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# $L_p$ DUAL MIXED GEOMINIMAL SURFACE AREAS FOR MULTIPLE STAR BODIES

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*Abstract.* In this article, by defining $L_p$ dual mixed quermassintegrals for multiple star bodies, we consider $L_p$ dual mixed geominimal surface areas for multiple star bodies. Further, some related inequalities for this concept are obtained.

## 1. Introduction

In this article, we let $\mathscr{K}^n$ denote the set of convex bodies, that is, compact, convex subsets with non-empty interiors in Euclidean space $\mathbb{R}^n$. For the set of convex bodies whose centroids lie at the origin in $\mathbb{R}^n$, we mark $\mathscr{K}_c^n$. We also use $\mathscr{S}_o^n$ to denote the set of star bodies (about the origin) in $\mathbb{R}^n$. In addition, the $n$-dimensional volume of a body $K$ is titled $V(K)$, for the standard unit ball $B$, we write its volume $V(B) = \omega_n$.

The development of geominimal surface area has a long history. In 1974, the notion of geominimal surface area was first proposed by Petty (see [14]). Afterwards, based on the notion of $L_p$ mixed volume (see [9]), Lutwak ([10]) considered the $L_p$ geominimal surface area and gave some related inequalities. The $L_p$ geominimal surface area is an important part in the $L_p$ Brunn Minkowski theory. However, for the $L_p$ geominimal surface area, it seems painful to find an integral expression. Until recently, Zhu, Zhou and Xu ([30]) gave a kind of integral expression for $L_p$ geominimal surface area and researched its mixed version, their work is stronger than classical counterparts. According to $L_p$ dual mixed volume (see [10]), Wang and Qi ([22]) put forward the notion of $L_p$ dual geominimal surface area, which is a key component of the $L_p$ dual Brunn Minkowski theory. For the $L_p$ dual geominimal surface area and its mixed version, Yan, Wang and Si ([24]) showed their integral expression and got some related inequalities.

In 2014, Li and Wang ([6]) defined the following $L_p$ dual mixed geominimal surface areas for multiple star bodies.

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DEFINITION 1.A. For $p \geqslant 1$, $K_1,\cdots,K_n \in \mathscr{S}_o^n$, the $L_p$ dual mixed geominimal surface areas, $\widetilde{G}_{-p}^{(j)}(K_1,\cdots,K_n)$ $(j=1,2)$, of $K_1,\cdots,K_n$ are defined by

$$\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}^{(1)}(K_1,\cdots,K_n) = \inf_{L\in\mathscr{K}_c^n}\{n\widetilde{V}_{-p}(K_1,\cdots,K_n;L,\cdots,L)V(L^*)^{-\frac{p}{n}}\}; \tag{1.1}$$

$$\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}^{(2)}(K_1,\cdots,K_n) = \inf_{L_i\in\mathscr{K}_c^n}\{n\widetilde{V}_{-p}(K_1,\cdots,K_n;L_1,\cdots,L_n)\prod_{i=1}^{n}V(L_i^*)^{-\frac{p}{n^2}}\}. \tag{1.2}$$

Here $\widetilde{V}_{-p}(K_1,\cdots,K_n;L_1,\cdots,L_n)$ denotes $L_p$ dual mixed volume which is defined by

$$\widetilde{V}_{-p}(K_1,\cdots,K_n;L_1,\cdots,L_n) = \frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\left[\rho(K_i,u)^{n+p}\rho(L_i,u)^{-p}\right]^{\frac{1}{n}}du. \tag{1.3}$$

We mark sequence $\mathbf{K}=(K_1,\cdots,K_n)$, $\mathbf{L}=(L_1,\cdots,L_n)$, $\mathbf{K}\in\mathscr{S}_o^n$ $(\mathscr{K}_c^n)$ means each $K_j\in\mathscr{S}_o^n$ $(\mathscr{K}_c^n)$ $(j=1,\cdots,n)$, $\mathbf{K}$, $\mathbf{L}$ are dilates means $K_j$ and $L_j$ are dilates (for $j=1,\cdots,n$). From this, (1.3) can be rewritten as follows:

$$\widetilde{V}_{-p}(\mathbf{K};\mathbf{L}) = \frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\left[\rho(K_i,u)^{n+p}\rho(L_i,u)^{-p}\right]^{\frac{1}{n}}du. \tag{1.4}$$

Obviously, if $K_1=\cdots=K_n=K$ and $L_1=\cdots=L_n=L$, by (1.4) then $\widetilde{V}_{-p}(\mathbf{K};\mathbf{L}) = \widetilde{V}_{-p}(K,L)$, where $\widetilde{V}_{-p}(K,L)$ is just the $L_p$ dual mixed volume of $K$ and $L$ which was defined by Lutwak (see [10]).

In this article, we will consider $i$-th $L_p$ dual mixed geominimal surface areas. Here, we first define the $L_p$ dual mixed quermassintegrals for multiple star bodies as follows.

DEFINITION 1.1. If $\mathbf{K}$, $\mathbf{L}\in\mathscr{S}_o^n$, real $i\neq n$ and $p\geqslant 1$, the $L_p$ dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L})$, of $\mathbf{K}$ and $\mathbf{L}$ are defined by

$$\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L}) = \frac{1}{n}\int_{S^{n-1}}\prod_{j=1}^{n}\left(\rho(K_j,u)^{n+p-i}\rho(L_j,u)^{-p}\right)^{\frac{1}{n}}du. \tag{1.5}$$

From (1.4) and (1.5), we easily get $\widetilde{W}_{-p,0}(\mathbf{K},\mathbf{L}) = \widetilde{V}_{-p}(\mathbf{K};\mathbf{L})$. When $K_1=\cdots=K_n=K$ and $L_1=\cdots=L_n=L$, we write $\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L}) = \widetilde{W}_{-p,i}(K,L)$ which is Wang and Leng's $L_p$ dual mixed quermassintegrals (see[19]).

Meanwhile, based on (1.5), we define $i$-th $L_p$ dual mixed geominimal surface areas for multiple star bodies as follows.

DEFINITION 1.2. If real $i\neq n$, $p\geqslant 1$ and $\mathbf{K}\in\mathscr{S}_o^n$, the $i$-th $L_p$ dual mixed geominimal surface areas, $\widetilde{G}_{-p,i}(\mathbf{K})$, of $\mathbf{K}$, are defined by

$$\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(\mathbf{K}) = \inf_{L_j\in\mathscr{K}_c^n}\left\{n\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L})\prod_{j=1}^{n}\widetilde{W}_i(L_j^*)^{-\frac{p}{n^2}}\right\}, \tag{1.6}$$

where $\widetilde{W}_i(L)$ is the dual quermassintegral of $L$ and $L^*$ denotes the polar body of $L$.

Next, for the $i$-th $L_p$ dual mixed geominimal surface areas, we obtain the following property of affine transformation.

THEOREM 1.1. *If real* $i \neq n$, $\mathbf{K} \in \mathscr{S}_o^n$, $p \geqslant 1$ *and* $\phi \in GL(n)$, *then for any* $u_1$, $u_2 \in S^{n-1}$, *we have*

$$\widetilde{G}_{-p,i}(\phi\mathbf{K}) = \mid det\phi \mid^{\frac{n+p}{n}} \| \phi^{-1}u_1 \|^i \| \phi u_2 \|^{-\frac{pi}{n}} \widetilde{G}_{-p,i}(\mathbf{K}). \tag{1.7}$$

REMARK. If $\phi \in SL(n)$ is an isometric transformation in Theorem 1.1, we obtain $\widetilde{G}_{-p,i}(\phi\mathbf{K}) = \widetilde{G}_{-p,i}(\mathbf{K})$, which demonstrates that $i$-th $L_p$ dual mixed geominimal surface areas $\widetilde{G}_{-p,i}$ is affine invariant under $SL(n)$.

Further, we give a monotonic inequality and establish related Brunn-Minkowski type inequalities, which can be demonstrated as follows.

THEOREM 1.2. *If* $\mathbf{K} \in \mathscr{S}_o^n$, $1 \leqslant p < q$ *and real* $i < n$, *then*

$$\left[\frac{\widetilde{G}_{-p,i}(\mathbf{K})}{n\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{p}} \leqslant \left[\frac{\widetilde{G}_{-q,i}(\mathbf{K})}{n\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{q}}, \tag{1.8}$$

*with equality if and only if* $\mathbf{K} \in \mathscr{K}_c^n$.

THEOREM 1.3. *For* $\mathbf{K}$, $\mathbf{L} \in \mathscr{S}_o^n$, $p \geqslant 1$, $n+p < i < n+2p$ *and* $\lambda$, $\mu \geqslant 0$ *(not both zero), then*

$$\widetilde{G}_{-p,i}(\lambda \times \mathbf{K} \widetilde{+}_{-p} \mu \times \mathbf{L})^{-\frac{p}{n+p-i}} \geqslant \lambda \widetilde{G}_{-p,i}(\mathbf{K})^{-\frac{p}{n+p-i}} + \mu \widetilde{G}_{-p,i}(\mathbf{L})^{-\frac{p}{n+p-i}}, \tag{1.9}$$

*with equality if and only if* $\mathbf{K}$ *and* $\mathbf{L}$ *are dilates. Here* $\lambda \times \mathbf{K} \widetilde{+}_{-p} \mu \times \mathbf{L} = (\lambda \times K_1 \widetilde{+}_{-p} \mu \times L_1, \cdots, \lambda \times K_n \widetilde{+}_{-p} \mu \times L_n)$ *and* $\lambda \times K \widetilde{+}_{-p} \mu \times L$ *denotes* $L_p$ *harmonic radial combination of* $K$ *and* $L$.

THEOREM 1.4. *For* $\mathbf{K}$, $\mathbf{L} \in \mathscr{S}_o^n$, $p \geqslant 1$, $n < i < n+p$ *and* $\lambda, \mu \geqslant 0$ *(not both zero ), then*

$$\widetilde{G}_{-p,i}(\lambda \circ \mathbf{K} \widetilde{+}_p \mu \circ \mathbf{L})^{\frac{p}{n+p-i}} \geqslant \lambda \widetilde{G}_{-p,i}(\mathbf{K})^{\frac{p}{n+p-i}} + \mu \widetilde{G}_{-p,i}(\mathbf{L})^{\frac{p}{n+p-i}}, \tag{1.10}$$

*with equality if and only if* $\mathbf{K}$ *and* $\mathbf{L}$ *are dilates. Here* $\lambda \circ \mathbf{K} \widetilde{+}_p \mu \circ \mathbf{L} = (\lambda \circ K_1 \widetilde{+}_p \mu \circ L_1, \cdots, \lambda \circ K_n \widetilde{+}_p \mu \circ L_n)$ *and* $\lambda \circ K \widetilde{+}_p \mu \circ L$ *denotes* $L_p$ *radial combination of* $K$ *and* $L$.

This paper belongs to the research realm of the affine and geominimal surface area, a large number of scholars have already studied a lot and have some achievements in recent years (see [1, 7, 8, 11, 12, 13, 16, 17, 18, 20, 21, 23, 25, 26, 27, 28, 29]).

## 2. Preliminaries

If $E$ denotes a nonempty subset in $\mathbb{R}^n$, the polar set, $E^*$, of $E$ is defined by (see [2, 15])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in E\}, \tag{2.1}$$

for $\phi \in GL(n)$, we have $(\phi K)^* = \phi^{-\tau} K^*$ and $\| \phi^{-\tau}(u) \| = \| \phi(u) \|^{-1}$.

If $K$ is a compact star-shaped (with respect to the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot)$: $\mathbb{R}^n \setminus \{0\} \to [0, +\infty)$, is defined by (see [2, 15])

$$\rho(K, x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

If $\rho_K$ is continuous and positive, then $K$ will be called a star body (respect to the origin). Two star bodies $K$ and $L$ are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. It is easy to check that if $\phi \in GL(n)$, $\rho_{\phi K}(x) = \rho_K(\phi^{-1}x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

For real $p \geqslant 1$, $K,\ L \in \mathscr{S}_o^n$, $\lambda,\ \mu \geqslant 0$ (not both zero ), $L_p$ harmonic radial combination, $\lambda \times K \widetilde{+}_{-p} \mu \times L$, of $K$ and $L$ is defined by (see [10])

$$\rho(\lambda \times K \widetilde{+}_{-p} \mu \times L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}, \tag{2.3}$$

where the mark "$\widetilde{+}_{-p}$" is called $L_p$ harmonic radial sum and $\lambda \times K$ is called $L_p$ harmonic radial scalar multiplication. It is obvious that $\lambda \times K = \lambda^{-\frac{1}{p}} K$.

For $K,\ L \in \mathscr{S}_o^n$, real $p > 0$, $\lambda,\ \mu \geqslant 0$ (not both zero ), $L_p$ radial combination, $\lambda \circ K \widetilde{+}_p \mu \circ L$, of $K$ and $L$ is defined by (see[3])

$$\rho(\lambda \circ K \widetilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p, \tag{2.4}$$

where the mark "$\widetilde{+}_p$" is called $L_p$ radial sum and $\lambda \circ K$ is called $L_p$ radial scalar multiplication. It is obvious that $\lambda \circ K = \lambda^{\frac{1}{p}} K$.

## 3. Proofs of Theorems

In this section, Theorems 1.1–1.4 will be proved.

*Proof of Theorem 1.1.* For $u_1 \in S^{n-1}$ and $\phi \in GL(n)$, we let $v_1 = \frac{\phi^{-1}u_1}{\|\phi^{-1}u_1\|}$, then $u_1 = \| \phi^{-1}u_1 \| \phi v_1$,

$$\rho(L_j, u_1) = \rho(L_j, \| \phi^{-1}u_1 \| \phi v_1) = \| \phi^{-1}u_1 \|^{-1} \rho(\phi^{-1}L_j, v_1).$$

In association with (1.5), we know

$$\begin{aligned}
&\widetilde{W}_{-p,i}(\phi \mathbf{K}, \mathbf{L}) \\
&= \frac{1}{n} \int_{S^{n-1}} \prod_{j=1}^{n} \left[ \rho(\phi K_j, u_1)^{n+p-i} \rho(L_j, u_1)^{-p} \right]^{\frac{1}{n}} du_1 \\
&= \frac{|\, det\phi \,|}{n} \|\phi^{-1}u_1\|^{-(n+p-i)+n+p} \int_{S^{n-1}} \prod_{j=1}^{n} \left[ \rho(K_j, v_1)^{n+p-i} \rho(\phi^{-1}L_j, v_1)^{-p} \right]^{\frac{1}{n}} dv_1 \\
&= |\, det\phi \,| \cdot \| \phi^{-1}u_1 \|^i \, \widetilde{W}_{-p,i}(\mathbf{K}, \phi^{-1}\mathbf{L}).
\end{aligned} \tag{3.1}$$

Similarly, for $u_2 \in S^{n-1}$ and $\phi \in GL(n)$, we let $v_2 = \frac{\phi^{-\tau}u_2}{\|\phi^{-\tau}u_2\|}$, then $u_2 = \| \phi^{-\tau}u_2 \| \phi^{\tau} v_2$, combining with (2.1), one has

$$\widetilde{W}_i((\phi^{-1}L_j)^*) = \widetilde{W}_i(\phi^{\tau} L_j{}^*). \tag{3.2}$$

Therefore, we have the following equality,

$$\begin{aligned}
\widetilde{W}_i((\phi^{-1}L_j)^*) &= \frac{1}{n} \int_{S^{n-1}} \rho(\phi^{\tau} L_j{}^*, u_2)^{n-i} du_2 \\
&= |\, det\phi^{\tau} \,| \frac{1}{n} \int_{S^{n-1}} \rho(L_j{}^*, \phi^{-\tau}u_2)^{n-i} d(\phi^{-\tau}u_2)
\end{aligned}$$

$$=|\, det\phi^{\tau} \,|\, \|\phi^{-\tau}u_2\|^{-(n-i)+n}\frac{1}{n}\int_{S^{n-1}}\rho(L_j{}^*,v_2)^{n-i}dv_2$$
$$=|\, det\phi \,|\, \|\phi u_2\|^{-i}\widetilde{W}_i(L_j{}^*). \tag{3.3}$$

From (1.6), (3.1) and (3.3), we know that

$$\begin{aligned}\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(\phi\mathbf{K}) &= \inf_{L_j\in\mathscr{K}_c^n}\left\{n\widetilde{W}_{-p,i}(\phi\mathbf{K},\mathbf{L})\prod_{j=1}^{n}\widetilde{W}_i(L_j^*)^{-\frac{p}{n^2}}\right\}\\ &=|\, det\phi \,|^{\frac{n+p}{n}}\|\,\phi^{-1}u_1\,\|^i\|\,\phi u_2\,\|^{-\frac{pi}{n}}\\ &\quad\times\inf_{L_j\in\mathscr{K}_c^n}\left\{n\widetilde{W}_{-p,i}(\mathbf{K},\phi^{-1}\mathbf{L})\prod_{j=1}^{n}\widetilde{W}_i((\phi^{-1}L_j)^*)^{-\frac{p}{n^2}}\right\}\\ &=|\, det\phi \,|^{\frac{n+p}{n}}\cdot\|\,\phi^{-1}u_1\,\|^i\|\,\phi u_2\,\|^{-\frac{pi}{n}}\;\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(\mathbf{K}).\end{aligned}$$

This yields equality (1.7). □

The proof of Theorem 1.2 requires the following lemma.

LEMMA 3.1. *For real $i\neq n$, $\mathbf{K}$, $\mathbf{L}\in\mathscr{S}_o^n$ and $1\leqslant p<q$, then*

$$\left[\frac{\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L})}{\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{p}}\leqslant\left[\frac{\widetilde{W}_{-q,i}(\mathbf{K},\mathbf{L})}{\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{q}}, \tag{3.4}$$

*with quality if and only if $\mathbf{K}$, $\mathbf{L}$ are dilates.*

*Proof.* Since $\frac{q}{p}>1$, we obtain the following inequality by using the Hölder's integral inequality (see [4]),

$$\begin{aligned}\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L}) &= \frac{1}{n}\int_{S^{n-1}}\prod_{j=1}^{n}\left[\rho(K_j,u)^{n+p-i}\rho(L_j,u)^{-p}\right]^{\frac{1}{n}}du\\ &=\frac{1}{n}\int_{S^{n-1}}\left[\prod_{j=1}^{n}(\rho(K_j,u)^{n+q-i}\rho(L_j,u)^{-q})^{\frac{1}{n}}\right]^{\frac{p}{q}}\left[\prod_{j=1}^{n}(\rho(K_j,u)^{n-i})^{\frac{1}{n}}\right]^{\frac{q-p}{q}}du\\ &\leqslant\left[\frac{1}{n}\int_{S^{n-1}}\prod_{j=1}^{n}(\rho(K_j,u)^{n+q-i}\rho(L_j,u)^{-q})^{\frac{1}{n}}du\right]^{\frac{p}{q}}\\ &\quad\times\left[\frac{1}{n}\int_{S^{n-1}}\prod_{j=1}^{n}\rho(K_j,u)^{\frac{n-i}{n}}du\right]^{\frac{q-p}{q}}\\ &=\widetilde{W}_{-q,i}(\mathbf{K},\mathbf{L})^{\frac{p}{q}}\widetilde{W}_i(\mathbf{K})^{\frac{q-p}{q}},\end{aligned}$$

that is,

$$\left[\frac{\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L})}{\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{p}}\leqslant\left[\frac{\widetilde{W}_{-q,i}(\mathbf{K},\mathbf{L})}{\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{q}},$$

this is just (3.4), with equality if and only if $\mathbf{K}$ and $\mathbf{L}$ are dilates. □

*Proof of Theorem 1.2.* From (1.6) and inequality (3.4), we know for $i < n$

$$\begin{aligned}
\omega_n^{-1}\left[\frac{\widetilde{G}_{-p,i}(\mathbf{K})}{n\widetilde{W}_i(\mathbf{K})}\right]^{\frac{n-i}{p}} &= \omega_n^{-1}\inf_{L_j\in\mathscr{K}_c^n}\left\{\frac{\omega_n n^{\frac{n-i}{p}}\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L})^{\frac{n-i}{p}}\prod_{j=1}^n\widetilde{W}_i(L_j^*)^{-\frac{p}{n^2}\cdot\frac{n-i}{p}}}{n^{\frac{n-i}{p}}\widetilde{W}_i(\mathbf{K})^{\frac{n-i}{p}}}\right\}\\
&= \inf_{L_j\in\mathscr{K}_c^n}\left\{\left[\frac{\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{L})}{\widetilde{W}_i(\mathbf{K})}\right]^{\frac{n-i}{p}}\prod_{j=1}^n\widetilde{W}_i({L_j}^*)^{-\frac{n-i}{n^2}}\right\}\\
&\leqslant \inf_{L_j\in\mathscr{K}_c^n}\left\{\left[\frac{\widetilde{W}_{-q,i}(\mathbf{K},\mathbf{L})}{\widetilde{W}_i(\mathbf{K})}\right]^{\frac{n-i}{q}}\prod_{j=1}^n\widetilde{W}_i({L_j}^*)^{-\frac{n-i}{n^2}}\right\}\\
&= \omega_n^{-1}\left[\frac{\widetilde{G}_{-q,i}(\mathbf{K})}{n\widetilde{W}_i(\mathbf{K})}\right]^{\frac{n-i}{q}},
\end{aligned}$$

i.e.,

$$\left[\frac{\widetilde{G}_{-p,i}(\mathbf{K})}{n\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{p}} \leqslant \left[\frac{\widetilde{G}_{-q,i}(\mathbf{K})}{n\widetilde{W}_i(\mathbf{K})}\right]^{\frac{1}{q}}.$$

Since each $L_j \in \mathscr{K}_c^n$ in (1.6), which means $\mathbf{L} \in \mathscr{K}_c^n$, together with the equality condition of (3.4), we know that equality holds in (1.8) if and only if $\mathbf{K} \in \mathscr{K}_c^n$. □

Theorems 1.3 and 1.4 show Brunn-Minkowski type inequalities of $L_p$ dual mixed geominimal surface areas, the following results are indispensable to the proof.

LEMMA 3.2. (see [5]) *If $a_k,\ b_k > 0$, then*

$$\left[\prod_{k=1}^n (a_k+b_k)\right]^{\frac{1}{n}} \geqslant \left[\prod_{k=1}^n a_k\right]^{\frac{1}{n}} + \left[\prod_{k=1}^n b_k\right]^{\frac{1}{n}}, \tag{3.5}$$

*with equality if and only if $a_k$ and $b_k$ are proportional.*

LEMMA 3.3. *If real $p \geqslant 1$, $\mathbf{K},\ \mathbf{L} \in \mathscr{S}_o^n$, $\lambda,\ \mu \geqslant 0$ (not both zero), $i < n+p$ or $n+p < i < n+2p$, then for any $\mathbf{Q} \in \mathscr{S}_o^n$, we have*

$$\widetilde{W}_{-p,i}(\lambda\times\mathbf{K}\widetilde{+}_{-p}\mu\times\mathbf{L},\mathbf{Q})^{-\frac{p}{n+p-i}} \geqslant \lambda\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q})^{-\frac{p}{n+p-i}} + \mu\widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q})^{-\frac{p}{n+p-i}}, \tag{3.6}$$

*equality holds if and only if* $\mathbf{K}$ *and* $\mathbf{L}$ *are dilates.*

*Proof.* From (2.3), (3.5) and Minkowski's integral inequality (see [4]) which enforces the condition $-\frac{n+p-i}{p} < 0$ or $0 < -\frac{n+p-i}{p} < 1$, it follows that, for any $\mathbf{Q} \in \mathscr{S}_o^n$,

$$\begin{aligned}
&\widetilde{W}_{-p,i}(\lambda\times\mathbf{K}\widetilde{+}_{-p}\mu\times\mathbf{L},\mathbf{Q})^{-\frac{p}{n+p-i}}\\
&= \left[\frac{1}{n}\int_{S^{n-1}}\prod_{j=1}^n\left(\rho(\lambda\times K_j\widetilde{+}_{-p}\mu\times L_j,u)^{\frac{n+p-i}{n}}\rho(Q_j,u)^{-\frac{p}{n}}\right)du\right]^{-\frac{p}{n+p-i}}\\
&= \left[\frac{1}{n}\int_{S^{n-1}}\left(\prod_{j=1}^n\rho(\lambda\times K_j\widetilde{+}_{-p}\mu\times L_j,u)^{-p}\rho(Q_j,u)^{\frac{p^2}{n+p-i}}\right)^{\frac{1}{n}\cdot(-\frac{n+p-i}{p})}du\right]^{-\frac{p}{n+p-i}}
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg[\frac{1}{n}\int_{S^{n-1}} \Bigg(\prod_{j=1}^{n}(\lambda\rho(K_j,u)^{-p}\rho(Q_j,u)^{\frac{p^2}{n+p-i}} \\
&\quad + \mu\rho(L_j,u)^{-p}\rho(Q_j,u)^{\frac{p^2}{n+p-i}})\Bigg)^{\frac{1}{n}\cdot(-\frac{n+p-i}{p})} du\Bigg]^{-\frac{p}{n+p-i}} \\
&\geqslant \Bigg[\frac{1}{n}\int_{S^{n-1}} \Bigg((\prod_{j=1}^{n}\lambda\rho(K_j,u)^{-p}\rho(Q_j,u)^{\frac{p^2}{n+p-i}})^{\frac{1}{n}} \\
&\quad + (\prod_{j=1}^{n}\mu\rho(L_j,u)^{-p}\rho(Q_j,u)^{\frac{p^2}{n+p-i}})^{\frac{1}{n}}\Bigg)^{-\frac{n+p-i}{p}} du\Bigg]^{-\frac{p}{n+p-i}} \\
&\geqslant \Bigg[\frac{1}{n}\int_{S^{n-1}} (\prod_{j=1}^{n}\lambda\rho(K_j,u)^{-p}\rho(Q_j,u)^{\frac{p^2}{n+p-i}})^{\frac{1}{n}\cdot(-\frac{n+p-i}{p})} du\Bigg]^{-\frac{p}{n+p-i}} \\
&\quad + \Bigg[\frac{1}{n}\int_{S^{n-1}} (\prod_{j=1}^{n}\mu\rho(L_j,u)^{-p}\rho(Q_j,u)^{\frac{p^2}{n+p-i}})^{\frac{1}{n}\cdot(-\frac{n+p-i}{p})} du\Bigg]^{-\frac{p}{n+p-i}} \\
&= \lambda\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q})^{-\frac{p}{n+p-i}} + \mu\widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q})^{-\frac{p}{n+p-i}}.
\end{aligned}
$$

This yields (3.6), according to the equality conditions of (3.5) and Minkowski's integral inequality, we see that equality in (3.6) holds if and only if $\rho(K_j,\cdot)$ and $\rho(L_j,\cdot)$ are positively proportional (for $j = 1,\cdots,n$). Therefore, equality holds in (3.6) if and only if $\mathbf{K}$ and $\mathbf{L}$ are dilates. $\square$

Similarly, by using (2.4), (3.5) and Minkowski's integral inequality (see [4]) which now enforces the condition $\frac{n+p-i}{p} < 0$ or $0 < \frac{n+p-i}{p} < 1$, we get the following inequality with the equality condition.

LEMMA 3.4. *If real $p \geqslant 1$, $\mathbf{K}$, $\mathbf{L} \in \mathscr{S}_o^n$, $\lambda$, $\mu \geqslant 0$ (not both zero), $n < i < n+p$ or $i > n+p$, then for any $\mathbf{Q} \in \mathscr{S}_o^n$, we have*

$$
\widetilde{W}_{-p,i}(\lambda\circ\mathbf{K}\widetilde{+}_p\mu\circ\mathbf{L},\mathbf{Q})^{\frac{p}{n+p-i}} \geqslant \lambda\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q})^{\frac{p}{n+p-i}} + \mu\widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q})^{\frac{p}{n+p-i}}, \tag{3.7}
$$

*equality holds if and only if $\mathbf{K}$ and $\mathbf{L}$ are dilates.*

*Proof.* If $n < i < n+p$ or $i > n+p$, then $0 < \frac{n+p-i}{p} < 1$ or $\frac{n+p-i}{p} < 0$. thus, for any $\mathbf{Q} \in \mathscr{S}_o^n$, we have

$$
\begin{aligned}
&\widetilde{W}_{-p,i}(\lambda\circ\mathbf{K}\widetilde{+}_p\mu\circ\mathbf{L},\mathbf{Q})^{\frac{p}{n+p-i}} \\
&= \Bigg[\frac{1}{n}\int_{S^{n-1}} \prod_{j=1}^{n}\Bigg(\rho(\lambda\circ K_j\widetilde{+}_p\mu\circ L_j,u)^{\frac{n+p-i}{n}}\rho(Q_j,u)^{-\frac{p}{n}}\Bigg)du\Bigg]^{\frac{p}{n+p-i}} \\
&= \Bigg[\frac{1}{n}\int_{S^{n-1}} \Bigg(\prod_{j=1}^{n}\rho(\lambda\circ K_j\widetilde{+}_p\mu\circ L_j,u)^{p}\rho(Q_j,u)^{-\frac{p^2}{n+p-i}}\Bigg)^{\frac{1}{n}\cdot\frac{n+p-i}{p}} du\Bigg]^{\frac{p}{n+p-i}}
\end{aligned}
$$

$$= \left[\frac{1}{n}\int_{S^{n-1}} \left( \prod_{j=1}^{n} (\lambda\rho(K_j,u)^p \rho(Q_j,u)^{-\frac{p^2}{n+p-i}} + \mu\rho(L_j,u)^p\rho(Q_j,u)^{-\frac{p^2}{n+p-i}}) \right)^{\frac{1}{n}\cdot\frac{n+p-i}{p}} du\right]^{\frac{p}{n+p-i}}$$

$$\geqslant \left[\frac{1}{n}\int_{S^{n-1}} \left( (\prod_{j=1}^{n} \lambda\rho(K_j,u)^p \rho(Q_j,u)^{-\frac{p^2}{n+p-i}})^{\frac{1}{n}} + (\prod_{j=1}^{n}\mu\rho(L_j,u)^p\rho(Q_j,u)^{-\frac{p^2}{n+p-i}})^{\frac{1}{n}} \right)^{\frac{n+p-i}{p}} du\right]^{\frac{p}{n+p-i}}$$

$$\geqslant \left[\frac{1}{n}\int_{S^{n-1}} (\prod_{j=1}^{n} \lambda\rho(K_j,u)^p \rho(Q_j,u)^{-\frac{p^2}{n+p-i}})^{\frac{n+p-i}{np}} du\right]^{\frac{p}{n+p-i}} + \left[\frac{1}{n}\int_{S^{n-1}} (\prod_{j=1}^{n} \mu\rho(L_j,u)^p \rho(Q_j,u)^{-\frac{p^2}{n+p-i}})^{\frac{n+p-i}{np}} du\right]^{\frac{p}{n+p-i}}$$

$$= \lambda\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q})^{\frac{p}{n+p-i}} + \mu\widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q})^{\frac{p}{n+p-i}},$$

which yields (3.7), with equality if and only if $\mathbf{K}$ and $\mathbf{L}$ are dilates. □

Now, Theorems 1.3 and 1.4 will be proved by using Lemmas 3.3 and 3.4, respectively.

*Proof of Theorem 1.3.* Since $p \geqslant 1$ and $n+p<i<n+2p$, thus $-\frac{p}{n+p-i} > 0$. Hence, by (1.6) and Lemma 3.3, we obtain

$$\lambda\widetilde{G}_{-p,i}(\mathbf{K})^{-\frac{p}{n+p-i}} + \mu\widetilde{G}_{-p,i}(\mathbf{L})^{-\frac{p}{n+p-i}}$$

$$= \inf_{Q_j\in\mathscr{K}_c^n}\left\{\lambda\left(\omega_n^{\frac{p}{n-i}} n\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q})\prod_{j=1}^{n}\widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}}\right)^{-\frac{p}{n+p-i}}\right\} + \inf_{Q_j\in\mathscr{K}_c^n}\left\{\mu\left(\omega_n^{\frac{p}{n-i}} n\widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q})\prod_{j=1}^{n}\widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}}\right)^{-\frac{p}{n+p-i}}\right\}$$

$$\leqslant \inf_{Q_j\in\mathscr{K}_c^n}\left\{\left(\omega_n^{\frac{p}{n-i}} n\prod_{j=1}^{n}\widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}}\right)^{-\frac{p}{n+p-i}} \times\left(\lambda\widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q})^{-\frac{p}{n+p-i}} + \mu\widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q})^{-\frac{p}{n+p-i}}\right)\right\}$$

$$\leqslant \inf_{Q_j\in\mathscr{K}_c^n}\left\{\omega_n^{\frac{p}{n-i}} n\prod_{j=1}^{n}\widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}}\widetilde{W}_{-p,i}(\lambda\times\mathbf{K}\widetilde{+}_{-p}\mu\times\mathbf{L},\mathbf{Q})\right\}^{-\frac{p}{n+p-i}}$$

$$= \widetilde{G}_{-p,i}(\lambda\times\mathbf{K}\widetilde{+}_{-p}\mu\times\mathbf{L})^{-\frac{p}{n+p-i}}.$$

We know that equality in (1.9) holds if and only if $\mathbf{K}$ and $\mathbf{L}$ are dilates. □

*Proof of Theorem 1.4.* If $p \geqslant 1$, $n < i < n+p$, then $\frac{p}{n+p-i} > 0$. From (1.6) and Lemma 3.4, we have

$$\begin{aligned}
&\lambda \widetilde{G}_{-p,i}(\mathbf{K})^{\frac{p}{n+p-i}} + \mu \widetilde{G}_{-p,i}(\mathbf{L})^{\frac{p}{n+p-i}} \\
&= \inf_{Q_j \in \mathscr{K}_c^n} \left\{ \lambda \left( \omega_n^{\frac{p}{n-i}} n \widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q}) \prod_{j=1}^{n} \widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}} \right)^{\frac{p}{n+p-i}} \right\} \\
&\quad + \inf_{Q_j \in \mathscr{K}_c^n} \left\{ \mu \left( \omega_n^{\frac{p}{n-i}} n \widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q}) \prod_{j=1}^{n} \widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}} \right)^{\frac{p}{n+p-i}} \right\} \\
&\leqslant \inf_{Q_j \in \mathscr{K}_c^n} \left\{ \left( \omega_n^{\frac{p}{n-i}} n \prod_{j=1}^{n} \widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}} \right)^{\frac{p}{n+p-i}} \times \right. \\
&\quad \left. \left( \lambda \widetilde{W}_{-p,i}(\mathbf{K},\mathbf{Q})^{\frac{p}{n+p-i}} + \mu \widetilde{W}_{-p,i}(\mathbf{L},\mathbf{Q})^{\frac{p}{n+p-i}} \right) \right\} \\
&\leqslant \inf_{Q_j \in \mathscr{K}_c^n} \left\{ \omega_n^{\frac{p}{n-i}} n \prod_{j=1}^{n} \widetilde{W}_i(Q_j^*)^{-\frac{p}{n^2}} \widetilde{W}_{-p,i}(\lambda \circ \mathbf{K} \widetilde{+}_p \mu \circ \mathbf{L}, \mathbf{Q}) \right\}^{\frac{p}{n+p-i}} \\
&= \widetilde{G}_{-p,i}(\lambda \circ \mathbf{K} \widetilde{+}_p \mu \circ \mathbf{L})^{\frac{p}{n+p-i}}.
\end{aligned}$$

We know that equality in (1.10) holds if and only if $\mathbf{K}$ and $\mathbf{L}$ are dilates. □

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