $n \geqslant 1$ $(g_U(n),\ n \geqslant 1)$. If $\{f_n(\cdot), n \geqslant 1\}$ are nondecreasing, then $\{f_n(X_n), n \geqslant 1\}$ are still WLOD (WUOD) with dominating coefficients $g_L(n)$, $n \geqslant 1$ $(g_U(n),\ n \geqslant 1)$; if $\{f_n(\cdot), n \geqslant 1\}$ are nonincreasing, then $\{f_n(X_n), n \geqslant 1\}$ are WUOD (WLOD) with dominating coefficients $g_L(n)$, $n \geqslant 1$ $(g_U(n),\ n \geqslant 1)$.*

LEMMA 2.2. [8] *Let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables with dominating coefficients $g_n = \max\{g_L(n), g_U(n)\}$. If $\{f_n, n \geqslant 1\}$ is a sequence of real nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \geqslant 1\}$ is still a sequence of WOD random variables with the same dominating coefficients $g(n)$.*

LEMMA 2.3. [8] *Let $p \geqslant 1$ and $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables with $EX_n = 0$, $E|X_n|^p < \infty$ for each $n \geqslant 1$ and dominating coefficients $g_n = \max\{g_L(n), g_U(n)\}$. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants. Then there exist positive constants $C_1(p)$ and $C_2(p)$ depending only on p such that*

$$E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|\right)^p \leqslant [C_1(p) + C_2(p)g(n)]\sum_{i=1}^{n} E|a_{ni}X_i|^p, \quad 1 < p \leqslant 2, \tag{2.1}$$

$$E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|\right)^p \leqslant C_1(p)\sum_{i=1}^{n} E|a_{ni}X_i|^p + C_2(p)g(n)\left(\sum_{i=1}^{n} E|a_{ni}X_i|^2\right)^{\frac{p}{2}}, \quad p > 2. \tag{2.2}$$

LEMMA 2.4. [10] *Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of random variables stochastically dominated by a random variable X. Then, for all $q > 0$ and $x > 0$,*

$$E|X_n|^q I(|X_n| \leqslant x) \leqslant C(E|X|^q I(|X| \leqslant x) + x^q P(|X| > x)), \tag{2.3}$$

$$E|X_n|^q I(|X_n| > x) \leqslant C(E|X|^q I(|X| > x)). \tag{2.4}$$

LEMMA 2.5. [11] *Let the function* $\psi(x)=1$ *or* $\psi(x)=lm(x)$. *Then the function* $\psi(x)$ *have the following properties:*

(1) for all $1\leqslant k\leqslant m$,

$$\sum_{n=k}^{m} n^{r-1}\psi(n)\leqslant Cm^{r}\psi(m),\ \ \textit{if}\ \ r>0, \tag{2.5}$$

$$\sum_{n=m}^{\infty} n^{r-1}\psi(n)\leqslant Cm^{r}\psi(m),\ \ \textit{if}\ \ r<0. \tag{2.6}$$

(2) for all $s>0$,

$$\psi(|x|^{s})\leqslant C(s)\psi(|x|)\leqslant C(s)\psi(1+|x|). \tag{2.7}$$

3. Main results and proofs

THEOREM 3.1. *Let* $\alpha>\frac{1}{2}$, $p>\frac{1}{\alpha}$, *and* $\{X_n,n\geqslant 1\}$ *be a sequence of WOD random variables which is mean zero and stochastically dominated by a random variable* X. *Assume that dominating coefficients* $g(n)=O(n^{\alpha t})$ *for some* $t>0$. *Let* $\{a_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ *be an array of constants satisfying* $\sum_{i=1}^{n}|a_{ni}|^{r}=O(n)$ *for some* $r>\max\left\{p,2(1+\frac{\alpha p}{\alpha p-1}),\frac{\alpha p+\alpha t-1}{\alpha-\frac{1}{2}}\right\}$. *If*

$$E|X|^{p}\psi(|X|)<\infty, \tag{3.1}$$

then for every $\varepsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\left|\sum_{i=1}^{n}a_{ni}X_{i}\right|>\varepsilon n^{\alpha}\right)<\infty. \tag{3.2}$$

Proof. Without loss of generality, we assume that $a_{ni}\geqslant 0$ (otherwise, we can note that $a_{ni}=a_{ni}^{+}-a_{ni}^{-}$). For fixed $n\geqslant 1$, $1\leqslant i\leqslant n$, define

$$X_{i}^{'}=-n^{\alpha}I(X_{i}<-n^{\alpha})+X_{i}I(|X_{i}|\leqslant n^{\alpha})+n^{\alpha}I(X_{i}>n^{\alpha}).$$

It is easy to check that

$$\left(\left|\sum_{i=1}^{n}a_{ni}X_{i}\right|>\varepsilon n^{\alpha}\right)\subset\left(\bigcup_{i=1}^{n}(|X_{i}|>n^{\alpha})\right)\bigcup\left(\left|\sum_{i=1}^{n}a_{ni}X_{i}^{'}\right|>\varepsilon n^{\alpha}\right),$$

which implies that

$$\begin{aligned}
&P\left(\left|\sum_{i=1}^{n}a_{ni}X_{i}\right|>\varepsilon n^{\alpha}\right)\\
&\leqslant P\left(\bigcup_{i=1}^{n}(|X_{i}|>n^{\alpha})\right)+P\left(\left|\sum_{i=1}^{n}a_{ni}X_{i}^{'}\right|>\varepsilon n^{\alpha}\right)\\
&\leqslant \sum_{i=1}^{n}P(|X_{i}|>n^{\alpha})+P\left(\left|\sum_{i=1}^{n}a_{ni}(X_{i}^{'}-EX_{i}^{'})\right|>\varepsilon n^{\alpha}-\left|\sum_{i=1}^{n}a_{ni}EX_{i}^{'}\right|\right).
\end{aligned} \tag{3.3}$$

First, we shall show that

$$\frac{1}{n^{\alpha}}\left|\sum_{i=1}^{n} a_{ni}EX_i^{'}\right| \to 0, \text{ as } n\to\infty. \tag{3.4}$$

From Lemma 2.2, it is easily seen that $\{X_i^{'}, 1\leqslant i\leqslant n\}$ is still a sequence of WOD random variables. Hence, it follows from the $EX_i=0$, $\sum_{i=1}^{n}|a_{ni}|^r=O(n)$, Markov's inequality and Lemma 2.4 that

$$\begin{aligned}\frac{1}{n^{\alpha}}\left|\sum_{i=1}^{n} a_{ni}EX_i^{'}\right| &\leqslant 2\frac{1}{n^{\alpha}}\sum_{i=1}^{n} a_{ni}E|X_i|I(|X_i|>n^{\alpha})\\ &\leqslant C\frac{1}{n^{\alpha}}\left(\sum_{i=1}^{n} a_{ni}^r\right)^{\frac{1}{r}} n^{1-\frac{1}{r}}E|X|I(|X|>n^{\alpha})\\ &\leqslant Cn^{1-\alpha}E|X|I(|X|>n^{\alpha}).\end{aligned} \tag{3.5}$$

When $p\geqslant 1$, from $\alpha p>1$, we have

$$\frac{1}{n^{\alpha}}\left|\sum_{i=1}^{n} a_{ni}EX_i^{'}\right| \leqslant Cn^{1-\alpha p}E|X|^p \to 0, \text{ as } n\to\infty.$$

When $0<p<1$, from $\alpha p>1$, we have $\alpha>1$, so

$$\frac{1}{n^{\alpha}}\left|\sum_{i=1}^{n} a_{ni}EX_i^{'}\right| \leqslant Cn^{1-\alpha}E|X|I(|X|>n^{\alpha}) \to 0, \text{ as } n\to\infty.$$

Hence, by (3.3) and (3.4),

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|>\varepsilon n^{\alpha}\right)\\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)\sum_{i=1}^{n}P(|X_i|>n^{\alpha})\\ &\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\left|\sum_{i=1}^{n} a_{ni}(X_i^{'}-EX_i^{'})\right|>\frac{\varepsilon n^{\alpha}}{2}\right).\end{aligned} \tag{3.6}$$

To prove (3.2), it suffices to show that

$$I_1=C\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)\sum_{i=1}^{n}P(|X_i|>n^{\alpha})<\infty, \tag{3.7}$$

$$I_2=C\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\left|\sum_{i=1}^{n} a_{ni}(X_i^{'}-EX_i^{'})\right|>\frac{\varepsilon n^{\alpha}}{2}\right)<\infty. \tag{3.8}$$

By (1.4) and (3.1), we can obtain that

$$
\begin{aligned}
I_1 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1}\psi(n)P(|X|>n^{\alpha}) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1}\psi(n)\sum_{j=n}^{\infty} P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha}) \\
&\leqslant C\sum_{j=1}^{\infty} P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha})\sum_{n=1}^{j} n^{\alpha p-1}\psi(n) \\
&\leqslant C\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha}) \\
&\leqslant CE|X|^p\psi(|X|)<\infty. \qquad (3.9)
\end{aligned}
$$

Following from Lemma 2.3, Markov's inequality and Jensen's inequality, there exists some $r>\max\left\{p,2(1+\frac{\alpha p}{\alpha p-1}),\frac{\alpha p+\alpha t-1}{\alpha-\frac{1}{2}}\right\}$ such that

$$
\begin{aligned}
I_2 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\psi(n)E\left(\left|\sum_{i=1}^{n} a_{ni}(X_i^{'}-EX_i^{'})\right|\right)^r \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\alpha t}\psi(n)\left(\sum_{i=1}^{n} a_{ni}^2 E|X_i^{'}-EX_i^{'}|^2\right)^{\frac{r}{2}} \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\psi(n)\sum_{i=1}^{n} a_{ni}^r E|X_i^{'}-EX_i^{'}|^r \\
&=C(I_{21}+I_{22}). \qquad (3.10)
\end{aligned}
$$

By Cr-inequality and $\sum_{i=1}^{n}|a_{ni}|^r=O(n)$, we can know that for all $r>2$,

$$
\left(\frac{1}{n}\sum_{i=1}^{n} a_{ni}^2\right)^{\frac{1}{2}}\leqslant\left(\frac{1}{n}\sum_{i=1}^{n} a_{ni}^r\right)^{\frac{1}{r}},
$$

so we can know that

$$
\left(\sum_{i=1}^{n} a_{ni}^2\right)^{\frac{r}{2}}\leqslant n^{\frac{r}{2}-1}\sum_{i=1}^{n} a_{ni}^r\leqslant Cn^{\frac{r}{2}}.
$$

When $p\geqslant 2$, by $r>\frac{\alpha p+\alpha t-1}{\alpha-\frac{1}{2}}$, Cr-inequality, Jensen's inequality and Lemma 2.4, we have that

$$\begin{aligned}
I_{21} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\alpha t}\psi(n)\left(\sum_{i=1}^{n} a_{ni}^{2}E|X_i'|^2\right)^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\alpha t}\psi(n)\left(\sum_{i=1}^{n} a_{ni}^{2}(E|X_i|^2 I(|X_i|\leqslant n^{\alpha})+n^{2\alpha}P(|X_i|>n^{\alpha}))\right)^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\alpha t+\frac{r}{2}}\psi(n)(EX^2I(|X|\leqslant n^{\alpha})+EX^2I(|X|>n^{\alpha}))^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\alpha t+\frac{r}{2}}\psi(n)\\
&<\infty. \qquad (3.11)
\end{aligned}$$

When $p<2$, by $r>\max\{p,2(1+\frac{\alpha p}{\alpha p-1}),\frac{\alpha p+\alpha t-1}{\alpha-\frac{1}{2}}\}$, we can know that $r>2$, it follows from Cr-inequality and Jensen's inequality and Lemma 2.4 that

$$\begin{aligned}
I_{21} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\alpha t+\frac{r}{2}}\psi(n)(n^{\alpha(2-p)}(EX^pI(|X|\leqslant n^{\alpha})+EX^pI(|X|>n^{\alpha})))^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\alpha t-\frac{(\alpha p-1)r}{2}}\psi(n)(EX^pI(|X|\leqslant n^{\alpha})+EX^pI(|X|>n^{\alpha}))^{\frac{r}{2}}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\alpha t-\frac{(\alpha p-1)r}{2}}\psi(n)(E|X|^p)^{\frac{r}{2}}\\
&<\infty. \qquad (3.12)
\end{aligned}$$

For I_{22}, by Lemma 2.4, (2.6) and (3.9), we get that

$$\begin{aligned}
I_{22} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\psi(n)\sum_{i=1}^{n} a_{ni}^{r}E|X|^rI(|X|\leqslant n^{\alpha})\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)\sum_{i=1}^{n} a_{ni}^{r}P(|X|>n^{\alpha})\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1}\psi(n)E|X|^rI(|X|\leqslant n^{\alpha})+C\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1}\psi(n)\sum_{j=1}^{n} E|X|^rI((j-1)^{\alpha}<|X|\leqslant j^{\alpha})+C\\
&\leqslant C\sum_{j=1}^{\infty} E|X|^rI((j-1)^{\alpha}<|X|\leqslant j^{\alpha})\sum_{n=j}^{\infty} n^{\alpha p-\alpha r-1}\psi(n)+C\\
&\leqslant C\sum_{j=1}^{\infty} E|X|^p\psi(j)I((j-1)^{\alpha}<|X|\leqslant j^{\alpha})+C\\
&\leqslant CE|X|^p\psi(|X|)+C\\
&<\infty. \qquad (3.13)
\end{aligned}$$

Hence from (3.6)–(3.13), the proof of Theorem 3.1 is completed. □

THEOREM 3.2. *Let $\alpha > \frac{1}{2}$, $p = \frac{1}{\alpha}$, $1 \leqslant p < 2$, and $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables which is mean zero and stochastically dominated by a random variable X. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^r = O(n^{\delta})$ for some $0 < \delta < 1$ and $r > 2$. Assume that dominating coefficients $g(n) = O(n^{\alpha t})$ for some $0 < t < \frac{1-\delta}{\alpha}$. If*

$$E|X|^p \psi(|X|) < \infty, \tag{3.14}$$

then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \psi(n) P\left(\left| \sum_{i=1}^{n} a_{ni} X_i \right| > \varepsilon n^{\alpha} \right) < \infty. \tag{3.15}$$

Proof. Similar to the proof of Theorem 3.1, without loss of generality, we assume that $a_{ni} \geqslant 0$ (otherwise, we can note that $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$). For fixed $n \geqslant 1$, $1 \leqslant i \leqslant n$, define

$$X_i^{'} = -n^{\alpha} I(X_i < -n^{\alpha}) + X_i I(|X_i| \leqslant n^{\alpha}) + n^{\alpha} I(X_i > n^{\alpha}).$$

It is easy to check that

$$\begin{aligned} P\left(\left| \sum_{i=1}^{n} a_{ni} X_i \right| > \varepsilon n^{\alpha} \right) \leqslant & \sum_{i=1}^{n} P(|X_i| > n^{\alpha}) \\ & + P\left(\left| \sum_{i=1}^{n} a_{ni} (X_i^{'} - EX_i^{'}) \right| > \varepsilon n^{\alpha} - \left| \sum_{i=1}^{n} a_{ni} EX_i^{'} \right| \right). \end{aligned} \tag{3.16}$$

First, we will show that

$$\frac{1}{n^{\alpha}} \left| \sum_{i=1}^{n} a_{ni} EX_i^{'} \right| \to 0, \text{ as } n \to \infty. \tag{3.17}$$

From Lemma 2.2, it obviously follows that $\{X_i^{'}, 1 \leqslant i \leqslant n\}$ is still a sequence of WOD random variables. Hence, it follows from the $EX_i = 0$, $\sum_{i=1}^{n} |a_{ni}|^r = O(n^{\delta})$, Markov's inequality and Lemma 2.4, for any $1 \leqslant p < 2$ that

$$\begin{aligned} \frac{1}{n^{\alpha}} \left| \sum_{i=1}^{n} a_{ni} EX_i^{'} \right| & \leqslant C \frac{1}{n^{\alpha}} \left(\sum_{i=1}^{n} a_{ni}^{r} \right)^{\frac{1}{r}} n^{1-\frac{1}{r}} E|X| I(|X| > n^{\alpha}) \\ & \leqslant C n^{\frac{\delta-1}{\alpha}} E|X|^p \\ & \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.18}$$

Hence, by (3.16) and (3.17),

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n}\psi(n)P\left(\left|\sum_{i=1}^{n}a_{ni}X_i\right|>\varepsilon n^{\alpha}\right)\\ \leqslant\ & C\sum_{n=1}^{\infty}\frac{1}{n}\psi(n)\sum_{i=1}^{n}P(|X_i|>n^{\alpha})\\ &+C\sum_{n=1}^{\infty}\frac{1}{n}\psi(n)P\left(\left|\sum_{i=1}^{n}a_{ni}(X_i^{'}-EX_i^{'})\right|>\frac{\varepsilon n^{\alpha}}{2}\right).\end{aligned} \tag{3.19}$$

To prove (3.15), it suffices to show that

$$I_1^{*}=C\sum_{n=1}^{\infty}\frac{1}{n}\psi(n)\sum_{i=1}^{n}P(|X_i|>n^{\alpha})<\infty. \tag{3.20}$$

$$I_2^{*}=C\sum_{n=1}^{\infty}\frac{1}{n}\psi(n)P\left(\left|\sum_{i=1}^{n}a_{ni}(X_i^{'}-EX_i^{'})\right|>\frac{\varepsilon n^{\alpha}}{2}\right)<\infty. \tag{3.21}$$

By Lemma 2.4 and (3.1), we can obtain that

$$\begin{aligned}I_1^{*}&\leqslant C\sum_{n=1}^{\infty}\psi(n)P(|X|>n^{\alpha})\\ &\leqslant C\sum_{j=1}^{\infty}P(j^{\alpha}<|X|\leqslant(j+1)^{\alpha})\sum_{n=1}^{j}\psi(n)\\ &\leqslant C\sum_{j=1}^{\infty}j\psi(j)P(j^{\alpha}<|X|\leqslant(j+1)^{\alpha})\\ &\leqslant CE|X|^{p}\psi(|X|)<\infty.\end{aligned} \tag{3.22}$$

Following from Lemma 2.3, Markov's inequality and Jensen's inequality, there exists some $r>2$ such that

$$\begin{aligned}I_2^{*}&\leqslant C\sum_{n=1}^{\infty}n^{-1-\alpha r}\psi(n)E\left(\left|\sum_{i=1}^{n}a_{ni}(X_i^{'}-EX_i^{'})\right|\right)^{r}\\ &\leqslant C\sum_{n=1}^{\infty}n^{-1-\alpha r+\alpha t}\psi(n)\left(\sum_{i=1}^{n}a_{ni}^{2}E|X_i^{'}-EX_i^{'}|^{2}\right)^{\frac{r}{2}}\\ &\quad+C\sum_{n=1}^{\infty}n^{-1-\alpha r}\psi(n)\sum_{i=1}^{n}a_{ni}^{r}E|X_i^{'}-EX_i^{'}|^{r}\\ &=C(I_{21}^{*}+I_{22}^{*}).\end{aligned} \tag{3.23}$$

By Cr-inequality and $\sum_{i=1}^{n}|a_{ni}|^{r}=O(n^{\delta})$, we can know that for all $r>2$,

$$\left(\frac{1}{n}\sum_{i=1}^{n}a_{ni}^{2}\right)^{\frac{1}{2}}\leqslant\left(\frac{1}{n}\sum_{i=1}^{n}a_{ni}^{r}\right)^{\frac{1}{r}},$$

so we can know that

$$\left(\sum_{i=1}^{n} a_{ni}^2\right)^{\frac{r}{2}} \leqslant n^{\frac{r}{2}-1}\sum_{i=1}^{n} a_{ni}^r \leqslant Cn^{\frac{r}{2}-1+\delta}.$$

By $1 \leqslant p < 2,\ r > 2,\ 0 < t < \frac{1-\delta}{\alpha}$, it follows from Cr-inequality, Jensen's inequality, and Lemma 2.4 that

$$\begin{aligned}
I_{21}^* &\leqslant C\sum_{n=1}^{\infty} n^{-1-\alpha r+\alpha t}\psi(n)\left(\sum_{i=1}^{n} a_{ni}^2 E|X_i^{'}|^2\right)^{\frac{r}{2}} \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-\alpha r+\alpha t}\psi(n)\left(\sum_{i=1}^{n} a_{ni}^2 (E|X_i|^2 I(|X_i| \leqslant n^{\alpha}) + n^{2\alpha}P(|X_i| > n^{\alpha}))\right)^{\frac{r}{2}} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha t+\delta-2}\psi(n)(E|X|^p I(|X| \leqslant n^{\alpha}) + E|X|^p I(|X| > n^{\alpha}))^{\frac{r}{2}} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha t+\delta-2}\psi(n)(E|X|^p)^{\frac{r}{2}} \\
&< \infty.
\end{aligned} \tag{3.24}$$

For I_{22}^*, by $0 < \delta < 1$, (2.3), (2.6) and (3.22), we can get that

$$\begin{aligned}
I_{22}^* &\leqslant C\sum_{n=1}^{\infty} n^{-1-\alpha r+\delta}\psi(n)E|X|^r I(|X| \leqslant n^{\alpha}) + C \\
&\leqslant C\sum_{j=1}^{\infty} E|X|^r I((j-1)^{\alpha} < |X| \leqslant j^{\alpha})\sum_{n=j}^{\infty} n^{-1-\alpha r+\delta}\psi(n) + C \\
&\leqslant C\sum_{j=1}^{\infty} j^{\delta-1}E|X|^p\psi(j)I((j-1)^{\alpha} < |X| \leqslant j^{\alpha}) + C \\
&\leqslant CE|X|^{p+\frac{\delta-1}{\alpha}}\psi(|X|) + C \\
&< \infty.
\end{aligned} \tag{3.25}$$

Hence from (3.19)–(3.25), the proof of Theorem 3.2 is completed. □

If $a_{ni} \equiv 1$ and $\psi(x) = 1$ in Theorem 3.2, we can get the following corollary immediately.

COROLLARY 3.3. *Let $\alpha > \frac{1}{2}$, $p = \frac{1}{\alpha}$, $1 \leqslant p < 2$, and $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables which is mean zero and stochastically dominated by a random variable X. Assume that dominating coefficients $g(n) = O(n^{\alpha t})$ for some $0 < t < \frac{1}{\alpha}$. If $E|X|^p < \infty$, then for every $\varepsilon > 0$, we have that*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\left|\sum_{i=1}^{n} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{3.26}$$

REMARK 3.4. This paper obtains the complete convergence for WOD random variables, which extends the corresponding results of Wang et al. [3] for the case of END random variables. If we take $\psi(x) = 1$ in Theorem 3.1, then we can obtain the result of Ding et al. [12] for WOD sequence without adding any extra condition. Furthermore, this paper obtains the complete convergence for WOD random variables when $\alpha p = 1$ in Theorem 3.2.

REMARK 3.5. The results of Theorems 3.1 and 3.2 hold true for $\psi(x) = 1$ and $\psi(x) = lm(x)$. In fact, Theorems 3.1 and 3.2 can be also obtained when $\psi(x)$ is a slowly varying function at infinite according to Bai and Su [22].

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