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COMPLETE MONOTONICITY PROPERTY FOR TWO FUNCTIONS RELATED TO THE q–DIGAMMA FUNCTION

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Abstract. In this paper, the complete monotonicity property for two functions involving the q-digamma function are proven for all positive real q and exploited to establish some sharp inequalities for the q-gamma, q-digamma and q-polygamma functions. Comparisons between our results with previous results are provided.

1. Introduction

The q-gamma function is defined for all positive real numbers x as

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \qquad 0 < q < 1, \tag{1.1}$$

and

$$\Gamma_q(x) = (q-1)^{1-x} q^{\frac{x(x-1)}{2}} \prod_{n=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \qquad q > 1. \tag{1.2}$$

From the previous definitions, for a positive real x and $q \geqslant 1$, we get

$$\Gamma_q(x) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{q^{-1}}(x). \tag{1.3}$$

The logarithmic derivative of the q-gamma function is the so-called q-digamma or q-psi function $\psi_q(x)$ which appeared in the work of Krattenthaler and Srivastava [1] (see also [2]). The nth derivatives of the q-digamma function are the so-called the q-polygamma functions denoted by $\psi_q^{(n)}(x)$; $n \in \mathbb{N}$. Therefore, the q-digamma function can be represented for all positive real x and $0 < q < 1$ as

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=1}^{\infty} \frac{q^{xk}}{1-q^k}, \tag{1.4}$$

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Ismail and Muldoon [3] provided an equivalent Stieltjes integral representation for the q-digamma function (1.4) as

$$\psi_q(x) = -\log(1-q) - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} d\gamma_q(t), \qquad x>0 \tag{1.5}$$

where $\gamma_q(t)$ is a discrete measure with positive masses $-\log q$ at the positive points $-k\log q$, $k\in\mathbb{N}$. For completeness, and economy of later statements, they include the value $q=1$ in the definition of $\gamma_q(t)$:

$$\gamma_q(t) = \begin{cases} -\log q \sum_{k=1}^{\infty} \delta(t+k\log q), & 0<q<1, \\ t, & q=1. \end{cases}$$

They used the identities

$$\frac{q^x \log q}{1-q^x} = -\int_0^\infty e^{-xt} d\gamma_q(t) \text{ and } \log(1-q^x) = -\int_0^\infty \frac{e^{-xt}}{t} d\gamma_q(t) \tag{1.6}$$

which follow easily from the definition of $d\gamma_q(t)$ for all $x>0$ and $0<q<1$.

With the Euler-Maclaurin formula, Moak [4] obtained the following asymptotic expansion for the q-digamma function

$$\psi_q(x) \sim \log[x]_q + \frac{1}{2}\frac{q^x\log q}{1-q^x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{1-\hat{q}^x}\right)^{2k} \hat{q}^x P_{2k-2}(\hat{q}^x), \quad x\to\infty \tag{1.7}$$

where B_k are the Bernoulli numbers, $[x]_q = (1-q^x)/(1-q)$,

$$\hat{q} = \begin{cases} q & \text{if } 0<q\leqslant 1 \\ q^{-1} & \text{if } q\geqslant 1, \end{cases},$$

and P_k is a polynomial of degree k satisfying

$$P_k(z) = (z-z^2)P'_{k-1}(z) + (kz+1)P_{k-1}(z), \qquad P_0 = P_{-1} = 1, \quad k\in\mathbb{N}.$$

Recently, the q-digamma function plays an important role in the framework of quantum statistical mechanics [5, 6, 7, 8, 9]. Also, numerous papers were published presenting remarkable inequalities involving the q-gamma and the q-digamma functions (see [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] and the extensive list of references given therein).

Motivated by these importance and applications, this paper is devoted to study and introduce some properties of the q-digamma function. Based on the approximation (1.7), the complete monotonicity property of the following two functions involving the q-digamma function will be investigated and exploited to provide sharp lower and upper bounds for the q-gamma, q-digamma and q-polygamma functions for all $q>0$:

$$F_a(x;q) = \psi_q(x+1) - \log[x+a]_q + \left(a-\frac{1}{2}\right)H(q-1)\log q \tag{1.8}$$

$$G_c(x;q) = \log[x]_q - \psi_q(x+1) - \frac{1}{2}\frac{q^{x+1/2}\log q}{1-q^{x+1/2}} + \frac{1}{6}\frac{q^{x+c}\log^2 q}{(1-q^{x+c})^2} \tag{1.9}$$

where $H(\cdot)$ denotes the Heaviside step function.

It is worth mentioning that a real-valued function f, defined on an interval I, is called completely monotonic, if f has derivatives of all orders which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geqslant 0, \qquad n \in \mathbb{N}_0; \quad x \in I. \tag{1.10}$$

If inequality (1.10) is strict for all $x \in I$ and for all integers $n \geqslant 0$, then f is said to be strictly completely monotonic. These functions have numerous applications in various branches of mathematics, such as probability theory, numerical analysis, and potential theory. Completely monotonic functions have attracted the attention of many authors who proved several results on these functions and gave many interesting examples. The study of monotonicity properties and complete monotonicity property are very useful to establish inequalities and sharp inequalities in various field and thus we study in the following sections the complete monotonicity property of the functions (1.8) and (1.9) to present best lower and upper bounds for the q-gamma, q-digamma and q-polygamma functions for all $q > 0$.

2. The first function (1.8)

This section is devoted to investigate the complete monotonicity property of the function $F_a(x;q)$ defined in (1.8) and how these results can be exploited to provide best lower and upper bounds for the q-gamma, q-digamma and q-polygamma functions for all $q > 0$. Before proving the main theorem in this section, we need the following lemma:

LEMMA 2.1. *Let the function*

$$a(t) = \frac{\log(e^t - 1) - \log t}{t} \tag{2.1}$$

be defined for all $t > 0$. Then, the function $a(t)$ is increasing on $(0,\infty)$ onto $(1/2, 1)$.

Proof. Differentiation gives $t^2 a'(t) = \lambda(t)$ where

$$\lambda(t) = \log t - \log(e^t - 1) - 1 + \frac{te^t}{e^t - 1}$$

Again, differentiating $\lambda(t)$ gives $t(e^t - 1)^2 \lambda'(t) = \delta(t)$ where

$$\delta(t) = e^{2t} + 1 - (t^2 + 2)e^t$$

which has the derivative

$$\delta'(t) = e^t(2e^t - 2 - 2t - t^2) > 0, \qquad t > 0.$$

Since $\delta(0) = 0$ and $\lim_{t\to 0} \lambda(t) = 0$, then $\lambda(t) > 0$ for all $t > 0$ which yields $a(t)$ is increasing on $(0,\infty)$. L'Hospital rule gives the limits $\lim_{t\to 0} a(t) = 1/2$ and $\lim_{t\to\infty} a(t) = 1$. $\square$

THEOREM 2.2. *Let x and q be reals with $q>0$, then the function $F_a(x;q)$ defined in (1.8) is strictly completely monotonic on $(-a,\infty)$ if and only if $a \leqslant a(-\log\hat{q}) \triangleq g(\hat{q})$ where $a(\cdot)$ defined in (2.1) and the function $-F_b(x;q)$ is strictly completely monotonic on $(-1,\infty)$ if $b \geqslant 1$.*

Proof. When $0<q<1$, (1.5) and (1.6) give

$$F_a(x;q) = \psi_q(x+1) - \log[x+a]_q = \int_0^\infty \frac{e^{-(x+a)t}}{t(e^t-1)} f(a,t)\,d\gamma_q(t).$$

Hence,

$$(-1)^n F_a^{(n)}(x;q) = \int_0^\infty \frac{t^{n-1}e^{-(x+a)t}}{e^t-1} f(a,t)\,d\gamma_q(t), \tag{2.2}$$

where

$$f(a,t) = e^t - 1 - te^{at} \tag{2.3}$$

According to formula (2.2) and the definition of the discrete measure $d\gamma_q(t)$, the function $F_a(x;q)$ is strictly completely monotonic on $(-a,\infty)$ if $f(a,t)d\gamma_q(t)>0$ for all $t>0$. That is, if $f(a,t)>0$ at the points $t=-k\log q$, $k\in\mathbb{N}$. Also, the function $-F_a(x;q)$ is strictly completely monotonic on $(-1,\infty)$ if $f(a,t)<0$ at the points $t=-k\log q$, $k\in\mathbb{N}$.

Clearly, the function $a\mapsto f(a,t)$ is decreasing on $\mathbb{R}$ and $f(0,t)>0$ and $f(1,t)<0$ for all $t>0$ which mean that $f(a,t)$ has a unique root function at $a=a(t)$ where $a(t)$ defined in (2.1). Thus the function $a\mapsto f(a,-k\log q)$ is decreasing on $\mathbb{R}$ and has only one root at $a=a(-k\log q)$, $k\in\mathbb{N}$ and $0<q<1$. From Lemma 2.1, the function $a(t)$ is increasing on $(0,\infty)$ and so the function $k\mapsto a(-k\log q)$ is increasing for all $k\in\mathbb{N}$. This reveals that

$$g(q) = a(-\log q) < a(-k\log q) < \lim_{t\to\infty} a(t) = 1.$$

Therefore, $f(a,t)>0$ if $a\leqslant g(q)$ and $f(a,t)<0$ if $a\geqslant 1$ at $t=-k\log q$, $k\in\mathbb{N}$ which conclude that $F_a(x;q)$ is strictly completely monotonic on $(-a,\infty)$ if $a\leqslant g(q)$ and $-F_a(x;q)$ is strictly completely monotonic on $(-1,\infty)$ if $a\geqslant 1$.

It is easy from logarithmic derivative of (1.3) to see that $F_a(x;q)=F_a(x;q^{-1})$ for all $q\geqslant 1$ which concludes that $F_a(x;q)$ is strictly completely monotonic on $(-a,\infty)$ if $a\leqslant g(\hat{q})$ and $-F_a(x;q)$ is strictly completely monotonic on $(-1,\infty)$ if $a\geqslant 1$ for all $q>0$.

Conversely, let $F_a(x;q)$ is strictly completely monotonic on $(-a,\infty)$ for all real $q>0$ which means that $\hat{q}^{-x}F_a(x;q)>0$. Based on approximation (1.7), we get

$$\begin{aligned}\lim_{x\to\infty}\hat{q}^{-x}F_a(x;q) &= \lim_{x\to\infty}\left[\hat{q}^{-x}\log\frac{1-\hat{q}^x}{1-\hat{q}^{x+a}} - \frac{1}{2}\frac{\log\hat{q}}{1-\hat{q}^x} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{1-\hat{q}^x}\right)^{2k} P_{2k-2}(\hat{q}^x)\right]\\ &= \hat{q}^a + \frac{\hat{q}\log\hat{q}}{1-\hat{q}} \geqslant 0\end{aligned}$$

which yields that $a \leqslant g(\hat{q})$. Here, we used L'Hospital rule, $P_k(0) = 1$, $k \in \mathbb{N}_0$, the well-known identity for the q-digamma function

$$\psi_q(x+1) = \psi_q(x) - \frac{q^x \log q}{1-q^x}$$

and the generating function of Bernoulli number

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} x^{2k} + B_0 + B_1 x.$$

Now, suppose that $-F_a(x;q)$, $a < 1$ is strictly completely monotonic on $(-1,\infty)$ for all real $q > 0$. This means that $F_a(x;q)$ is negative on $(-1,\infty)$. But, this contradicts

$$F_a(-a;q) = \psi_q(1-a) - \lim_{x \to -a} \log[x+a]_q + \left(a - \frac{1}{2}\right) H(q-1) \log q = \infty, \ \ a < 1.$$

This ends the proof. □

COROLLARY 2.3. *Let x and q be reals with $q > 0$. Then, the inequalities*

$$\log[x+a]_q + \left(\frac{1}{2} - a\right) H(q-1)\log q < \psi_q(x+1) < \log[x+b]_q + \left(\frac{1}{2} - b\right) H(q-1)\log q \tag{2.4}$$

hold true for all real $x > -a$, $a \leqslant g(\hat{q})$ and $b \geqslant 1$ with the best possible constants $a = g(\hat{q})$ and $b = 1$.

Also, for all positive integer n, the class of inequalities

$$\begin{aligned} (-1)^{n+1} \left(\frac{\log q}{1-q^{x+a}}\right)^n q^{x+a} P_{n-2}(q^{x+a}) &< (-1)^n \psi_q^{(n)}(x+1) \\ &< (-1)^{n+1} \left(\frac{\log q}{1-q^{x+b}}\right)^n q^{x+b} P_{n-2}(q^{x+b}) \end{aligned} \tag{2.5}$$

holds for all $x > -a$, $a \leqslant g(\hat{q})$ and $b \geqslant 1$ with best possible constants $a = g(\hat{q})$ and $b = 1$.

Proof. Theorem 2.2 tells that $F_b(x;q) < 0 < F_a(x;q)$ which is equivalent (2.4), and

$$(-1)^n F_b^{(n)}(x;q) < 0 < (-1)^n F_a^{(n)}(x;q), \qquad n \in \mathbb{N}$$

which is equivalent (2.5) with using the identity

$$\frac{d^n}{dx^n}\left[\frac{q^x \log q}{1-q^x}\right] = \left(\frac{\log q}{1-q^x}\right)^{n+1} q^x P_{n-1}(q^x), \qquad n \in \mathbb{N}_0$$

which was proved by Moak [4]. □

THEOREM 2.4. *Let x be non-negative real and q be positive real. Then, the inequalities*

$$[a]_q^{-a}[x+a]_q^{x+a}q^{(\frac{1}{2}-a)xH(q-1)}\exp\left(\frac{Li_2(1-q^{x+a})-Li_2(1-q^a)}{\log q}\right)\leqslant\Gamma_q(x+1)$$
$$\leqslant[b]_q^{-b}[x+b]_q^{x+b}q^{(\frac{1}{2}-b)xH(q-1)}\exp\left(\frac{Li_2(1-q^{x+b})-Li_2(1-q^b)}{\log q}\right)\tag{2.6}$$

hold true for all $0<a\leqslant g(\hat{q})$ and $b\geqslant 1$ with best possible constants $a=g(\hat{q})$ and $b=1$.

Proof. Let the function

$$f_a(x;q)=\log\Gamma_q(x+1)-(x+a)\log[x+a]_q-\frac{\mathrm{Li}_2(1-q^{x+a})}{\log q}$$
$$+\left(a-\frac{1}{2}\right)xH(q-1)\log q\tag{2.7}$$

be defined for all $x>-a$ and $q>0$. Differentiation gives $f_a'(x;q)=F_a(x;q)$ where $F_a(x;q)$ defined as in (1.8). By virtue of Theorem 2.2, we get the function $f_a(x;q)$ is strictly increasing on $(-a,\infty)$ if $a\leqslant g(\hat{q})$ and the function $f_b(x;q)$ strictly decreasing on $(-1,\infty)$ if $b\geqslant 1$. Therefore, for $x\geqslant 0$, we have

$$f_a(x;q)\geqslant f_a(0;q)=-a\log[a]_q-\frac{\mathrm{Li}_2(1-q^a)}{\log q},\qquad 0<a\leqslant g(\hat{q})$$

and

$$G_b(x;q)\leqslant G_b(0;q)=-b\log[b]_q-\frac{\mathrm{Li}_2(1-q^b)}{\log q},\qquad b\geqslant 1$$

which are equivalent (2.6). This ends the proof. □

THEOREM 2.5. *Let x be non-negative real and q be positive real. Then, the double inequality*

$$[a]_q^{-a}[x+a]_q^{x+a}q^{(\frac{1}{2}-a)H(q-1)}\exp\left(\frac{Li_2(1-q^{x+a})-Li_2(1-q^a)}{\log q}\right)\leqslant\Gamma_q(x+1)$$
$$<\sqrt{2\pi}|1-q|^{a-\frac{1}{2}}q^{(\frac{1}{2}(1-a^2)+(\frac{1}{2}-a)x)H(q-1)}S_{\hat{q}}[x+a]_q^{x+a}\exp\left(\frac{Li_2(1-q^{x+a})}{\log q}\right)\tag{2.8}$$

holds true for $a=g(\hat{q})$. In particular, if $a=1/2$

$$\sqrt{1+\sqrt{q}}[x+1/2]_q^{x+1/2}\exp\left(\frac{Li_2(1-q^{x+1/2})-Li_2(1-q^{1/2})}{\log q}\right)\leqslant\Gamma_q(x+1)$$
$$<\sqrt{2\pi}q^{\frac{3}{8}H(q-1)}S_{\hat{q}}[x+1/2]_q^{x+1/2}\exp\left(\frac{Li_2(1-q^{x+1/2})}{\log q}\right)\tag{2.9}$$

holds for all $x \geqslant 0$ *and* $q > 0$*. Also, the inequality*

$$\sqrt{2\pi|1-q|}q^{-\frac{1}{2}xH(q-1)}S_{\hat{q}}[x+1]_q^{x+1}\exp\left(\frac{Li_2(1-q^{x+1})}{\log q}\right) \leqslant \Gamma_q(x+1)$$
$$< q^{-\frac{1}{2}xH(q-1)}[x+1]_q^{x+1}\exp\left(\frac{Li_2(1-q^{x+1})-Li_2(1-q)}{\log q}\right) \tag{2.10}$$

holds for all $x \geqslant 0$ *and* $q > 0$*, where* S_q *is defined as*

$$S_q = q^{\frac{-1}{24}}\sqrt{\frac{q-1}{\log q}}\sum_{m=-\infty}^{\infty}\left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right), \quad r = e^{\frac{4\pi^2}{\ln q}}. \tag{2.11}$$

Proof. Since the function $f_a(x;q)$ defined in (2.4) is strictly increasing on $(-a,\infty)$ if $a \leqslant g(\hat{q})$, then the function $f_a(x;q)$, $a = g(\hat{q})$ is increasing on $(-a,\infty)$ and can be rewritten in the form $f_a(x;q) = \mu_q(x) + \nu_q(x)$ where

$$\mu_q(x) = \log\Gamma_q(x) - \left(x - \frac{1}{2}\right)\log[x]_q - \frac{\mathrm{Li}_2(1-q^x)}{\log q} \tag{2.12}$$
$$\nu_q(x) = x\left(\log(1-q^x) - \log(1-q^{x+a})\right) + \frac{\mathrm{Li}_2(1-q^x) - \mathrm{Li}_2(1-q^{x+a})}{\log q}$$
$$+ \frac{1}{2}\log[x]_q - a\log[x+a]_q + \left(a - \frac{1}{2}\right)xH(q-1)\log q$$

The relation (2.5) in [23] shows that

$$\lim_{x\to\infty}\mu_q(x) = \log\sqrt{2\pi} + \log S_{\hat{q}} + \frac{1}{2}H(q-1)\log q, \qquad q > 0. \tag{2.13}$$

Using the well known identity for dilogarithm function [29]

$$\mathrm{Li}_2\left(\frac{z-1}{z}\right) = -\mathrm{Li}_2(1-z) - \frac{1}{2}\log^2 z$$

to rewrite $\nu_q(x)$ for all $q > 0$ as

$$\nu_q(x) = x\left(\log(1-\hat{q}^x) - \log(1-\hat{q}^{(x+a)})\right) + \frac{\mathrm{Li}_2(1-\hat{q}^x) - \mathrm{Li}_2(1-\hat{q}^{x+a})}{\log\hat{q}}$$
$$+ \frac{1}{2}\log[x]_{\hat{q}} - a\log[x+a]_{\hat{q}} - \frac{1}{2}(1-a)^2\log q$$

Using L'Hospital's rule would yield

$$\lim_{x\to\infty} x\log(1-q^{x+a}) = 0, \qquad \text{for all} \quad a \in \mathbb{R}, \quad 0 < q < 1$$

Hence, it is easy to see that

$$\lim_{x\to\infty}\nu_q(x) = \left(a - \frac{1}{2}\right)\log|1-q| - \frac{1}{2}a^2H(q-1)\log q, \qquad q > 0.$$

In view of the previous, we conclude that

$$\lim_{x\to\infty} f_a(x;q) = \log(\sqrt{2\pi}S_{\hat{q}}) + \left(a-\frac{1}{2}\right)\log|1-q| + \frac{1}{2}(1-a^2)H(q-1)\log q,\ q>0.$$

The increasing of the function $f_a(x;q);\ a=g(\hat{q})$ on $(-a,\infty)$ yields

$$f_a(0;q) \leqslant f_a(x;q) < \lim_{x\to\infty} f_a(x;q)$$

which is equivalent (2.8). Also, the decreasing of the function $f_1(x;q)$ on $(-1,\infty)$ yields

$$\lim_{x\to\infty} f_1(x;q) < f_1(x;q) \leqslant f_1(0;q)$$

which is equivalent (2.10) This completes the proof. $\square$

REMARK 2.6. In Theorem 2.1 of Batir [11], it was proved, for all positive reals x and q, that

$$\log[x+\alpha]_q < \psi_q(x+1) < \log[x+\beta]_q \tag{2.14}$$

with the best possible constant $\alpha = \begin{cases} g(q),\ 0<q<1 \\ \dfrac{1}{2},\ q>1 \end{cases}$ and

$$\beta = \frac{\log(1-(1-q)e^{\psi_q(1)})}{\log q}.$$

Since $\psi_q(1)<0$ for all $q>0$, then $\beta<1$ for all $q>0$.

When $0<q<1$, it is clear that:

1. The lower bound of (2.14) is the same lower bound of (2.4),
2. The upper bound of (2.14) is better than the upper bound of (2.4).

When $q>1$, we have two cases:

1. To compare the lower bounds of (2.4) and (2.14), we have

$$\log[x+a]_q + \left(\frac{1}{2}-a\right)\log q - \log[x+1/2]_q = \log\left(1+\frac{1-q^{\frac{1}{2}-a}}{q^{x+\frac{1}{2}}-1}\right) > 0$$

   where $1/2 \leqslant a = g(q^{-1}) \leqslant 1$, which emphasizes that the lower bound of (2.4) is bigger (better) than the lower bound of (2.14).

2. To compare the upper bounds of (2.4) and (2.14), let the function

$$t(x) = \log[x+1]_q - \frac{1}{2}\log q - \log[x+\beta]_q$$

be defined for all $x > 0$. Differentiation gives

$$t'(x) = \frac{q^{\beta} - q}{(q^{x+1}-1)(q^{x+\beta}-1)} q^x \log q \leqslant 0, \qquad \beta \leqslant 1$$

which yields that there exists a unique root depending on q at $x = x(q)$ where

$$x(q) = \frac{\log(\sqrt{q}-1) - \log(q^{\beta+\frac{1}{2}} - q)}{\log q}.$$

Therefore, the function $t(x) > 0$ if $x < x(q)$ and $t(x) < 0$ if $x > x(q)$ which conclude that the upper bound of (2.14) is better than the upper bound of (2.4) if $x < x(q)$ and the reverse is true if $x > x(q)$. Our numerical experiments carried out with the packet program Mathematica show that $x(2) = 2.54626$, $x(3) = 1.44311$, $x(10) = 0.485145$, $x(100) = 0.110505$, $x(1000) = 0.0309689$. It is noting that $x(q)$ is decreasing and approaches zero for large q.

3. The second function (1.9)

In this section, we investigate the complete monotonicity property of the function $G_c(x;q)$ defined in (1.9) and how these results can be exploited to provide best lower and upper bounds for the q-gamma, q-digamma and q-polygamma functions for all $q > 0$. Before proving the main theorem in this section, we need the following lemmas:

LEMMA 3.1. *Let the function*

$$c(t) = \frac{\log(t^2(e^t-1)) - \log\left(6(e^t-t-1) - 3te^{-\frac{1}{2}t}(e^t-1)\right)}{t} \tag{3.1}$$

be defined for all $t > 0$. Then, the function $c(t)$ is decreasing on $(0,\infty)$ onto $(0,3/8)$.

Proof. Differentiation gives $t^2 c'(t) = d(t)$ where

$$\begin{aligned} d(t) = {} & \log\left(6(e^t-t-1) - 3te^{-\frac{1}{2}t}(e^t-1)\right) - \log(t^2(e^t-1)) \\ & - \frac{t(t+2)(e^t-1)^2 - 4e^{\frac{1}{2}t}(2e^{2t} - (t^2+t+4)e^t + t + 2)}{2(e^t-1)(2e^{\frac{1}{2}t}(e^t-t-1) - t(e^t-1))}. \end{aligned}$$

Hence,

$$d'(t) = \frac{E(t)}{2t(e^t-1)^2(2e^{\frac{1}{2}t}(e^t-t-1) - t(e^t-1))^2}$$

where

$$\begin{aligned} E(t) = {} & -16e^{5t} + 2(4t^3 - 9t^2 + 16t + 32)e^{4t} + 32(t^2 - 3t - 3)e^{3t} \\ & - 4(2t^4 + 2t^3 + 3t^2 - 24t - 16)e^{2t} - 16(2t+1)e^t - 2t^2 \\ & + t(t^2 - 4t + 16)e^{\frac{9}{2}t} - t(t^3 + 4t^2 - 8t + 64)e^{\frac{7}{2}t} - t(5t^3 - 6t^2 - 96)e^{\frac{5}{2}t} \\ & + t(5t^3 - 4t^2 - 8t - 64)e^{\frac{3}{2}t} + t(t^3 + t^2 + 4t + 16)e^{\frac{1}{2}t} \end{aligned}$$

which can be represented in series form as

$$E(t)=\sum_{n=10}^{\infty}\frac{t^n}{n!}\Theta(n)$$

where

$$\begin{aligned}\Theta(n)=&-16\times 5^n+(n^3-12n^2+75n+512)2^{2n-3}+32(n^2-10n-27)3^{n-2}\\&-(n^4-4n^3+11n^2-104n-128)2^{n-1}-16(2n+1)\\&+n(n^2-21n+344)\left(\frac{9}{2}\right)^{n-3}-n(n^3+8n^2-129n+2864)\left(\frac{7}{2}\right)^{n-4}\\&-2n(n^3-9n^2+20n-312)\left(\frac{5}{2}\right)^{n-3}+n(5n^3-36n^2+55n-240)\left(\frac{3}{2}\right)^{n-4}\\&+n(2n^3-11n^2+21n-8)\left(\frac{1}{2}\right)^{n-3}\\ \triangleq&\,\theta_1(n)+\theta_2(n)+\theta_3(n)+\theta_4(n)+\theta_5(n)\end{aligned}$$

where

$$\begin{aligned}\theta_1(n)&=-7\times 5^n+(n^3-12n^2+75n+512)2^{2n-3}-(n^4-4n^3+11n^2-104n-128)2^{n-1}\\\theta_2(n)&=-0.2\times 5^n+32(n^2-10n-27)3^{n-2}\\\theta_3(n)&=-8.8\times 5^n+n(n^2-21n+344)\left(\frac{9}{2}\right)^{n-3}+n(5n^3-36n^2+55n-240)\left(\frac{3}{2}\right)^{n-4}\\\theta_4(n)&=n(2n^3-11n^2+21n-8)\left(\frac{1}{2}\right)^{n-3}-n(n^3+8n^2-129n+2864)\left(\frac{7}{2}\right)^{n-4}\\\theta_5(n)&=-16(2n+1)-2n(n^3-9n^2+20n-312)\left(\frac{5}{2}\right)^{n-3}\end{aligned}$$

In order to prove the negativity of $\theta_1(n)$, rewrite it as $8\theta_1(n))=5^n\phi(n)$ where

$$\phi(n)=-56+(n^3-12n^2+75n+512)\left(\frac{4}{5}\right)^n-(n^4-4n^3+11n^2-104n-128)\left(\frac{2}{5}\right)^n.$$

Forward shift operator gives

$$5\Delta\phi(n)=-\left(\frac{2}{5}\right)^n\left[(n^3-24n^2+159n+256)2^n-4(3n^4-20n^3+45n^2-30n-192)\right].$$

Mathematical induction yields $2^n>n^3$ for all $n\geqslant 10$ which can be used to show that

$$5\Delta\phi(n)<-\left(\frac{2}{5}\right)^n\left[n^6-24n^5+147n^4+336n^3-180n^2+136n+768)\right]<0,\ n\geqslant 10$$

which means that $\phi(n)$ is decreasing for all integer $n \geqslant 10$. Since $\phi(21) \sim -0.146073$, then the function $\phi(n) < 0$ for all integer $n \geqslant 21$ and so does the function $\theta_1(n)$. Similarly, we can deduce that $\theta_2(n) < 0$ for all $n \geqslant 10$, $\theta_3(n) < 0$ for all $n \geqslant 12$ and $\theta_4(n) < 0$ for all $n \geqslant 10$. By substituting $n = 10, 11, \cdots, 20$ into $\Theta(n)$, we find that $\Theta(n) < 0$ for all $n \in \{10, 11, \cdots, 20\}$. In view of these, we can declare that the function $\Theta(n) < 0$ for all $n \geqslant 10$ which reveals that $E(t) < 0$ for all $t > 0$. Thus, the function $d(t)$ is decreasing on $(0, \infty)$. L'Hospital rule leads to $\lim_{t \to 0} d(t) = 0$ which yields the function $d(t) < 0$ for all $t > 0$ and so does the function $c'(t)$. Therefore, the function $c(t)$ is decreasing on $(0, \infty)$. Again, L'Hospital rule gives $\lim_{t \to 0} c(t) = 3/8$ and $\lim_{t \to \infty} c(t) = 0$. □

LEMMA 3.2. *Let the function*

$$g(t,c) = t^2(e^t - 1)e^{-ct} + 3te^{-\frac{1}{2}t}(e^t - 1) - 6(e^t - t - 1) \tag{3.2}$$

be defined for all $t > 0$ *and* $c \geqslant 0$. *Then, the function* $g(t,c)$ *has a unique root function depending on* t *at* $c = c(t)$ *where* $c(t)$ *defined in (3.1).*

Proof. The exponential expansion can be used to rewrite $g(t,c)$ as

$$g(t,c) = \sum_{n=4}^{\infty} \frac{t^n}{n!} \Lambda(n,c)$$

where

$$\Lambda(n,c) = n(n-1)\left[(1-c)^{n-2} - (-1)^n c^{n-2}\right] + 3n\left(\frac{1}{2}\right)^{n-1}(1 + (-1)^n) - 6$$

When $c = 0$, we have

$$\Lambda(2n,0) = 2n(2n-1) - 6 + 12n\left(\frac{1}{2}\right)^n > 0, \qquad n \geqslant 2,$$
$$\Lambda(2n+1,0) = 2n(2n+1) - 6 > 0, \qquad n \geqslant 2.$$

which conclude that $\Lambda(n,0) > 0$ for all $n \geqslant 4$ and thus the function $g(t,0) > 0$ for all $t > 0$.

When $c = 1/2$ with using $2^n \geqslant 2n$ for all $n \geqslant 2$, we have

$$\Lambda(2n,1/2) = 12n\left(\frac{1}{2}\right)^{2n} - 6 < \frac{3}{n} - 6 < 0, \qquad n \geqslant 2,$$
$$\Lambda(2n+1,1/2) = 8n(2n+1)\left(\frac{1}{2}\right)^{2n} - 6 < \frac{4n+2}{n} - 6 < 0, \qquad n \geqslant 2.$$

which conclude that $\Lambda(n,1/2) < 0$ for all $n \geqslant 4$ and thus the function $g(t,1/2) < 0$ for all $t > 0$.

In view of the previous and the fact that the function $c \mapsto g(t,c)$ is decreasing on $[0, \infty)$ for all $t > 0$, we arrive at the desired result. □

THEOREM 3.3. *Let x and q be positive reals and $c \geqslant 0$. Then, the function $G_c(x;q)$ defined in (1.9) is strictly completely monotonic function if and only if $c = 0$ and the function $-G_c(x;q)$ is strictly completely monotonic function if and only if $c > c(-\log\hat{q}) \triangleq h(\hat{q})$ where $c(\cdot)$ defined in (3.1).*

Proof. When $0 < q < 1$, the relations (1.5) and (1.6) give

$$G_c(x;q) = \frac{1}{6}\int_0^\infty \frac{e^{-xt}}{t(e^t-1)} g(t,c)\,d\gamma_q(t)$$

where $g(t,c)$ defined in (3.2). Hence

$$(-1)^n G_c^{(n)}(x;q) = \frac{1}{6}\int_0^\infty \frac{t^{n-1}e^{-xt}}{e^t-1} g(t,c)\,d\gamma_q(t)$$

According to the former formula and the definition of the discrete measure $d\gamma_q(t)$, the function $G_c(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $g(c,t)d\gamma_q(t) > 0$ for all $t > 0$. That is, if $g(c,t) > 0$ at the points $t = -k\log q$, $k \in \mathbb{N}$. Also, the function $-G_c(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $g(c,t) < 0$ at the points $t = -k\log q$, $k \in \mathbb{N}$.

From Lemma 3.1, the function $c(t)$ is decreasing on $(0,\infty)$ and so the function $k \mapsto c(-k\log q)$ is also decreasing for all $k \in \mathbb{N}$. This reveals that

$$0 = \lim_{t\to\infty} c(t) < c(-k\log q) < c(-\log q) = h(q).$$

Therefore, $g(t,c) < 0$ if $c \geqslant h(q)$ and $g(t,c) > 0$ if $c = 0$ at $t = -k\log q$, $k \in \mathbb{N}$ which conclude that $-G_c(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $c \geqslant h(q)$ and $G_c(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $c = 0$.

It is not difficult from logarithmic derivative of (1.3) to show that $G_c(x;q) = G_c(x;q^{-1})$ for all $q \geqslant 1$ which concludes that $-G_c(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $c \geqslant h(\hat{q})$ and $G_c(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $c = 0$ for all $q > 0$.

Conversely, let $-G_c(x;q)$ is strictly completely monotonic on $(0,\infty)$ for all real $q > 0$ which means that $\hat{q}^{-x}G_c(x;q) \leqslant 0$. Based on the well-known identity for the q-digamma function and the generating function of Bernoulli number mentioned in the proof of Theorem 2.2 and the approximation (1.7), we get

$$\begin{aligned}\lim_{x\to\infty}\hat{q}^{-x}G_c(x;q) &= \lim_{x\to\infty}\Bigg[\frac{1}{2}\frac{\log\hat{q}}{1-\hat{q}^x} - \frac{1}{2}\frac{\hat{q}^{\frac{1}{2}}\log\hat{q}}{1-\hat{q}^{x+\frac{1}{2}}} + \frac{1}{6}\frac{\hat{q}^c\log^2\hat{q}}{(1-\hat{q}^{x+c})^2} \\ &\qquad + \sum_{k=1}^\infty \frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{1-\hat{q}^x}\right)^{2k} P_{2k-2}(\hat{q}^x)\Bigg] \\ &= \frac{1}{6}\left[\hat{q}^c\log^2\hat{q} - \frac{3\hat{q}^{\frac{1}{2}}(1-\hat{q})\log\hat{q} + 6(1-\hat{q}+\hat{q}\log\hat{q})}{1-\hat{q}}\right] \leqslant 0\end{aligned}$$

which yields that $c \geqslant h(\hat{q})$.

Now, suppose that $G_c(x;q)$, $c>0$ is strictly completely monotonic on $(0,\infty)$ for all real $q>0$. This means that $G_c(x;q)$, $c>0$ is positive on $(0,\infty)$. But, this contradicts

$$\lim_{x\to 0} G_c(x;q) = -\infty, \quad c>0.$$

This ends the proof. $\square$

COROLLARY 3.4. *Let x and q be positive reals. Then, the inequalities*

$$\frac{1}{6}\frac{q^{x+\alpha}\log^2 q}{(1-q^{x+\alpha})^2} < \psi_q(x+1) - \log[x]_q + \frac{1}{2}\frac{q^{\frac{x+1}{2}}\log q}{1-q^{x+\frac{1}{2}}} < \frac{1}{6}\frac{q^{x+\beta}\log^2 q}{(1-q^{x+\beta})^2} \tag{3.3}$$

hold true for all $\alpha \geqslant h(\hat{q})$ and $\beta = 0$ with the best possible constants $\alpha = h(\hat{q})$ and $\beta = 0$.

Also, for all positive integer n, the class of inequalities

$$\begin{aligned}
&(-1)^n\frac{1}{6}\left(\frac{\log q}{1-q^{x+\alpha}}\right)^{n+2} q^{x+\alpha}P_n(q^{x+\alpha}) < (-1)^n\psi_q^{(n)}(x+1)\\
&+(-1)^n\left(\frac{\log q}{1-q^x}\right)^n q^x P_{n-2}(q^x) + (-1)^n\frac{1}{2}\left(\frac{\log q}{1-q^{x+\frac{1}{2}}}\right)^{n+1} q^{x+\frac{1}{2}}P_n(q^{x+\frac{1}{2}})\\
&< (-1)^n\frac{1}{6}\left(\frac{\log q}{1-q^{x+\beta}}\right)^{n+2} q^{x+\beta}P_n(q^{x+\beta})
\end{aligned} \tag{3.4}$$

holds true for all $\alpha \geqslant h(\hat{q})$ and $\beta = 0$ with best possible constants $\alpha = h(\hat{q})$ and $\beta = 0$.

Proof. Theorem 3.3 tells that $G_\alpha(x;q) < 0 < G_\beta(x;q)$ which is equivalent (3.3), and

$$(-1)^n G_\alpha^{(n)}(x;q) < 0 < (-1)^n G_\beta^{(n)}(x;q), \qquad n \in \mathbb{N}$$

which is equivalent (3.4) with using the identity mentioned in the proof of Corollary 2.3. $\square$

THEOREM 3.5. *Let x and q be positive real numbers. Then, the inequalities*

$$\begin{aligned}
&\sqrt{2\pi}S_{\hat{q}}q^{\frac{5}{12}H(q-1)}[x]_q^x\sqrt{[x+1/2]_q}\exp\left(\frac{Li_2(1-q^x)}{\log q} + \frac{1}{6}\frac{q^{x+d}\log q}{1-q^{x+d}}\right) < \Gamma_q(x+1)\\
&< \sqrt{2\pi}S_{\hat{q}}q^{\frac{5}{12}H(q-1)}[x]_q^x\sqrt{[x+1/2]_q}\exp\left(\frac{Li_2(1-q^x)}{\log q} + \frac{1}{6}\frac{q^{x+c}\log q}{1-q^{x+c}}\right)
\end{aligned} \tag{3.5}$$

hold true for all $c \geqslant h(\hat{q})$ and $d = 0$ with best possible constants $c = h(\hat{q})$ and $d = 0$.

Proof. Let $c \geqslant 0$ and the function

$$\begin{aligned}
T_c(x;q) &= x\log[x]_q - \log\Gamma_q(x+1) + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2}\log[x+1/2]_q\\
&\quad + \frac{1}{6}\frac{q^{x+c}\log q}{1-q^{x+c}} + \log\sqrt{2\pi} + \log S_{\hat{q}} + \frac{5}{12}H(q-1)\log q
\end{aligned} \tag{3.6}$$

be defined for all positive reals x and q. Differentiation gives $T_c'(x;q) = G_c(x;q)$ where $G_c(x;q)$ defined in (1.9) which means $-T_c'(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $c \geqslant h(\hat{q})$ and $T_c'(x;q)$ is strictly completely monotonic on $(0,\infty)$ if $c = 0$. The function $T_c(x;q)$ can be represented as $T_c(x;q) = u_q(x) - \mu_q(x)$ where $\mu_q(x)$ defined in (2.12) and

$$u_q(x) = \frac{1}{2}\log[x+1/2]_q - \frac{1}{2}\log[x]_q + \frac{1}{6}\frac{q^{x+c}\log q}{1-q^{x+c}} + \log(\sqrt{2\pi}S_{\hat{q}}) + \frac{5}{12}H(q-1)\log q.$$

Using L'Hospital rule gives

$$\lim_{x\to\infty} u_q(x) = \log\sqrt{2\pi} + \log S_{\hat{q}} + \frac{1}{2}H(q-1)\log q$$

which means with using (2.13) that $\lim_{x\to\infty} T_c(x;q) = 0$. Therefore, for all positive reals x and q, we have $T_c(x;q) > 0$ for all $c \geqslant h(\hat{q})$ and $T_c(x;q) < 0$ if $c = 0$. That is

$$T_0(x;q) < 0 < T_c(x;q), \qquad c \geqslant h(\hat{q})$$

which is equivalent (3.5). □

COROLLARY 3.6. *Let x and q be positive real numbers. Then, the inequality*

$$\begin{aligned} &\alpha_q [x]_q^x \sqrt{[x+1/2]_q}\exp\left(\frac{Li_2(1-q^x)}{\log q} + \frac{1}{6}\frac{q^{x+c}\log q}{1-q^{x+c}}\right) \leqslant \Gamma_q(x+1) \\ &< \beta_q [x]_q^x \sqrt{[x+1/2]_q}\exp\left(\frac{Li_2(1-q^x)}{\log q} + \frac{1}{6}\frac{q^{x+c}\log q}{1-q^{x+c}}\right) \end{aligned} \tag{3.7}$$

holds true, where $c = h(\hat{q})$ and

$$\begin{aligned} \alpha_q &= \frac{1}{\sqrt{[1/2]_q}}\exp\left(-\frac{1}{6}\frac{q^c\log q}{1-q^c}\right), \\ \beta_q &= \sqrt{2\pi}S_{\hat{q}}q^{\frac{5}{12}H(q-1)} \end{aligned}$$

are the best possible constants.

Proof. The proof of this corollary comes immediately from the decreasing monotone of the function $T_c(x;q)$, that is

$$\begin{aligned} 0 = \lim_{x\to\infty} T_c(x;q) &< T_c(x;q) \leqslant T_c(0;q) \\ &= \log(\sqrt{2\pi[1/2]_q}S_{\hat{q}}) + \frac{1}{6}\frac{q^c\log q}{1-q^c} + \frac{5}{12}H(q-1)\log q \end{aligned}$$

which is equivalent (3.7). □

REMARK 3.7. Batir [30] proved the complete monotonicity property of the function $T_c(x;q)$ defined in (3.6) when $q \to 1$ and $c = 3/8$ and exploited this result to provide lower and upper bounds for the gamma function and so some results in this section generalize and refine some results of Batir [30] for all $q > 0$.

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