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SHARP BOUNDS FOR SÁNDOR–YANG MEANS IN TERMS OF QUADRATIC MEAN

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Abstract. In the article, we find the best possible parameters α, β, λ, $\mu \in (1/2,1)$ such that the double inequalities

$$Q[\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a] < R_{QA}(a,b) < Q[\beta a+(1-\beta)b, \beta b+(1-\beta)a],$$

$$Q[\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a] < R_{AQ}(a,b) < Q[\mu a+(1-\mu)b, \mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a \neq b$, where $Q(a,b)=\sqrt{(a^2+b^2)/2}$ is the quadratic mean, and $R_{QA}(a,b)$ and $R_{AQ}(a,b)$ are two Sándor-Yang means.

1. Introduction

Let $a,b>0$ with $a \neq b$. Then the Schwab-Borchardt mean $SB(a,b)$ [1–5] is defined by

$$SB(a,b)=\begin{cases} \frac{\sqrt{b^2-a^2}}{\arccos(a/b)} & (a<b), \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & (a>b), \end{cases}$$

where $\cosh^{-1}(x)=\log(x+\sqrt{x^2-1})$ is the inverse hyperbolic cosine function.

It is well known that the Schwab-Borchardt mean $SB(a,b)$ is strict increasing, nonsymmetric and homogeneous of degree one with respect to its variables a and b. Many classical bivariate means are the special cases of the Schwab-Borchardt mean. For example,

$$P(a,b)=\frac{a-b}{2\arcsin\left(\frac{a-b}{a+b}\right)}=SB(G(a,b),A(a,b)),$$

$$T(a,b)=\frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)}=SB(A(a,b),Q(a,b)), \tag{1.1}$$

$$NS(a,b)=\frac{a-b}{2\sinh^{-1}\left(\frac{a-b}{a+b}\right)}=SB(Q(a,b),A(a,b)), \tag{1.2}$$

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$$L(a,b) = \frac{a-b}{\log a - \log b} = SB(A(a,b), G(a,b))$$

are respectively the first Seiffert mean [6–11], second Seiffert mean [12–18], Neuman-Sándor mean [19–21] and logarithmic mean [22–30], where $G(a,b) = \sqrt{ab}$ is the geometric mean, $A(a,b) = (a+b)/2$ is arithmetic mean, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ is the quadratic mean and $\sinh^{-1}(x) = \log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function.

The bivariate means have many important applications in the theory of special functions. For instance, the modulus of the plane Grötzsch ring and the complete elliptic integral $\mathscr{K}(r)$ [31–39] of the first kind can be expressed in terms of the Gaussian arithmetic-geometric mean AGM [40–44], and the Toader mean [45–47]

$$TD(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2 t + b^2 \sin^2(t)}dt$$

give the formula of the perimeter of an ellipse and it can be expressed by the complete elliptic integral $\mathscr{E}(r)$ [48–57] of the second kind. Recently, the bivariate means have attracted the attention of many researchers.

Let $X = X(a,b)$ and $Y = Y(a,b)$ be two symmetric bivariate means of a and b. Then the Sándor-Yang mean $R_{XY}(a,b)$ is defined by

$$R_{XY}(a,b) = Y(a,b)e^{\frac{X(a,b)}{SB(X(a,b),Y(a,b))}-1}.$$

Neuman [58] proved that the inequalities

$$X(a,b) < R_{XY}(a,b) < R_{YX}(a,b) < Y(a,b) \tag{1.3}$$

for all $a,b>0$ with $a \neq b$ if $X(a,b) < Y(a,b)$.

In [59], Yang gave the explicit formulas for the Sándor-Yang means $R_{QA}(a,b)$ and $R_{AQ}(a,b)$ as follows:

$$R_{QA}(a,b) = A(a,b)e^{\frac{Q(a,b)}{NS(a,b)}-1}, \tag{1.4}$$

$$R_{AQ}(a,b) = Q(a,b)e^{\frac{A(a,b)}{T(a,b)}-1}. \tag{1.5}$$

Let $f(x) = Q[xa+(1-x)b, xb+(1-x)a]$ for $x \in [1/2,1]$. Then from (1.3) and $A(a,b) < Q(a,b)$ we clearly see that

$$f(1/2) = A(a,b) < R_{AQ}(a,b) < R_{QA}(a,b) < Q(a,b) = f(1) \tag{1.6}$$

for all $a,b>0$ with $a \neq b$.

Note that, the function $f(x)$ is strictly increasing on interval $[1/2,1]$ for fixed $a,b>0$ with $a \neq b$.

Motivated by inequality (1.6) and the monotonicity of the function $f(x)$ on interval $[1/2,1]$, it is natural to ask what are the best possible parameters $\alpha,\beta,\lambda,\mu \in (1/2,1)$ such that the double inequalities

$$Q[\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a] < R_{QA}(a,b) < Q[\beta a+(1-\beta)b, \beta b+(1-\beta)a],$$

$$Q[\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a] < R_{AQ}(a,b) < Q[\mu a+(1-\mu)b, \mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a \neq b$. The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results, we need two lemmas which we present in this section.

LEMMA 2.1. *Let* $p \in (1/2, 1)$ *and* $f(x)$ *be defined by*

$$f(x) = \sinh^{-1}(x) - \frac{x\sqrt{1+x^2}}{1+(1-2p)^2x^2}. \tag{2.1}$$

The the following statements are true:

(1) $f(x) > 0$ *for all* $x \in (0,1)$ *if* $p = 1/2 + \sqrt{6}/6 = 0.9082\cdots$.

(2) *There exists* $\lambda_1 \in (0,1)$ *such that* $f(x) < 0$ *for* $x \in (0, \lambda_1)$ *and* $f(x) > 0$ *for* $x \in (\lambda_1, 1)$ *if* $p = 1/2 + \sqrt{(3+2\sqrt{2})^{\sqrt{2}} - e^2}/(2e) = 0.8990\cdots$.

Proof. It follows from (2.1) that

$$f(0) = 0, \tag{2.2}$$

$$f'(x) = \frac{x^2}{[1+(1-2p)^2x^2]^2\sqrt{1+x^2}} f_1(x), \tag{2.3}$$

where

$$f_1(x) = (1-2p)^4x^2 + 12p^2 - 12p + 1. \tag{2.4}$$

(1) If $p = 1/2 + \sqrt{6}/6$, then (2.4) leads to

$$f_1(x) = \frac{4}{9}x^2 > 0 \tag{2.5}$$

for $x \in (0,1)$.

Therefore, Lemma 2.1(1) follows easily from (2.2), (2.3) and (2.5).

(2) If $p = 1/2 + \sqrt{(3+2\sqrt{2})^{\sqrt{2}} - e^2}/(2e)$, then (2.1) and (2.4) lead to

$$f(1) = \log(1+\sqrt{2}) - \frac{\sqrt{2}e^2}{(3+2\sqrt{2})^{\sqrt{2}}} = 0.0174\cdots > 0, \tag{2.6}$$

$$f_1(0) = 12p^2 - 12p + 1 = -0.0895\cdots < 0, \tag{2.7}$$

$$f_1(1) = 16p^4 - 32p^3 + 36p^2 - 20p + 2 = 0.3159\cdots > 0, \tag{2.8}$$

$$f_1'(x) = 2(1-2p)^4 > 0 \tag{2.9}$$

for $x \in (0,1)$.

From (2.3) and (2.7)–(2.9) we clearly see that there exists $\lambda_0 \in (0,1)$ such that $f(x)$ is strictly decreasing on $(0, \lambda_0]$ and strictly increasing on $[\lambda_0, 1)$.

We divide the proof into two cases.

Case 1. $x \in (0, \lambda_0]$. Then $f(x) < 0$ follows from (2.2) and the monotonicity of $f(x)$ on the interval $(0, \lambda_0]$.

Case 2. $x \in [\lambda_0, 1)$. Then it follows from Case 1 that

$$f(\lambda_0) < 0. \tag{2.10}$$

Therefore, there exists $\lambda_1 \in (\lambda_0, 1)$ such that $f(x) < 0$ for $x \in [\lambda_0, \lambda_1)$ and $f(x) > 0$ for $x \in (\lambda_1, 1)$ follows from (2.6) and (2.10) together with the monotonicity of the function $f(x)$ on the interval $[\lambda_0, 1)$. □

LEMMA 2.2. *Let $p \in (1/2, 1)$ and $g(x)$ be defined by*

$$g(x) = \arctan(x) - \frac{x}{1+(1-2p)^2x^2}. \tag{2.11}$$

Then the following statements are true:

(1) *$g(x) > 0$ for all $x \in (0,1)$ if $p = 1/2+\sqrt{3}/6 = 0.7886\cdots$.*

(2) *There exists $\mu_1 \in (0,1)$ such that $g(x) < 0$ for $x \in (0, \mu_1)$ and $g(x) > 0$ for $x \in (\mu_1, 1)$ if $p = 1/2+\sqrt{2e^{\pi/2-2}-1}/2 = 0.7747\cdots$.*

Proof. From (2.11) we get

$$g(0) = 0, \tag{2.12}$$

$$g'(x) = \frac{2x^2}{\left[1+(1-2p)^2x^2\right]^2(1+x^2)} g_1(x), \tag{2.13}$$

where

$$g_1(x) = \left(8p^4 - 16p^3 + 14p^2 - 6p + 1\right)x^2 + 6p^2 - 6p + 1. \tag{2.14}$$

(1) If $p = 1/2+\sqrt{3}/6$, then (2.14) becomes

$$g_1(x) = \frac{2}{9}x^2. \tag{2.15}$$

Therefore, Lemma 2.2(1) follows easily from (2.12), (2.13) and (2.15).

(2) If $p = 1/2+\sqrt{2e^{\pi/2-2}-1}/2$, then (2.11) and (2.14) lead to

$$g(1) = \frac{\pi}{4} - \frac{e^2}{2e^{\pi/2}} = 0.0713\cdots > 0, \tag{2.16}$$

$$g_1(0) = 6p^2 - 6p + 1 = -0.0469\cdots < 0, \tag{2.17}$$

$$g_1(1) = 8p^4 - 16p^3 + 20p^2 - 12p + 2 = 0.9245\cdots > 0. \tag{2.18}$$

Note that

$$8p^4 - 16p^3 + 14p^2 - 6p + 1 = 0.1966\cdots > 0. \tag{2.19}$$

From (2.13), (2.14) and (2.17)–(2.19) we know that there exists $\mu_0 \in (0,1)$ such that $g(x)$ is strictly decreasing on $(0, \mu_0]$ and strictly increasing on $[\mu_0, 1)$.

We divide the proof into two cases.

Case 1. $x \in (0, \mu_0]$. Then $g(x) < 0$ follows from (2.12) and the monotonicity of the function $g(x)$ on the interval $(0, \mu_0]$.

Case 2. $x \in [\mu_0, 1)$. Then it follows from Case 1 that

$$g(\mu_0) < 0. \tag{2.20}$$

From (2.16) and (2.20) together with the monotonicity of $g(x)$ on the interval $[\mu_0, 1)$ we clearly see that there exists $\mu_1 \in (\mu_0, 1)$ such that $g(x) < 0$ for $x \in [\mu_0, \mu_1)$ and $g(x) > 0$ for $x \in (\mu_1, 1)$. □

3. Main results

THEOREM 3.1. *Let* $\alpha,\beta\in(1/2,1)$. *Then the double inequality*

$$Q[\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a]<R_{QA}(a,b)<Q[\beta a+(1-\beta)b,\beta b+(1-\beta)a]$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only* $\alpha\leqslant 1/2+\sqrt{(3+2\sqrt{2})^{\sqrt{2}}-e^2}/(2e)$ *and* $\beta\geqslant 1/2+\sqrt{6}/6$.

Proof. Since $R_{QA}(a,b)$ and $Q(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a>b>0$. Let $x=(a-b)/(a+b)\in(0,1)$ and $p\in(1/2,1)$. Then (1.2) and (1.4) lead to

$$\log\frac{Q[pa+(1-p)b,pb+(1-p)a]}{R_{QA}(a,b)} \tag{3.1}$$

$$=\frac{1}{2}\log\left[1+(1-2p)^2x^2\right]-\frac{\sqrt{1+x^2}\sinh^{-1}(x)}{x}+1.$$

Let

$$F(x)=\frac{1}{2}\log\left[1+(1-2p)^2x^2\right]-\frac{\sqrt{1+x^2}\sinh^{-1}(x)}{x}+1. \tag{3.2}$$

Then elaborated computations lead to

$$F(0)=0, \tag{3.3}$$

$$F(1)=\frac{1}{2}\log\left[1+(1-2p)^2\right]-\sqrt{2}\log(1+\sqrt{2})+1, \tag{3.4}$$

$$F'(x)=\frac{1}{x^2\sqrt{1+x^2}}f(x), \tag{3.5}$$

where $f(x)$ is defined by (2.1).

We divide the proof into four cases.

Case 1. $p=1/2+\sqrt{6}/6$. Then Lemma 2.1(1), (3.3) and (3.5) lead to the conclusion that

$$F(x)>0 \tag{3.6}$$

for all $x\in(0,1)$, and

$$R_{QA}(a,b)<Q[pa+(1-p)b,pb+(1-p)a]$$

follows from (3.1), (3.2) and (3.6).

Case 2. $1/2<p<1/2+\sqrt{6}/6$. Then (3.2) and the power series expansion lead to

$$F(x)=2\left[p-\left(\frac{1}{2}+\frac{\sqrt{6}}{6}\right)\right]\left[p-\left(\frac{1}{2}-\frac{\sqrt{6}}{6}\right)\right]x^2+o\left(x^2\right)\quad(x\to 0^+). \tag{3.7}$$

Equations (3.1), (3.2) and (3.7) imply that there exists $\delta_1\in(0,1)$ such that

$$R_{QA}(a,b)>Q[pa+(1-p)b,pb+(1-p)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_1)$.

Case 3. $p = 1/2 + \sqrt{(3+2\sqrt{2})^{\sqrt{2}} - e^2}/(2e)$. Then it follows from (3.5) and Lemma 2.1(2) that there exists $\lambda_1 \in (0,1)$ such that $F(x)$ is strictly decreasing on $(0, \lambda_1)$ and strictly increasing on $(\lambda_1, 1)$.

Note that (3.4) becomes

$$F(1) = 0. \tag{3.8}$$

Equations (3.3) and (3.8) together with the piecewise monotonicity of the function $F(x)$ on the interval $(0,1)$ lead to the conclusion that

$$F(x) < 0 \tag{3.9}$$

for all $x \in (0,1)$.

Therefore,

$$R_{QA}(a,b) > Q[pa + (1-p)b, pb + (1-p)a]$$

follows from (3.1), (3.2) and (3.9).

Case 4. $1/2 + \sqrt{(3+2\sqrt{2})^{\sqrt{2}} - e^2}/(2e) < p < 1$. Then (3.4) leads to

$$F(1) > 0. \tag{3.10}$$

Equations (3.1) and (3.2) together with inequality (3.10) imply that there exists $\delta_2 \in (0,1)$ such that

$$R_{QA}(a,b) < Q[pa + (1-p)b, pb + (1-p)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. $\square$

THEOREM 3.2. *Let $\lambda, \mu \in (1/2, 1)$. Then the double inequality*

$$Q[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a] < R_{AQ}(a,b) < Q[\mu a + (1-\mu)b, \mu b + (1-\mu)a]$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leqslant 1/2 + \sqrt{2e^{\pi/2-2} - 1}/2$ and $\mu \geqslant 1/2 + \sqrt{3}/6$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$ and $p \in (1/2, 1)$. Then (1.1) and (1.5) lead to

$$\log \frac{Q[pa + (1-p)b, pb + (1-p)a]}{R_{AQ}(a,b)} \tag{3.11}$$
$$= \frac{1}{2} \log\left[1 + (2p-1)^2 x^2\right] - \frac{1}{2} \log\left(1 + x^2\right) - \frac{\arctan(x)}{x} + 1.$$

Let

$$G(x) = \frac{1}{2} \log\left[1 + (2p-1)^2 x^2\right] - \frac{1}{2} \log\left(1 + x^2\right) - \frac{\arctan(x)}{x} + 1. \tag{3.12}$$

Then elaborated computations lead to

$$G(0) = 0, \tag{3.13}$$

$$G(1) = \frac{1}{2}\log\left[1+(2p-1)^2\right] - \frac{\pi}{4} + 1 - \frac{1}{2}\log 2, \tag{3.14}$$

$$G'(x) = \frac{1}{x^2\sqrt{1+x^2}}g(x), \tag{3.15}$$

where $g(x)$ is defined by (2.11).

Next, we divide the proof into four cases.

Case 1. $p = 1/2+\sqrt{3}/6$. Then it follows from Lemma 2.2(1) and (3.13) together with (3.15) that

$$G(x) > 0 \tag{3.16}$$

for all $x \in (0,1)$. Therefore,

$$R_{AQ}(a,b) < Q[pa+(1-p)b, pb+(1-p)a]$$

follows from (3.11), (3.12) and (3.16).

Case 2. $1/2 < p < 1/2+\sqrt{3}/6$. Then (3.12) and the power series expansion lead to

$$G(x) = 2\left[p - \left(\frac{1}{2}+\frac{\sqrt{3}}{6}\right)\right]\left[p - \left(\frac{1}{2}-\frac{\sqrt{3}}{6}\right)\right]x^2 + o\left(x^2\right) \quad \left(x \to 0^+\right). \tag{3.17}$$

Equations (3.11) and (3.12) together with (3.17) imply that there exists $\delta_3 \in (0,1)$ such that

$$R_{AQ}(a,b) > Q[pa+(1-p)b, pb+(1-p)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0,\delta_3)$.

Case 3. $p = 1/2+\sqrt{2e^{\pi/2-2}-1}/2$. Then Lemma 2.2(2) and (3.15) lead to the conclusion that there exists $\mu_1 \in (0,1)$ such that $G(x)$ is strictly decreasing on $(0,\mu_1]$ and strictly increasing on $[\mu_1,1)$.

Note that (3.14) becomes

$$G(1) = 0. \tag{3.18}$$

It follows from (3.13) and (3.18) together with the piecewise monotonicity of $G(x)$ on the interval $(0,1)$ that

$$G(x) < 0 \tag{3.19}$$

for all $x \in (0,1)$. Therefore,

$$R_{AQ}(a,b) > Q[pa+(1-p)b, pb+(1-p)a]$$

for all $a,b > 0$ with $a \neq b$ follows from (3.11) and (3.12) together with (3.19).

Case 4. $1/2+\sqrt{2e^{\pi/2-2}-1}/2 < p < 1$. Then (3.14) leads to

$$G(1) > 0. \tag{3.20}$$

Equations (3.11) and (3.12) together with inequality (3.20) imply that there exists $\delta_4 \in (0,1)$ such that

$$R_{AQ}(a,b) < Q[pa+(1-p)b, pb+(1-p)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_4,1)$. $\square$

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