Journal of Mathematical Inequalities
Volume 12, Number 4 (2018), 1173–1188 doi:10.7153/jmi-2018-12-89

COMMUTATORS OF MARCINKIEWICZ INTEGRALS WITH ROUGH KERNELS ON HERZ–TYPE HARDY SPACES WITH VARIABLE EXPONENT

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(*Communicated by S. Li*)

Abstract. Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s \geqslant 1$ be homogeneous functions of degree zero and b be BMO functions. In this paper, we obtain some boundedness of the Marcinkiewicz integral operator μ_Ω and its commutator $[b,\mu_\Omega]$ on Herz-type Hardy spaces with variable exponent.

1. Introduction

The theory of function spaces with variable exponent has extensively studied by researchers since the work of Kováčik and Rákosník [9] appeared in 1991, see [2, 4] and the references therein. In [17], the authors introduced the Herz-type Hardy spaces with variable exponent and gave their atomic decomposition characterizations. In [1–4] and [16], the authors proved the boundedness of some integral operators on variable L^p spaces, respectively.

Suppose that S^{n-1} denote the unit sphere in $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in \mathrm{Lip}_\beta(\mathrm{S}^{n-1})$ for $0 < \beta \leqslant 1$ be homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. In 1958, Stein [14] introduced the Marcinkiewicz integral related to the Littlewood-Paley g function on $\mathbb{R}^n$ as follows

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It was shown that μ_Ω is of type (p,p) for $1 < p \leqslant 2$ and of weak type $(1,1)$.

Mathematics subject classification (2010): 42B20, 42B35.
Keywords and phrases: Herz-type Hardy space, variable exponent, commutator, Marcinkiewicz integral.

Let b be a locally integrable function on $\mathbb{R}^n$, the commutator generated by the Marcinkiewicz integral μ_Ω and b is defined by

$$[b,\mu_\Omega](f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}.$$

Motivated by [10, 15], we will study the boundedness for the Marcinkiewicz integral operator μ_Ω and its commutator $[b,\mu_\Omega]$ on the Herz-type Hardy spaces with variable exponent, where $\Omega\in L^s(\mathrm{S}^{n-1})$ for $s\geqslant 1$.

Let us explain the outline of this article. In Section 2, we first briefly recall some standard notations and lemmas in variable function spaces. In Section 3, we will prove the boundedness of the Marcinkiewicz integral operators μ_Ω on Herz-type Hardy spaces with variable exponent. We will establish the boundedness of the commutator $[b,\mu_\Omega]$ on Herz-type Hardy spaces with variable exponent in Section 4. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A\subset\mathbb{R}^n$ by $|A|$ and χ_A respectively. The notation $f\approx g$ means that there exist constants $C_1,C_2>0$ such that $C_1g\leqslant f\leqslant C_2g$.

2. Variable function spaces

Given an open set $E\subset\mathbb{R}^n$, and a measurable function $p(\cdot):E\longrightarrow[1,\infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda>0$,

$$\int_E\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx<\infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)}=\inf\left\{\lambda>0:\int_E\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx\leqslant 1\right\}.$$

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x)=p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega):=\{f:f\in L^{p(\cdot)}(E)\text{ for all compact subsets }E\subset\Omega\}.$$

Define $\mathscr{P}^0(E)$ to be set of $p(\cdot):E\longrightarrow(0,\infty)$ such that

$$p^-=\operatorname{ess\,inf}\{p(x):x\in E\}>0,\quad p^+=\operatorname{ess\,sup}\{p(x):x\in E\}<\infty.$$

Define $\mathscr{P}(E)$ to be set of $p(\cdot):E\longrightarrow[1,\infty)$ such that

$$p^-=\operatorname{ess\,inf}\{p(x):x\in E\}>1,\quad p^+=\operatorname{ess\,sup}\{p(x):x\in E\}<\infty.$$

Denote $p'(x)=p(x)/(p(x)-1)$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$. Let $\mathscr{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In variable L^p spaces there are some important lemmas as follows.

LEMMA 2.1. *If $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x)-p(y)| \leqslant \frac{C}{-\log(|x-y|)}, \quad |x-y| \leqslant 1/2 \tag{2.1}$$

and

$$|p(x)-p(y)| \leqslant \frac{C}{\log(|x|+e)}, \quad |y| \geqslant |x|, \tag{2.2}$$

then $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, that is the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

LEMMA 2.2. [9] *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

LEMMA 2.3. [7] *Let $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leqslant C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1}$$

and

$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Throughout this paper δ_2 is the same as in Lemma 2.3.

LEMMA 2.4. [7] *Suppose $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leqslant C.$$

Next we recall the definitions of the Herz spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

DEFINITION 2.1. [7] Let $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$ and $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

In [17], the authors gave the definition of Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and the atomic decomposition characterizations. $\mathscr{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathscr{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathscr{S}(\mathbb{R}^n)$. Let $G_N(f)(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N(f)(x) = \sup_{\phi \in \mathscr{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathscr{A}_N = \{\phi \in \mathscr{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leqslant N} |x^\alpha D^\beta \phi(x)| \leqslant 1\}$ and $N > n+1$, ϕ^*_∇ is the non-tangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

DEFINITION 2.2. [17] Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $N > n+1$.

(i) The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : G_N(f)(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Herz-type Hardy space with variable exponent $HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)=\left\{f\in\mathscr{S}'(\mathbb{R}^n):G_N(f)(x)\in K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)\right\}$$

and $\|f\|_{HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}=\|G_N(f)\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}$.

For $x\in\mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x.

DEFINITION 2.3. [17] Let $n\delta_2\leqslant\alpha<\infty$, $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and non-negative integer $s\geqslant[\alpha-n\delta_2]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,q(\cdot))$-atom, if it satisfies

(1) supp $a\subset B(0,r)=\{x\in\mathbb{R}^n:|x|<r\}$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leqslant|B(0,r)|^{-\alpha/n}$.

(3) $\int_{\mathbb{R}^n}a(x)x^\beta dx=\int_{\mathbb{R}^n}a(x)b(x)x^\beta dx=0,\ |\beta|\leqslant s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,q(\cdot))$-atom of restricted type, if it satisfies the conditions (2), (3) above and

$(1)'$ supp $a\subset B(0,r),\ r\geqslant 1$.

If $r=2^k$ for some $k\in\mathbb{Z}$ in Definition 1.3, then the corresponding central $(\alpha,q(\cdot))$-atom is called a dyadic central $(\alpha,q(\cdot))$-atom.

LEMMA 2.5. [17] *Let $n\delta_2\leqslant\alpha<\infty$, $0<p<\infty$ and $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. Then $f\in H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$) if and only if*

$$f=\sum_{k=-\infty}^{\infty}\lambda_k a_k\left(\text{or}\sum_{k=0}^{\infty}\lambda_k a_k\right),\quad\text{in the sense of }\mathscr{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha,q(\cdot))$-atom (or central $(\alpha,q(\cdot))$-atom of restricted type) with support contained in B_k and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$ (or $\sum_{k=0}^{\infty}|\lambda_k|^p<\infty$). Moreover,

$$\|f\|_{H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}\approx\inf\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}\left(\text{or}\quad\|f\|_{HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}\approx\inf\left(\sum_{k=0}^{\infty}|\lambda_k|^p\right)^{1/p}\right),$$

where the infimum is taken over all above decompositions of f.

3. Estimate for the Marcinkiewicz integral operator

In this section we will prove the boundedness of the Marcinkiewicz integral operators μ_Ω on Herz-type Hardy spaces with variable exponent.

A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to A_p $(1<p<\infty)$, if there is a constant $C>0$ such that

$$\sup_Q\left(\frac{1}{|Q|}\int_Q\omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q\omega(x)^{1-p'}dx\right)^{p-1}\leqslant C<\infty,$$

where $p' = p/(p-1)$, Q denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

The weighted (L^p, L^p) boundedness of μ_Ω have been proved by Ding, Fan and Pan [5].

LEMMA 3.1. [5] *Suppose that $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s>1)$ satisfying (1.1). If $\omega \in A_{p/s'}$, $s' < p < \infty$, then there is a constant C, independent of f, such that*

$$\int_{\mathbb{R}^n} |\mu_\Omega(f)(x)|^p \omega(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx.$$

LEMMA 3.2. [3] *Let $\mathscr{F}$ denote a family of ordered pairs of non-negative, measurable functions (f,g). Given a family $\mathscr{F}$ and an open set $E \subset \mathbb{R}^n$, assume that for some p_0, $0 < p_0 < \infty$ and for every $\omega \in A_\infty$,*

$$\int_E f(x)^{p_0}\omega(x)dx \leqslant C_0 \int_E g(x)^{p_0}\omega(x)dx, \quad (f,g) \in \mathscr{F}.$$

Given $p(\cdot) \in \mathscr{P}^0(E)$ such that $p(\cdot)$ satisfies (2.1) and (2.2) in Lemma 2.1. Then for all $(f,g) \in \mathscr{F}$ such that $f \in L^{p(\cdot)}(E)$,

$$\|f\|_{L^{p(\cdot)}(E)} \leqslant C\|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{p/s'} \subset A_\infty$, by Lemma 3.1 and Lemma 3.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the Marcinkiewicz integral operators μ_Ω.

Before stating our result, let us recall the definition of the L^s-Dini condition. We say that satisfies the L^s-Dini condition if $\Omega(x') \in L^s(\mathrm{S}^{n-1})$ with $s \geqslant 1$ is homogeneous of degree zero on $\mathbb{R}^n$, and

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta < \infty,$$

where $\omega_s(\delta)$ denotes the integral modulus of continuity of order s of Ω defined by

$$\omega_s(\delta) = \sup_{\|\rho\| \leqslant \delta} \left(\int_{\mathrm{S}^{n-1}} \left|\Omega(\rho x') - \Omega(x')\right|^s d\sigma(x') \right)^{1/s}$$

and ρ is a rotation on S^{n-1} and $\|\rho\| = \sup_{x' \in \mathrm{S}^{n-1}} |\rho x' - x'|$.

Next, we will give the corresponding result about the operator μ_Ω on Herz-type Hardy spaces with variable exponent.

THEOREM 3.1. *Suppose that $0 < \beta \leqslant 1$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfies conditions (2.1) and (2.2) in Lemma 2.1 with $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > q^+)$ and satisfies*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\beta}} d\delta < \infty.$$

Let $0 < p_1 \leqslant p_2 < \infty$ and $n\delta_2 \leqslant \alpha < n\delta_2 + \beta$ (or $0 < \max(n\delta_2, \alpha_2) \leqslant \alpha_1 < n\delta_2 + \beta$). Then μ_Ω is bounded from $H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha_1,p_1}_{q(\cdot)}(\mathbb{R}^n)$) to $\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha_2,p_2}_{q(\cdot)}(\mathbb{R}^n)$).

In the proof of Theorem 3.1, we also need the following lemmas.

LEMMA 3.3. [2] *Given E and $p(\cdot)\in\mathscr{P}(E)$, let $f:E\times E\to\mathbb{R}$ be a measurable function (with respect to product measure) such that for almost every $y\in E$, $f(\cdot,y)\in L^{p(\cdot)}(E)$. Then*

$$\left\|\int_E f(\cdot,y)dy\right\|_{L^{p(\cdot)}(E)}\leqslant C\int_E \|f(\cdot,y)\|_{L^{p(\cdot)}(E)}dy.$$

LEMMA 3.4. [13] *Define a variable exponent $\tilde{q}(\cdot)$ by $\frac{1}{p(x)}=\frac{1}{\tilde{q}(x)}+\frac{1}{q}\quad(x\in\mathbb{R}^n)$. Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$$

for all measurable functions f and g.

LEMMA 3.5. [4] *Let $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ satisfies conditions (2.1) and (2.2) in Lemma 2.1. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\approx\begin{cases}|Q|^{\frac{1}{p(x)}} & \text{if}\,|Q|\leqslant 2^n \text{ and } x\in Q,\\ |Q|^{\frac{1}{p(\infty)}} & \text{if}\,|Q|\geqslant 1\end{cases}$$

for every cube (or ball) $Q\subset\mathbb{R}^n$, where $p(\infty)=\lim\limits_{x\to\infty}p(x)$.

LEMMA 3.6. [11] *Suppose that Ω satisfies the L^s-Dini condition ($1\leqslant s<\infty$). Then for any $R>0$ and $x\in\mathbb{R}^n$, when $|y|<R/2$, there is a constant $C>0$ such that*

$$\left(\int_{R<|x|<2R}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\right|^s dx\right)^{1/s}\leqslant CR^{\frac{n}{s}-n}\left\{\frac{|y|}{R}+\int_{|y|/2R<\delta<|y|/R}\frac{\omega_s(\delta)}{\delta}d\delta\right\}.$$

Proof of Theorem 3.1. We only prove homogeneous case. In [18], the authors proved $K^{\alpha_1,p_2}_{q(\cdot)}(\mathbb{R}^n)\subset K^{\alpha_2,p_2}_{q(\cdot)}(\mathbb{R}^n)$ for $0<\alpha_2\leqslant\alpha_1$. So the non-homogeneous case can be proved in the same way. Let $f\in H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$. By Lemma 2.5 we get $f=\sum\limits_{j=-\infty}^{\infty}\lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}\approx\inf(\sum\limits_{j=-\infty}^{\infty}|\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha,q(\cdot))$-atom with the support B_j. Note that $p_1\leqslant p_2$, we have

$$\begin{aligned}\|\mu_\Omega(f)\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)}&=\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_2}\|\mu_\Omega(f)\chi_k\|^{p_2}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{p_1/p_2}\\&\leqslant\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|\mu_\Omega(f)\chi_k\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\leqslant C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|\mu_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\&\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|\mu_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\&=:I_1+I_2.\end{aligned}\tag{3.1}$$

Since μ_Ω is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, we have

$$\begin{aligned} I_2 &\leqslant C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\ &\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{\infty}|\lambda_j|2^{(k-j)\alpha}\right)^{p_1} \\ &\leqslant C\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}. \end{aligned} \tag{3.2}$$

Now we estimate I_1. We consider

$$\begin{aligned} |\mu_\Omega(a_j)(x)| &\leqslant \left(\int_0^{|x|}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\ &\quad+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\ &=: I_{11}+I_{12}. \end{aligned}$$

Note that $x\in A_k$, $y\in A_j$ and $j\leqslant k-2$. So we know that $|x-y|\sim|x|$, and by mean value theorem we have

$$\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|\leqslant\frac{|y|}{|x-y|^3}. \tag{3.3}$$

By (3.3), the Minkowski inequality and the vanishing moments of a_j we have

$$\begin{aligned} I_{11} &\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||a_j(y)|\left(\int_{|x-y|}^{|x|}\frac{dt}{t^3}\right)^{1/2}dy \\ &\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||a_j(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{1/2}dy \\ &\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||a_j(y)|\frac{|y|^{1/2}}{|x-y|^{3/2}}dy \\ &\leqslant C\frac{2^{j/2}}{|x|^{1/2}}\int_{A_j}\left|\frac{\Omega(x-y)}{|x-y|^{n}}-\frac{\Omega(x)}{|x|^{n}}\right||a_j(y)|dy \\ &\leqslant C2^{(j-k)/2}\int_{A_j}\left|\frac{\Omega(x-y)}{|x-y|^{n}}-\frac{\Omega(x)}{|x|^{n}}\right||a_j(y)|dy. \end{aligned}$$

Similarly, we consider I_{12}. Noting that $|x-y|\sim|x|$, by the Minkowski inequality and the vanishing moments of a_j we have

$$\begin{aligned} I_{12} &\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||a_j(y)|\left(\int_{|x|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy \\ &\leqslant C\int_{A_j}\left|\frac{\Omega(x-y)}{|x-y|^{n}}-\frac{\Omega(x)}{|x|^{n}}\right||a_j(y)|dy. \end{aligned}$$

So we have

$$|\mu_\Omega(a_j)(x)| \leqslant C\int_{A_j}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\right||a_j(y)|dy.$$

By Lemma 3.3 and the generalized Hölder inequality we have

$$\begin{aligned}\|\mu_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant C\int_{A_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}|a_j(y)|dy\\ &\leqslant C\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Noting $s>q^+$, we denote $\tilde{q}(\cdot)>1$ and $\frac{1}{q(x)}=\frac{1}{\tilde{q}(x)}+\frac{1}{s}$. By Lemma 3.4 we have

$$\begin{aligned}&\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leqslant\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\|\chi_k(\cdot)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

When $|B_k|\leqslant 2^n$ and $x_k\in B_k$, by Lemma 3.5 we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\approx|B_k|^{\frac{1}{\tilde{q}(x_k)}}\approx\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}}.$$

When $|B_k|\geqslant 1$ we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\approx|B_k|^{\frac{1}{\tilde{q}(\infty)}}\approx\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}}.$$

So we obtain $\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\approx\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}}$.

Meanwhile, by Lemma 3.6 we have

$$\begin{aligned}&\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\\ &\leqslant 2^{(k-1)(\frac{n}{s}-n)}\left\{\frac{|y|}{2^k}+\int_{|y|/2^k}^{|y|/2^{k-1}}\frac{\omega_s(\delta)}{\delta}d\delta\right\}\\ &\leqslant 2^{(k-1)(\frac{n}{s}-n)}\left(2^{j-k+1}+2^{(j-k+1)\beta}\int_0^1\frac{\omega_s(\delta)}{\delta}d\delta\right)\\ &\leqslant C2^{(k-1)(\frac{n}{s}-n)}2^{(j-k)\beta}.\end{aligned}$$

So by Lemma 2.3 and Lemma 2.4 we have

$$\begin{aligned}\|\mu_\Omega(a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant C2^{(k-1)(\frac{n}{s}-n)}2^{(j-k)\beta}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-kn+(j-k)\beta}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-kn+(j-k)\beta}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left(|B_k|\|\chi_{B_k}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\right)\\ &\leqslant C2^{(j-k)\beta}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\ &\leqslant C2^{(j-k)(n\delta_2+\beta)}\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
I_1 &\leqslant C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2+\beta)}|\lambda_j|\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\
&\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)(n\delta_2+\beta-\alpha)}\right)^{p_1}.
\end{aligned}
$$

If $1<p_1<\infty$, take $1/p_1+1/p_1'=1$. Since $n\delta_2+\beta-\alpha>0$, by the Hölder inequality we have

$$
\begin{aligned}
I_1 &\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(n\delta_2+\beta-\alpha)p_1/2}\right)\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2+\beta-\alpha)p_1'/2}\right)^{p_1/p_1'}\\
&\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(n\delta_2+\beta-\alpha)p_1/2}\right)\\
&= C\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2+\beta-\alpha)p_1/2}\right)\\
&\leqslant C\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.
\end{aligned}
\tag{3.4}
$$

If $0<p_1\leqslant 1$, then we have

$$
\begin{aligned}
I_1 &\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(n\delta_2+\beta-\alpha)p_1}\right)\\
&= C\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2+\beta-\alpha)p_1}\right)\\
&\leqslant C\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.
\end{aligned}
\tag{3.5}
$$

Therefore, by (3.1)–(3.5) we complete the proof of Theorem 3.1.

4. BMO estimate for the commutator of Marcinkiewicz integral operator

Let us first recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$
\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|dx<\infty,
$$

where $f_Q=|Q|^{-1}\int_Q f(y)dy$, the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

Let $b\in\mathrm{BMO}(\mathbb{R}^n)$. The weighted (L^p,L^p) boundedness of $[b,\mu_\Omega]$ have been proved by Ding, Lu and Yabuta[6].

LEMMA 4.1. [6] *Suppose that* $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s>1)$ *satisfying (1.1). If* $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $\omega \in A_{p/s'}$, $s'<p<\infty$, *then there is a constant* C, *independent of* f, *such that*

$$\int_{\mathbb{R}^n} |[b,\mu_\Omega](f)(x)|^p \omega(x)dx \leqslant C\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx.$$

By Lemma 4.1 and Lemma 3.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b,\mu_\Omega]$.

Next, we will give the corresponding result about the commutator $[b,\mu_\Omega]$ on Herz-type Hardy spaces with variable exponent.

THEOREM 4.1. *Suppose that* $b\in \mathrm{BMO}(\mathbb{R}^n)$, $0<\beta\leqslant 1$, $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ *satisfies conditions (2.1) and (2.2) in Lemma 2.1 with* $\Omega\in L^s(\mathrm{S}^{n-1})$ $(s>q^+)$ *and satisfies*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\beta}}d\delta<\infty.$$

Let $0<p_1\leqslant p_2<\infty$ *and* $n\delta_2\leqslant\alpha<n\delta_2+\beta$ *(or* $0<\max(n\delta_2,\alpha_2)\leqslant\alpha_1<n\delta_2+\beta$*). Then* $[b,\mu_\Omega]$ *is bounded from* $H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ *(or* $HK^{\alpha_1,p_1}_{q(\cdot)}(\mathbb{R}^n)$*) to* $\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$ *(or* $K^{\alpha_2,p_2}_{q(\cdot)}(\mathbb{R}^n)$*).*

In the proof of Theorem 4.1, we also need the following lemma.

LEMMA 4.2. [8] *Let* $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, k *be a positive integer and* B *be a ball in* $\mathbb{R}^n$. *Then we have that for all* $b\in\mathrm{BMO}(\mathbb{R}^n)$ *and all* $j,i\in\mathbb{Z}$ *with* $j>i$,

$$\frac{1}{C}\|b\|_*^k \leqslant \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)^k\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|_*^k,$$

$$\|(b-b_{B_i})^k\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C(j-i)^k\|b\|_*^k\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i=\{x\in\mathbb{R}^n:|x|\leqslant 2^i\}$ *and* $B_j=\{x\in\mathbb{R}^n:|x|\leqslant 2^j\}$.

Proof of Theorem 4.1. Similar to Theorem 3.1, we only prove homogeneous case. Let $f\in H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$. By Lemma 2.5 we get $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}\approx \inf(\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha,q(\cdot))$-atom with the support B_j. Note that $p_1\leqslant p_2$, we have

$$\begin{aligned}\|[b,\mu_\Omega](f)\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_2}\|[b,\mu_\Omega](f)\chi_k\|^{p_2}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{p_1/p_2}\\ &\leqslant \sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|[b,\mu_\Omega](f)\chi_k\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|[b,\mu_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\
&\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|[b,\mu_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\
&=: J_1+J_2. \qquad (4.1)
\end{aligned}
$$

Since $[b,\mu_\Omega]$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, we have

$$
\begin{aligned}
J_2 &\leqslant C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\
&\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{\infty}|\lambda_j|2^{(k-j)\alpha}\right)^{p_1} \qquad (4.2)\\
&\leqslant C\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.
\end{aligned}
$$

Now we estimate J_1. We consider

$$
\begin{aligned}
|[b,\mu_\Omega](a_j)(x)| &\leqslant \left(\int_0^{|x|}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&=: J_{11}+J_{12}.
\end{aligned}
$$

Note that $x\in A_k$, $y\in A_j$ and $j\leqslant k-2$. So we know that $|x-y|\sim|x|$. By (2.3), the Minkowski inequality and the vanishing moments of a_j we have

$$
\begin{aligned}
J_{11} &\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||b(x)-b(y)||a_j(y)|\left(\int_{|x-y|}^{|x|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||b(x)-b(y)||a_j(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{1/2}dy\\
&\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||b(x)-b(y)||a_j(y)|\frac{|y|^{1/2}}{|x-y|^{3/2}}dy\\
&\leqslant C2^{(j-k)/2}\int_{A_j}\left|\frac{\Omega(x-y)}{|x-y|^{n}}-\frac{\Omega(x)}{|x|^{n}}\right||b(x)-b(y)||a_j(y)|dy.
\end{aligned}
$$

Similarly, we consider J_{12}. Noting that $|x-y|\sim|x|$, by the Minkowski inequality and the vanishing moments of a_j we have

$$
\begin{aligned}
J_{12} &\leqslant C\int_{\mathbb{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\right||b(x)-b(y)||a_j(y)|\left(\int_{|x|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leqslant C\int_{A_j}\left|\frac{\Omega(x-y)}{|x-y|^{n}}-\frac{\Omega(x)}{|x|^{n}}\right||b(x)-b(y)||a_j(y)|dy.
\end{aligned}
$$

So we have

$$|[b,\mu_\Omega](a_j)(x)| \leqslant C\int_{A_j}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\right||b(x)-b(y)||a_j(y)|dy.$$

By Lemma 3.3 and the Minkowski inequality we have

$$\begin{aligned}
\|[b,\mu_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant C\int_{A_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right||b(\cdot)-b(y)|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}|a_j(y)|dy\\
&\leqslant C\int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right||b(\cdot)-b_{B_j}|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}|a_j(y)|dy\\
&\quad+C\int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}|b_{B_j}-b(y)||a_j(y)|dy\\
&=: J_{13}+J_{14}.
\end{aligned}$$

For J_{13}, noting $s>q^+$, we denote $\tilde{q}(\cdot)>1$ and $\frac{1}{q(x)}=\frac{1}{\tilde{q}(x)}+\frac{1}{s}$. By Lemma 4.2 and Lemma 3.4 we have

$$\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right||b(\cdot)-b_{B_j}|\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant \left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\left\||b(\cdot)-b_{B_j}|\chi_k(\cdot)\right\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $|B_k|\leqslant 2^n$ and $x_k\in B_k$, by Lemma 3.5 we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\approx|B_k|^{\frac{1}{\tilde{q}(x_k)}}\approx\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}}.$$

When $|B_k|\geqslant 1$ we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\approx|B_k|^{\frac{1}{\tilde{q}(\infty)}}\approx\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}}.$$

So we obtain $\|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\approx\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}}$.

Meanwhile, by Lemma 3.6 we have

$$\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^n}-\frac{\Omega(\cdot)}{|\cdot|^n}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\\
&\leqslant 2^{(k-1)(\frac{n}{s}-n)}\left\{\frac{|y|}{2^k}+\int_{|y|/2^k}^{|y|/2^{k-1}}\frac{\omega_s(\delta)}{\delta}d\delta\right\}\\
&\leqslant 2^{(k-1)(\frac{n}{s}-n)}\left(2^{j-k+1}+2^{(j-k+1)\beta}\int_0^1\frac{\omega_s(\delta)}{\delta}d\delta\right)\\
&\leqslant C2^{(k-1)(\frac{n}{s}-n)}2^{(j-k)\beta}.
\end{aligned}$$

So by the generalized Hölder inequality we have

$$\begin{aligned} J_{13} &= \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| |b(\cdot) - b_{B_j}| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\ &\leqslant C(k-j) \|b\|_* 2^{(k-1)(\frac{n}{s}-n)} 2^{(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s}} \int_{B_j} |a_j(y)| dy \\ &\leqslant C(k-j) \|b\|_* 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \end{aligned} \tag{4.3}$$

For J_{14}, similar to the method of J_{13} we have

$$\begin{aligned} &\left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \|\chi_k(\cdot)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{s}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C 2^{(k-1)(\frac{n}{s}-n)} 2^{(j-k)\beta} \|\chi_{B_k}\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

So by Lemma 4.2 and the generalized Hölder inequality we have

$$\begin{aligned} J_{14} &= \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |b_{B_j} - b(y)| |a_j(y)| dy \\ &\leqslant C 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \int_{B_j} |b_{B_j} - b(y)| |a_j(y)| dy \\ &\leqslant C 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|(b_{B_j} - b)\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C \|b\|_* 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \end{aligned} \tag{4.4}$$

By (4.3), (4.4), Lemma 2.3 and Lemma 2.4 we have

$$\begin{aligned} &\|[b,\mu_\Omega](a_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C(k-j) \|b\|_* 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C(k-j) \|b\|_* 2^{(j-k)\beta} \|a_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\ &\leqslant C(k-j) 2^{-j\alpha+(j-k)(\beta+n\delta_2)} \|b\|_*. \end{aligned}$$

So we have

$$\begin{aligned} J_1 &\leqslant C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| (k-j) 2^{-j\alpha+(j-k)(\beta+n\delta_2)} \right)^{p_1} \\ &= C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| (k-j) 2^{(j-k)(\beta+n\delta_2-\alpha)} \right)^{p_1}. \end{aligned}$$

When $1<p_1<\infty$, take $1/p_1+1/p_1'=1$. Since $\beta+n\delta_2-\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}J_1 &\leqslant C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right)\\ &\quad\times\left(\sum_{j=-\infty}^{k-2}(k-j)^{p_1'}2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2}\right)^{p_1/p_1'}\\ &\leqslant C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right)\\ &= C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right)\\ &\leqslant C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.\end{aligned}\tag{4.5}$$

When $0<p_1\leqslant 1$, we have

$$\begin{aligned}J_1 &\leqslant C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}(k-j)^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\right)\\ &= C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}(k-j)^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\right)\\ &\leqslant C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.\end{aligned}\tag{4.6}$$

Thus, by (4.1)–(4.6) we complete the proof of Theorem 4.1.

Acknowledgements. The authors are very grateful to the referees for their valuable comments. This work was supported by China Postdoctoral Science Foundation Funded Project (Grant No. 2016M601105), Shandong Provincial Natural Science Foundation, China (Grant No. ZR2017MA041), a Project of Shandong Province Higher Educational Science and Technology Program (Grant No. J18KA225) and NSF of China (Grant No. 11761026).

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(Received January 3, 2018)

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